EUNOIA JUNIOR COLLEGE
JC1 Mid-Year Examination 2023
General Certificate of Education Advanced Level
Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1 [80 marks]

28 June 2023

2 hours 30 minutes

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **1** blank page(s).

1 ~~Prove by mathematical induction that $2^{n+1} > n^2$ for all positive integers $n \geq 3$.~~ [6]

2 A curve E has polar equation $r = \dfrac{3}{\sqrt{\cos^2\theta + (\sin\theta - \cos\theta)^2}}$ for $0 \leq \theta < 2\pi$.

(a) Taking the polar axis as the positive x-axis, find the cartesian equation of E, leaving your answer in the form $ax^2 + bxy + cy^2 = 9$, where a, b and c are constants to be determined. [3]

(b) Hence or otherwise, find the exact cartesian coordinates of the point(s) of intersection between E and the graph with polar equation $r = \dfrac{1}{\sin\theta - \cos\theta}$ for $\dfrac{\pi}{4} < \theta < \dfrac{5\pi}{4}$. [3]

3 The function f is defined by

$$\mathrm{f}(x) = 2 - \frac{1}{x}, \ x \in \mathbb{R}, \ x \geq 1.$$

(a) Write down expressions for $\mathrm{f}^2(x), \mathrm{f}^3(x)$ and $\mathrm{f}^4(x)$ in the form $\dfrac{ax+b}{cx+d}$, where a, b, c and d are integers. Hence make a conjecture for $\mathrm{f}^n(x)$ in terms of n. [3]

(b) Prove your conjecture for all positive integers n. [5]

(c) Let A be the largest subset of the real numbers such that when the domain of f is replaced with A, $\mathrm{f}^n(x)$ is defined for all positive integers n. State A. [1]

4 A curve T has polar equation $r = \sqrt{\dfrac{2}{\sin 3\theta}}$ where $-\pi < \theta \leq \pi$.

(a) Determine the range of values of θ for which the value of r is undefined. Hence state the equations of the asymptotes of T in polar form. [3]

(b) Hence sketch T, indicating clearly, in polar form, the equations of the asymptotes and any lines of symmetry, and the polar coordinates of any points where r attains stationary values. [6]

5 The general equation of the family of all quadratic curves, which includes the conic sections, is

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

It is given that a hyperbola H has $B = 2$ and $C = 0$.

(a) State a necessary condition on the value of A. [1]

It is given further that $D = 0$, $E = -4\sqrt{3}$, $F = 4$ and that the hyperbola H has eccentricity $e = \sqrt{3}$ and a focus at the origin O.

(b) State the exact coordinates of the other focus, O' and determine the value of A. [3]

It is given that the derivative of a curve with equation $Ax^2 + By^2 + Ey + F = 0$ at a given point, (x_0, y_0) can be written as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-2Ax_0}{2By_0 + E}$.

(c) Let T be a tangent to H at the point P with coordinates (x_0, y_0). Show that the angles made by the line segments OP and $O'P$ with the tangent T are equal. [7]

6 **(a)** The sequence $\{X_n\}$ is given by $X_0 = 6$ and

$$X_n = \frac{1}{4}X_{n-1} + 2^{1-n}, \quad n \geq 1$$

By multiplying the recurrence relation throughout by 2^n, use a suitable substitution to determine X_n as a function of n. [4]

(b) The sequence of real numbers $\{u_n\}$ is defined by $u_1 = a$, and the recurrence relation

$$u_{n+1} = \frac{u_n^{\,2} + 5}{2u_n + 4}, \quad n \geq 1.$$

(i) Given that the sequence converges to a limit l, find all possible values of l. [1]

(ii) With the aid of a graphing calculator, determine the long-term behaviour of the sequence when $a = -2.01$ and when $a = -1.99$. [2]

(iii) Show that $u_{n+1} > -2$ if $u_n > -2$, and $u_{n+1} < -2$ if $u_n < -2$. Hence explain the difference in the behaviour of the sequence when $a = -2.01$ and $a = -1.99$. [4]

7 After the Omega variant of a new virus emerged in 2023, a group of epidemiologists sought to model the spread of the virus in Singapore. In their model, the increase in cases from week $n-1$ to n is modelled as k times the increase in cases from week $n-2$ to $n-1$, where k is known as the weekly infection growth rate. Let x_n be the total number of Omega variant cases in Singapore within the first n weeks of the outbreak.

(a) Show that x_n is defined by the recurrence relation $x_n = ax_{n-1} + bx_{n-2}$, where a and b are constants to be determined in terms of k. [1]

8 Omega variant cases were reported in the first week of the outbreak, while 15 new cases were reported the week after.

(b) Given that $k \neq 0$ and $k \neq 1$, solve the recurrence relation and obtain an expression for x_n of the form $\alpha + \beta\left(k^{n-1} - 1\right)$, where α and β are constants to be determined in terms of k. [4]

(c) Determine the long-term behaviour of x_n for the cases where $k > 1$ and $0 < k < 1$, and hence explain in context what the long-term spread of the Omega variant in Singapore will be for each case. [3]

(d) State a limitation of using this model to predict the spread of the Omega variant in Singapore. [1]

(e) Show that x_n is an arithmetic progression if the weekly infection growth rate is 1, and state the value of its common difference. [2]

In one simulation, the weekly infection growth rate is estimated to be 4. After the implementation of a mass vaccination programme in the fifth week of the outbreak, the weekly infection growth rate is estimated to decrease to 1, such that the increase in cases from week 5 to 6 is equal to the increase from week 4 to 5. The weekly infection growth rate then remains unchanged for the rest of the outbreak. Alert Orange is triggered when the 10000^{th} case is reported.

(f) Under this simulation, determine the week in which Alert Orange will be triggered. [3]

8 A comet, C travels along a parabolic path with the Sun as the focus. The position of C is recorded with respect to a fixed polar axis where the Sun is at the pole. The polar axis does not coincide with the axis of symmetry for the parabola.

The polar equation of the path taken by C is given by $r = \dfrac{d}{1+\cos(\theta-\alpha)}$, where d is a constant and α is an acute angle in radians. The unit of measure used for distances is Astronomical Units (A.U.) where 1 A.U. is approximately 150 million kilometres.

The comet was first observed at the point with polar coordinates $\left(\dfrac{8\sqrt{2}}{3}, 0\right)$ and after a certain period of time, it was observed at the point with polar coordinates $\left(2\sqrt{2}, \dfrac{\pi}{3}\right)$.

(a) Show that $2+5\cos\alpha = 3\sqrt{3}\sin\alpha$. [3]

(b) Hence find the exact value of α. [3]

(c) Determine the exact value of d. [1]

(d) Sketch the path taken by C for $0 \le \theta \le \pi$, stating clearly the exact cartesian equation of the line of symmetry and the exact cartesian coordinates of the start and end points, and the point where the comet is closest to the Sun. [3]

It is known that when C was at its closest point to the Sun, a piece of the comet with negligible mass was dislodged and it flies off along a straight path that is tangential to the path of C.

(e) Find an exact cartesian equation of the path of the dislodged piece. [2]

(f) Hence find, in standard form, the distance in kilometres from the Sun to the point where the dislodged piece passes the line with polar equation $\theta = \dfrac{\pi}{2}$. [2]

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