General Tips:

- **Draw**ing a diagram helps massively. Regardless of whether it's abstract vectors or application.
- **Practice** the standard methods and get good at them (Foot of perpendicular, line of intersection, finding angles between vectors, etc.)
- Be familiar with the different forms of lines and planes (**Cartesian, Parametric, Vector, Scalar product**)
- Understand the geometric significance of the dot/cross products
  - **Dot:** Test for perpendicular, length of projection, angle, etc.
  - **Cross:** Area swept out, normal vector, length of opposite etc.

A summary of Basic Formulae:

Products:

$a \bullet b = |a||b|cos(\Theta) = a1b1 + a2b2 + a3b3\ (in\ 3D)$

$a \times b = |a||b|sin(\Theta)n$, where n is the unit vector perpendicular to both vectors, whose direction is given by the right hand rule. (Component-wise formula given in MF26, although if I'm not mistaken, they removed it from MF27.)

In particular:
If a and b are perpendicular, $a \bullet b = 0$
If a and b are parallel, $a \times b = 0$
$|a \times b|$ gives the area of the parallelogram whose sides are a and b.

Equation(s) of a line:

$r = a + \lambda d$, where a is the position vector of a point on the line, and d is the direction vector of the line. (A line is defined by a point and a direction.)

By writing the above equation out component-wise and equating λ in each, we get:

$\frac{x-a1}{d1} = \frac{y-a2}{d2} = \frac{z-a3}{d3}$, provided that none of $d1, d2, d3$ are zero. If one is 0, one the coordinates is fixed.

E.g. $r = (2, 1, 0) + \lambda(1, 1, 0) \leftrightarrow x - 2 = y - 1,\ z = 0$.

Equation(s) of a plane:

$r \bullet n = a \bullet n$, where n is the normal vector to the plane, and a is the position vector of a given point in the plane. (A plane is defined by a point and a normal). We can also get the Cartesian equation by writing $r = (x, y, z),\ n = (n1, n2, n3)$ and evaluating the right hand side. Note that if n is a unit vector, $a \bullet n$ represents the length of projection of a onto n, which is the perpendicular distance from the origin to the plane.

A plane can also be defined in terms of 2 linearly independent(i.e. not parallel) vectors parallel to it, or equivalently, three non-collinear points it contains. The difference vectors between the 3 points give you the 2 vectors it contains.

$r = a + \lambda b + \mu c$, where a is a point in the plane, b and c are vectors parallel to the plane, with $b \neq kc$ for any real scalar k, and λ,μ real parameters.

E.g. $r = (1, 1, 1) + \lambda(1, 0, 0) + \mu(0, 1, 0)$ defines a plane.
$r = (1, 2, 3) + \lambda(1, 0, 0) + \mu(-1, 0, 0)$ does not define a plane, as the two vectors given are parallel.

Taking the cross product of the two vectors gives you the normal to the plane, thereby letting you convert from this parametric form to the cartesian form. Converting from the cartesian to the parametric is simply a matter of finding 3 non-collinear points on the plane, but I have never seen a question ask for this.

Angles:

For any two vectors a, b, the angle Θ between them is given by:

$$cos(\Theta) = \frac{(a \bullet b)}{|a||b|}$$

For any two planes with normal vectors $n1, n2$, the acute angle between the planes is given by:

$$cos(\Theta) = \frac{|(n1 \bullet n2)|}{|n1||n2|}$$

For a line with direction vector $d1$ intersecting a plane with normal $n1$, the angle it makes with the plane is given by:

$$sin(\Theta) = \frac{|(n1 \bullet d1)|}{|n1||d1|}$$

Distances/Lengths:

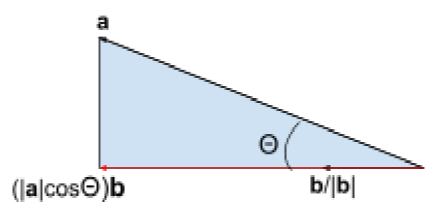
For two parallel planes, $r \bullet n = d1$ and $r \bullet n = d2$, the distance between them is given by:

$$D = |d1 - d2|/|n|.$$

(This formula can be derived by considering the projection of position vectors of points on the two planes onto their common normal.)

