2022 ACJC JC1 H2 Promotional Exam

1 **(i)** If $y=\ln\left(\frac{e^{\sqrt{x}}}{\cos^3 x}\right)$, find $\frac{dy}{dx}$ in terms of x, where $0<x<\frac{\pi}{2}$. [2]

(ii) Given that $y^{\frac{1}{x}}=x^{\ln x}$, find $\frac{dy}{dx}$ in terms of x and y. [3]

2 **(i)** Without using a calculator, solve the inequality

$$51-\frac{88}{x+2}\le 10x\ .$$ [3]

(ii) Hence, without using a calculator, solve the inequality

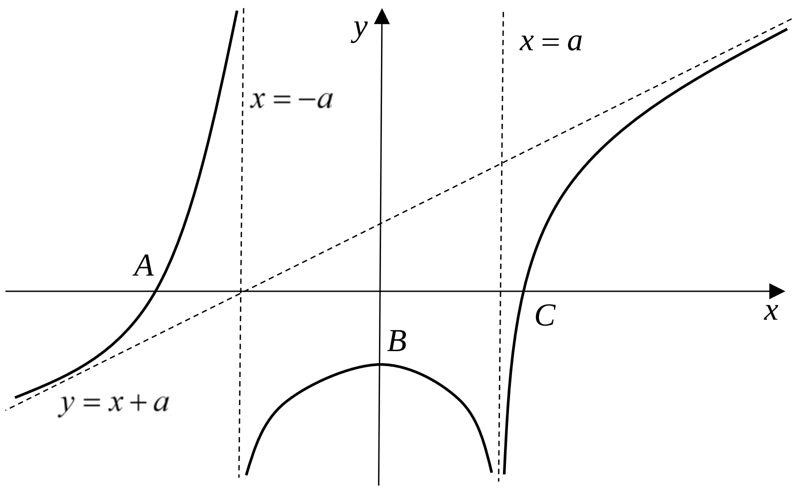
$$51|x|-\frac{88x^2}{1+2|x|}\le 10\ .$$ [2]

3 The curve C with equation $y=\frac{ax^2+bx+c}{3x+1}$ passes through the point with coordinates $(-1,-4)$ and has a turning point at $(-3,-2)$.

(i) Find the values of a, b and c. [3]

(ii) Hence sketch the graph of curve C, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the axes. [3]

4 The diagram below shows the graph of $y=\mathrm{f}(x)$. There are two vertical asymptotes with equations $x=-a$ and $x=a$, where a is a positive real constant, $a>1$. There is an oblique asymptote with equation $y=x+a$. The points A, B and C have coordinates $(-2,0)$, $(0,-2)$ and $(1.5,0)$ respectively, where B is also a maximum turning point.



Sketch the following curves and state the equations of the asymptotes, the coordinates of the turning points and of points where the curve crosses the axes, if any. Leave your answers in terms of a where necessary.

(i) $y=\frac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}(1-2x)$. [3]

5 The line L has vector equation

$$L:\ \mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

(i) Find the position vector of the point *N*, the foot of perpendicular from the point *A* with coordinates $(3, 6, 1)$ to the line L. [3]

(ii) Find the position vector of the point A', the reflection of point *A* in the line L. [2]

(iii) Find the coordinates of the points on the line *L* that are $3\sqrt{3}$ units away from point *A*. [2]

6 The function f is such that $\mathrm{f}(r) = \cos r\theta$.

(i) Show that $\mathrm{f}(2(r-1)) - \mathrm{f}(2r) = k \sin[(2r-1)\theta] \sin\theta$, where *k* is a constant to be determined. [2]

(ii) Using your result in **(i)**, show that $\sum_{r=1}^{n} \sin[(2r-1)\theta] = \frac{\sin^2 n\theta}{\sin\theta}$ using the method of difference. [3]

(iii) Hence find $\sum_{r=1}^{n} \sin[(2r+1)\theta]$ in terms of *n* and θ. [2]

7 **(a)** Referred to the origin *O*, points *A* and *B* have position vectors **a** and **b** respectively, where **a** and **b** are non-zero and non parallel vectors. Point *C* lies on *OA*, between *O* and *A*, such that *OC* : *CA* = 2 : 1. Point *D* lies on *OB* produced such that $BD = 5OB$.

