National Junior College
2016 – 2017 H2 Mathematics
Complex Numbers Tutorial

Basic Mastery Questions

1. Simplify the following complex numbers in the Cartesian form, $x+\mathrm{i}y$:

   (a) $(3-8\mathrm{i})(5+7\mathrm{i})$ (b) $\dfrac{7+5\mathrm{i}}{4-3\mathrm{i}}$

2. Find the modulus and argument of the following complex numbers, leaving the arguments in exact radians. Hence express the complex numbers in polar form and exponential form.

   (a) $-2\mathrm{i}$ (b) $-1+\sqrt{3}\mathrm{i}$

   (c) $(-1-\mathrm{i})\left(-1+\sqrt{3}\mathrm{i}\right)$ (d) $\dfrac{-1+\sqrt{3}\mathrm{i}}{-1-\mathrm{i}}$

   (e) $1+\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ (f) $3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

   (g) -100 (h)* $-3\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$

3. Convert the following complex numbers into Cartesian form.

   (a) $2\left(\cos\left(-\dfrac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\dfrac{5\pi}{6}\right)\right)$ (b) $2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$

4. See the following proof:

$$-1=\mathrm{i}^2=(\mathrm{i})(\mathrm{i})=\sqrt{-1}\sqrt{-1}=\sqrt{1}=1$$

   $-1=1$??? What went wrong?

Practice Questions

1. Find the exact values of the modulus and argument of $z=-\sqrt{3}+\mathrm{i}$ and $w=4+4\mathrm{i}$.

   Hence evaluate (i) $-\dfrac{1}{z}$, (ii) $\dfrac{1}{z^*}$, (iii) $(w^*)^3$, (iv) $\dfrac{z^*}{w}$, (v) z^2w^3,
   leaving your answers in exact polar form.

2. The complex numbers p and w are such that

$$|p|=3,\quad \arg(p)=\frac{7\pi}{8},\qquad \text{and}\qquad |w|=2,\quad \arg(w)=-\frac{5\pi}{8}.$$

   (i) Find the exact values of the modulus and argument of $\dfrac{2p}{w^2}$.

   (ii) Hence find the smallest positive value of n such that $\left(\dfrac{2p}{w^2}\right)^n$ is purely imaginary.

   (2011/MJC/P1/Q11a modified)

3. (i) The complex number w has modulus r and argument θ, where $0 < \theta < \frac{1}{2}\pi$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = \frac{w}{w^*}$.

   (ii) Given p^5 is real and positive, find the possible values of θ.

(GCE 2008/P2/Q3 modified)

4. The complex number a has modulus r and argument θ, where $0 < r < 1$ and $0 < \theta < \frac{\pi}{2}$. The complex number b is such that $\left|\frac{b}{a}\right| = 1$ and $\arg(a) + \arg(b) = \pi$.

   Let the points A, B, C, D and E represent the complex numbers a, b, $a + b$, $\frac{b}{a^*}$ and ia respectively, where a^* denotes the conjugate of a.

   On a single clearly labelled Argand diagram, illustrate these five points.

5. Show that $\mathrm{e}^{2\alpha \mathrm{i}} + \mathrm{e}^{-2\alpha \mathrm{i}}$ is a real number for all α.

   The complex number w is given by $w = \frac{2}{1 + \mathrm{e}^{4\alpha \mathrm{i}}}$, where $0 < \alpha < \pi$.

   Show that Re(w) = 1.

6. The complex number w is such that $ww^* + 2w = 3 + 4i$, where w^* is the complex conjugate of w. Find w in the form $a + \mathrm{i}b$, where a and b are real. (GCE 2007/P1/Q3b)

7. Solve the simultaneous equations

$$\mathrm{i}z + 2w = 1 \text{ and } 4z + (3 - \mathrm{i})w^* = -6,$$

   giving z and w in the form a + bi, where a and b are real.

8. Given that $(x + \mathrm{i}y)^2 = 12\mathrm{i} - 5$, where x and y are real numbers, find the set of possible values of $x + \mathrm{i}y$. Hence solve the equation $z^2 + 4z = 12\mathrm{i} - 9$.

9. (i) Given that $1 + \mathrm{i}$ is a root of the equation $2w^3 + aw^2 + bw - 2 = 0$, find the values of the real numbers a and b.

   (ii) For these values of a and b, solve the equation in part (ii) without the use of a calculator.

10. One of the roots of the equation $z^3 - 2z^2 + az + 1 + 3\mathrm{i} = 0$ is $z = \mathrm{i}$. Find the complex number a and the other roots. (2014/HCI/P1/Q3)

11. The complex number z is defined by $z = \cos\theta + \mathrm{i}\sin\theta$, where $-\pi < \theta \le \pi$.

    Find, in Cartesian form, $x + \mathrm{i}y$, the complex number z such that $3 + \mathrm{i}(3z + 1) = \mathrm{e}^{\mathrm{i}(\pi + \theta)}$.