Length of projection of a onto b is given by:

$$a \bullet \frac{b}{|b|} = a \bullet \hat{b}$$



Given two points A and B with position vectors a and b, the position vector of the point C, that has the property AC:CB = λ: μ , is given by:

$$OC = (\lambda OB + \mu OA)/(\lambda + \mu)$$

Finding the shortest distance/foot of perpendicular between X and Y.

Most of the time in an Applied Vectors question, you will be asked to find the shortest distance between two objects, X and Y. Most of the time, they fall into one of the following categories:

Shortest Distance between:

1) A point, X, and a line, **l**.
2) A point X, and a plane, **P**.

In addition to this, they may ask you to find the foot of perpendicular/points at either object where they are closest to each other. Solving for one tends to make solving for the other quite simple, if you haven't already found it. I will outline some methods to solve these types of questions below.

Case 1:

Suppose we have a point X, with position vector **OX**, and a line **l**. Let **d** be the direction vector of the line. We wish to find the shortest distance between X and **l**.

Method 1:

We will solve for the foot of perpendicular, F, from X to **l**, then find the length of XF. Note that the vector **XF** satisfies:

$$XF \bullet d = 0$$

That is, XF is perpendicular to the line **l**. Since F lies on **l**, we may sub in the general equation for the position vector of a point on **l**, and solve for the parameter that corresponds to the position vector **OF**. We can then solve for **XF** and hence the shortest distance.

Example:

Find the shortest distance between the point A, given by coordinates (2,2,-6) and the line given by: $x - 1 = 2 - y = z + 6$

We first convert the line equation from Cartesian to Parametric. If you're not confident in doing this, please review your notes. We obtain:

$$OR = (1 + \lambda, 2 - \lambda, -6 + \lambda)$$

Thus: $XR = (-1 + \lambda, -\lambda, \lambda)$

And hence: $XF = (-1 + \lambda, -\lambda, \lambda)$, for some λ.

Taking the dot product:

$$XF \bullet d = (-1 + \lambda, -\lambda, \lambda) \bullet (1, -1, 1) = 0$$

Where the last equality follows from the two vectors being perpendicular.

We get: $3\lambda - 1 = 0$ and thus, $\lambda = 1/3$

Subbing in $\lambda = 1/3$, we get:

$XF = (-2/3, -1/3, 1/3)$

And so the shortest distance is $|XF| = \sqrt{6} / 3$

<u>Method 2:</u>

Let **OR** be the position vector of an arbitrary point on **l**. We will find **XR** and hence obtain an expression for $|XR|$ in terms of the parameter λ. This will always be the square root of a quadratic function. Since the square root is an increasing function, it is minimised when the quadratic inside it is minimised. This will give us the minimising value of λ, and hence the minimal value of $|XR|$.

Example:

Find the shortest distance between the point A, given by coordinates (2,2,-6) and the line given by: $x - 1 = 2 - y = z + 6$

From above: $XR = (-1 + \lambda, -\lambda, \lambda)$

And thus:

$$|XR| = \sqrt{(\lambda - 1)^2 + \lambda^2 + \lambda^2}$$

$$= \sqrt{3\lambda^2 - 2\lambda + 1}$$

Recall from E-math that a smiley parabola is minimised at $-\ b/2a$:

$$\lambda =-\ (-\ 2)/2(3)\ =\ 1/3$$

Which is the same value of λ we obtained via Method 1, and hence we arrive at the same answer.

Strictly speaking, you *can* differentiate $|XR|$ with the square root and solve for stationary points/check 2nd derivative etc, but I believe the argument outlined above should suffice. That said, I am not a teacher, and I do not know how individual JCs allot marks for working.

<u>Method 3:</u>
Let **d** be the **unit** direction vector of **l**. Take any point R, on **l**. Then, the perpendicular distance from A to **l** is given by:

$$D\ =\ |AR\ \times\ d|$$

This method works well if you are not asked to find the coordinates of the foot of the perpendicular, only the distance.

Example:
Find the shortest distance between the point A, given by coordinates (2,2,-6) and the line given by: $x\ -\ 1\ =\ 2\ -\ y\ =\ z\ +\ 6$

$d\ =\ (1/\sqrt{3},-\ 1\sqrt{3},1\sqrt{3})$
$(1,2,-\ 6)$ lies on **l**.