Find, in terms of **a** and **b**, the position vector of the point *E* where the lines *AB* and *CD* meet. [3]

(b) The position vectors of the points *D* and *E* relative to the origin *O* are **d** and **e** respectively, where **d** and **e** are non-zero and non parallel vectors. It is given that the length *OE* is 2 units and $|\mathbf{d} \cdot \mathbf{e}| = 3$.

The point *F*, with position vector **f**, is the reflection of the point *D* in the line *OE*.

(i) Express **f** in terms of vectors **d** and **e**. [3]

(ii) Show that the area of triangle *ODF* can be expressed as $k|\mathbf{d} \times \mathbf{e}|$, where *k* is a constant to be determined. [2]

8 A curve C has parametric equations $x=\dfrac{a}{t}$, $y=\ln t^{a}$, where a is a positive constant and $t>0$.

(i) Find the equations of the tangent and normal to the curve at the point P with parameter p. [3]

(ii) The tangent at P and the normal at P meet the y-axis at point A and point B respectively. Find the area of triangle APB in terms of a and p. [2]

(iii) Sketch curve C and state, in terms of a, the coordinates of any point(s) where the curve crosses the axes. [1]

(iv) State the equation of the tangent to C when $p=1$. Hence state the range of values of m such that the line $y=mx+a$ cuts curve C at two points. [2]

9 It is given that

$$\mathrm{f}(x)=x^2-6x+5, \qquad \text{for } 1<x\le 5,$$
$$\mathrm{g}(x)=x^2-6x+5, \qquad \text{for } 1<x<3,$$
$$\mathrm{h}(x)=\begin{cases} \ln x & \text{for } 0<x\le 1, \\ x^2-6x+5 & \text{for } 1<x\le 5, \end{cases}$$

and that $\mathrm{h}(x)=\mathrm{h}(x-5)$ for all real values of x.

(i) Find the range of f. [1]

(ii) Show that g^{-1} exists. [1]

(iii) Find $\mathrm{g}^{-1}(x)$ and state the domain of g^{-1}. [3]

(iv) Find the exact value of $\mathrm{h}\left(\dfrac{11}{2}\right)+\mathrm{h}(-2)$. [2]

(v) Sketch the graph of $y=\mathrm{h}(x)$ for $-4\le x\le 6$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

10 The plane P_1 has the equation $-6x-4y+2z=4$.

(i) Find the vector equations of the planes such that the perpendicular distance from each plane to P_1 is 10 units. [2]

The plane P_2 has the equation $-x-y+2z=k$, where k is a constant.

(ii) Find the angle between P_1 and P_2. [2]

(iii) The planes P_1 and P_2 intersect in the line L. Show that a possible vector equation of L is $\mathbf{r} = \begin{pmatrix} 2k-2 \\ 2-3k \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$. [3]

The plane P_3 has the equation $5x + \beta y + 5z = \mu$, where $\beta, \mu \in \mathbb{R}$.

(iv) Given that the line L is contained in the plane P_3, find β and μ, giving your answer in terms of k if necessary. [2]

(v) Given instead that the line L does not intersect P_3, what can be said about the values of β and μ? [1]

11 Antonio used a credit card to buy a diamond ring to propose to his girlfriend. On 1st October 2022, he charged \$7500 to the credit card. The credit card charges a 2% monthly interest rate that is applied on the last day of every month. From 1st November 2022, Antonio started paying off his credit card debt by paying \$$x$ at the start of each month, where x is a constant.