Challenging Questions

1. The complex numbers $2e^{\frac{1}{12}\pi i}$ and $2e^{\frac{5}{12}\pi i}$ are represented by points *A* and *B* respectively in an Argand diagram with origin *O*. Show that triangle *OAB* is equilateral.

2. If $z = i$ is a root of the equation $z^3 + (1-3i)z^2 - (2+3i)z - 2 = 0$, determine the other roots.
   Hence find the roots of the equation $w^3 + (1+3i)w^2 + (3i-2)w - 2 = 0$.

3. A complex number *w* is such that $ww^* - 16\sqrt{3}i + 8iw = 0$ and $\text{Im}(w) < 5$, where w^* is the conjugate of w.
   (i) Find *w* in the form $x + yi$, where $x,\ y \in \mathbb{R}$.
   (ii) Find the integer values of *n* such that w^n is real.
   (iii) Evaluate $1 + \left(\frac{w}{4}\right)^3 + \left(\frac{w}{4}\right)^6 + \left(\frac{w}{4}\right)^9 + \ldots + \left(\frac{w}{4}\right)^{21}$.

   Another complex number *z* has modulus 4 and satisfies $\arg\left(\frac{zi}{1+i}\right) = \frac{3}{4}\pi$.
   (iv) Express *z* in the form of $a + bi$, where $a,\ b \in \mathbb{R}$.
   (v) Find the area of the triangle *ZWO* where *Z* and *W* are points on the Argand diagram that represent the complex numbers *z* and *w* respectively, and *O* is the origin.

(2009/HCI/P2/Q3)

Numerical Answers to Basic Mastery Questions

1(a) 71–19i (b) $\frac{13}{25} + \frac{41}{25}i$

2(a) $2, -\frac{\pi}{2}, 2e^{i\left(-\frac{\pi}{2}\right)}, 2\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$ (b) $2, \frac{2\pi}{3}, 2e^{i\left(\frac{2\pi}{3}\right)}, 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(c) $2\sqrt{2}, -\frac{\pi}{12}, 2\sqrt{2}e^{i\left(-\frac{\pi}{12}\right)}, 2\sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$

(d) $\sqrt{2}, -\frac{7}{12}\pi, \sqrt{2}e^{i\left(-\frac{7\pi}{12}\right)}, \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$

(e) $\sqrt{3}$, $\frac{\pi}{6}$, $\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $\sqrt{3}e^{i\left(\frac{\pi}{6}\right)}$ (f) $3, \frac{\pi}{3}, 3e^{i\left(\frac{\pi}{3}\right)}, 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(g) $3, \pi, 3e^{i(\pi)}, 3(\cos\pi + i\sin\pi)$ (h) $3, -\frac{\pi}{6}, 3e^{i\left(-\frac{\pi}{6}\right)}, 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

3(a) $-\sqrt{3} - i$ (b) $1 - i\sqrt{3}$

Numerical Answers to Practice Questions

1.(i) $\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ (ii) $\frac{1}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(iii) $128\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$ (iv) $\frac{\sqrt{2}}{4}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

(v) $512\sqrt{2}\left(\cos\frac{5\pi}{12}+\mathrm{i}\sin\frac{5\pi}{12}\right)$

2. (i) 3/2, $\frac{\pi}{8}$ (ii) 4

3. (i) 1, 2θ (ii) $\frac{\pi}{5},\frac{2\pi}{5}$

6. $-1+2\mathrm{i}$

7. $w=1+\mathrm{i},\ \ z=-2+\mathrm{i}$

8. $2+3\mathrm{i}$ or $-2-3\mathrm{i},\ z=3\mathrm{i},\ \ z=-4-3\mathrm{i}$

9. (i) a = -5, b = 6 (ii) $w=1+\mathrm{i},\quad 1-\mathrm{i},\quad \frac{1}{2}$

10. $a=-2+3\mathrm{i}$; $z=-1$ or $3-\mathrm{i}$

11. $z=-0.6+0.8\mathrm{i}$

Numerical Answers to Challenging Questions

2(i) $-1, 2\mathrm{i}$; $-\mathrm{i},\ -2\mathrm{i},\ -1$

3(i) $w=2\sqrt{3}+2\mathrm{i}=2(\sqrt{3}+\mathrm{i})$ (ii) $n=6k$ where $k\in\mathbb{Z}$ (iii) 0 (iv) $4\mathrm{i}$ (v) $4\sqrt{3}$ unit^2