Thus: $AR\ =\ (-\ 1,0,0)$
Hence: $D\ =\ |(-\ 1,0,0)\ \times\ (1/\sqrt{3},-\ 1/\sqrt{3},1/\sqrt{3})|\ =\sqrt{6}/3$

<u>Case 2:</u>

Suppose A is a point, and **P** a plane. We want to find the shortest distance from A to **P**.

Method 1:
We will solve for the foot of perpendicular, F, from A to **P**, and then take the length AF.

Since F is in the plane, **OF** satisfies the plane equation for **P**. Additionally, since F is the foot of perpendicular from A, **AF** is some multiple of the normal vector for **P**. This will be enough to solve for the coordinates of F.

Example:
Find the shortest distance from the point A, with coordinates $(5, -6, 5)$*, to the plane given by* $2x - 3y + z = 5$*.*

Method 1:
Let F be the foot of perpendicular from A to **P**.
$OF = OA + AF = (5, -6, 5) + \lambda(2, -3, 1)$, for some λ.

Since F lies in **P**: $OF \bullet (2, -3, 1) = 5$

Thus: $((5, -6, 5) + \lambda(2, -3, 1)) \bullet (2, -3, 1) = 5$

$13 + 14\lambda = 5$

$\lambda = -4/7$

And hence: $|AF| = |-4/7 * (2, -3, 1)| = 4\sqrt{14}/7$

Note that if the question had asked for the coordinates of F, we simply remove the absolute value sign from the last line. Thus, this method works the same way regardless of which they ask for.

Method 2:
We will construct a plane parallel to **P** at A, then find the distance between the two planes.

Example:
Find the shortest distance from the point A, with coordinates $(5, -6, 5)$*, to the plane given by* $2x - 3y + z = 5$*.*

Our constructed plane has equation:

$$2x - 3y + z = (5, -6, 5) \bullet (2, -3, 1) = 13$$

Thus: $Distance = |13 - 5|/|(2, -3, 1)| = 8/\sqrt{14} = 4\sqrt{14}/7$

Note: This method is faster at getting the shortest distance(in my opinion), but when it comes to finding the foot of perpendicular, a little bit of care is required. In this case, we know the distance AF, and that **AF** is parallel to the normal, $(2, -3, 1)$, but in which direction?

Here are two ways of reasoning this out:

The brainless approach is to simply try both of:

$$OF = (5, -6, 5) \pm 4\sqrt{14}/7 * (2/\sqrt{14}, -3/\sqrt{14}, 1/\sqrt{14}), \qquad —(\dagger)$$

And see which vector satisfies the plane equation of **P**.

A more methodical approach is to consider the two plane equations:

P: $2x - 3y + z = 5$
Constructed Plane: $2x - 3y + z = 13$

Note that the right hand side of both equations being positive implies that both planes are in the same direction as the normal vector, $(2, -3, 1)$. And since $13 > 5$, the constructed plane is further away from O than **P**. In other words, **P** lies between O and the constructed plane, so F lies between O and A.

Thus, we need to take the minus sign in equation $(\dagger)$.

$$OF = (5, -6, 5) - 4\sqrt{14}/7 * (2/\sqrt{14}, -3/\sqrt{14}, 1/\sqrt{14}) = 1/7 * (-43, 30, -31)$$

Note that the coordinates of F obtained this way agree with that of Method 1.

Shortest distance between skew lines(non-examinable):

This section is explicitly excluded according to the syllabus document, but I think it serves as a useful example to further illustrate the techniques employed thus far.

Suppose **l** and **m** are two lines that do not intersect. We wish to find the shortest distance between the two of them.

Method 1:

If **OR** and **OX** are the position vectors of arbitrary points on **l** and **m** respectively, we will find **RX**, and utilise the fact that at the two points of closest approach between **l** and **m**, RX is perpendicular to both **l** and **m**. Taking the dot product between **RX** and the two direction vectors of the lines will let us solve for **OX** and **OR**.