(i) Find the amount of Antonio's credit card debt left after his first repayment, giving your answer in terms of x. [1]

(ii) Show that the amount of Antonio's credit card debt after his nth repayment is

$$(7500 - 50x)(1.02)^n + 50x.$$ [3]

(iii) Find the minimum amount that Antonio must consistently pay each monthly repayment for him to fully pay off his credit card debt with his 60th repayment. Find also the amount of credit card interest he would have paid in this case. [4]

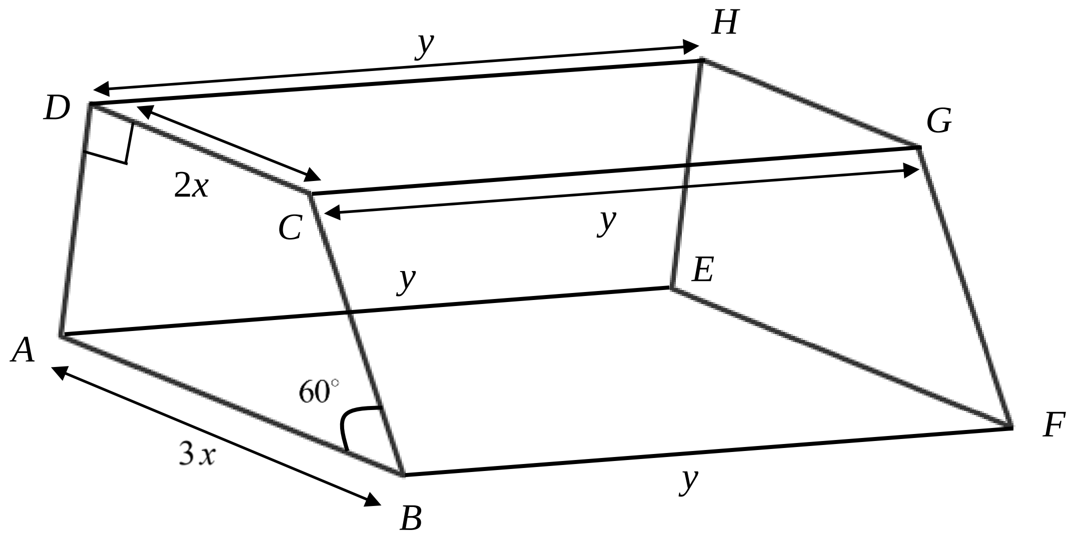
(iv) Suppose that up to and including 1st October 2023, Antonio had faithfully repaid \$500 a month to service his credit card debt. As a promotional reward, the credit card company decided to lower Antonio's monthly interest rate to 1% with immediate effect from October 2023, provided he maintains his \$500 monthly repayments. Find the date of Antonio's last repayment to the credit card company with this new interest rate, and the amount of his last repayment. [4]

Antonio and his fiancée are deciding between two venues for their wedding banquet by comparing the costs of the catering at each venue. At the Gomez Hotel, the catering is priced at \$1500 for each table of 10 guests. At the Grande Hotel, the catering is priced as follows:

GRANDE HOTEL BANQUET PRICE LIST	
Number of Tables (Each table sits 10 guests, venue can hold up to 40 tables)	**Price**
For first 10 tables	\$2000 per table
For more than 10 tables	Each additional table will be \$50 cheaper than the previous. (So the 11th table will cost \$1950, the 12th table will cost \$1900, and so on.)

(iv) How many tables will the couple need at least, for it to be cheaper to hold their wedding banquet at the Grande Hotel? [3]

12 Mr See wants to build a fish tank with a fixed capacity $\left(5\sqrt{3}\right)k$ m^3. The following diagram shows the dimensions of the fish tank he desires, with variables x and y.



The cross-section of the fish tank *ABCD* is a right trapezoid with *AB* being $3x$ m long, *DC* being $2x$ m long, and $\angle ABC = 60^\circ$. The cross-section *ABCD* is perpendicular to both the rectangular base *ABFE* and the rectangular opening *DCGH*, with $AE = BF = DH = CG = y$ m.

(i) Show that the total surface area A m^2 of the fish tank in contact with water when it is fully filled is $A = 5\sqrt{3}x^2 + \dfrac{2\left(5+\sqrt{3}\right)k}{x}$. [4]

(ii) Mr See would like to minimise the amount of material required to construct the walls and base of the fish tank. Using differentiation, find the value of x in terms of k when A is minimum. [4]

(iii) Mr See decided to build his tank with $k = \frac{3}{160}$ and $y = 0.6$. After which, he used a hose to fill it with water at a constant rate of $0.015\,\text{m}^3$ per minute. Find the rate at which the depth of the water is increasing when the fish tank is 50% filled with water. [5]