Example:
Find the shortest distance between the lines:
$r1 = (3, 0, 1) + \lambda(2, 1, 1)$ $\quad r2 = (-1, 2, 1) + \mu(-3, 2, 4)$

Taking **OX** as $r1$, and **OR** as $r2$, we get:
$RX = (4 + 2\lambda + 3\mu, -2 + \lambda - 2\mu, \lambda - 4\mu)$

Dotting with direction vectors and setting to 0, we get:
$RX \bullet (2, 1, 1) = 6 + 6\lambda = 0$
$RX \bullet (-3, 2, 4) = -16 - 29\mu = 0$

Thus:
$\lambda = -1$
$\mu = -16/29$
(In general, just push G.C. to solve for unknowns.)

Subbing into RX, we get:
$RX = (10/29, -55/29, 35/29)$

And hence:
Shortest Distance=$|RX| = \sqrt{150/29}$

Method 2:
We will cross the direction vectors of **l** and **m** to obtain a vector normal to both, and hence two parallel planes that contain **l** and **m**. Then, the distance between the two planes is the shortest distance between the two lines.

Example:
Find the shortest distance between the lines:
$r1 = (3, 0, 1) + \lambda(2, 1, 1)$ $\quad r2 = (-1, 2, 1) + \mu(-3, 2, 4)$

Taking the cross product:
$(2, 1, 1) \times (-3, 2, 4) = (2, -11, 7)$

We thus construct two planes, both normal to $(2, - 11, 7)$, one passing through $(3, 0, 1)$ and the other passing through $(- 1, 2, 1)$. If you find it difficult to get the equations of the planes with this information, please review your notes.

Denoting the planes P1 and P2, we get:

P1: $2x - 11y + 7y = 13$
P2: $2x - 11y + 7y =- 17$

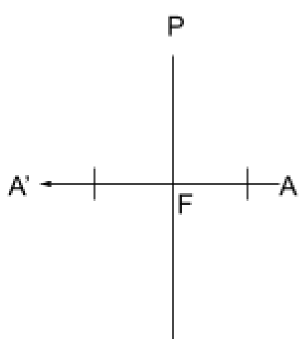
Using the formula for the distance between parallel planes, we get:

Shortest Distance $= | - 17 - 13|/\sqrt{2^2 + 11^2 + 7^2}$
$= 30/\sqrt{174}$
$= \sqrt{900/174} = \sqrt{150/29}$

Note: For two parallel planes, $r \bullet n = d1$ and $r \bullet n = d2$, the distance between them is given by: $D = |d1 - d2|/|n|$.

Reflections

This section is just here as a short add-on. Questions involving reflections come up every now and then, so they're worth some discussion. Once you understand the geometry, they shouldn't be difficult.



Let **P** be a plane, A a point, F the foot of perpendicular, A' its reflection across **P**. It should be clear that **FA**'=**AF**. Thus, if you want to find the coordinates of A', simply find AF, then use:

OA'=**OA**+2**AF**.

A similar approach works for reflections across lines, as you just need to find the length of the foot of perpendicular from A to the line, which is the same as finding the shortest distance from the point to the line.

If you want to find the reflection of a line across a plane, it suffices to take any 2 points on the line, A and B, and find their reflections about P, A' and B'. The equation of the reflected line is the line through A' and B'. Of course, if you choose one of those points to be the point of intersection between the line and the plane, C, then C'=C and your work is halved(in principle).

Example:
Find the coordinates of the reflection of the point A, given by coordinates $(3, 1, 2)$*,* *across the plane* ***P*** *given by the equation* $x + 2y + z = 1$

Construct a plane parallel to **P** through A: $x + 2y + z = 7$

Distance between planes: $6/\sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$
Since $7 > 1$, our constructed plane is further from the origin than **P**.

Hence: $AF =- \sqrt{6}(1/\sqrt{6},\ 2\sqrt{6},\ 1/\sqrt{6})$

Thus: $OA' = OA - 2\sqrt{6}(1/\sqrt{6},\ 2\sqrt{6},\ 1/\sqrt{6}) = (3, 1, 2) - 2(1, 2, 1) = (1, - 3, 0)$

If the bit about obtaining the sign of AF is not clear, I would suggest looking at the section above regarding solving for the foot of perpendicular.