Name:____________________________________ **Class:**_____________

JURONG PIONEER JUNIOR COLLEGE

JC1 Year End Examination 2023

FURTHER MATHEMATICS 9649/01
Higher 2

28 September 2023

Paper 1 3 hours

Additional materials: List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

[Turn over

1 A boy was trying to colour a 130 cm by 5 cm banner with a crayon. He coloured an area of 125 cm^2 at the end of the first hour. After each subsequent hour, the area he coloured is $\frac{4}{5}$ of the area he coloured during the previous hour.

(i) The area he coloured at the end of the nth hour is p cm^2. Show that

$$\ln p = (An + B)\ln 2 - (Cn + D)\ln 5,$$

where constants A, B, C and D are to be determined. **[3]**

(ii) Show that the boy will never finish colouring the banner. **[2]**

2 **(i)** Sketch the roots of the 8th roots of unity on an Argand diagram. **[2]**

(ii) The roots represented by z_1 and z_2 are such that $0 \le \arg(z_1) < \arg(z_2) < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z - z_1| = |z - z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its exact cartesian equation. **[3]**

3 The terms in the sequence $u_0, u_1, u_2, \ldots$ satisfy the recurrence relation

$$u_n = u_{n-1} + \frac{1}{2}u_{n-2}.$$

(i) Show that the general solution of this recurrence relation is

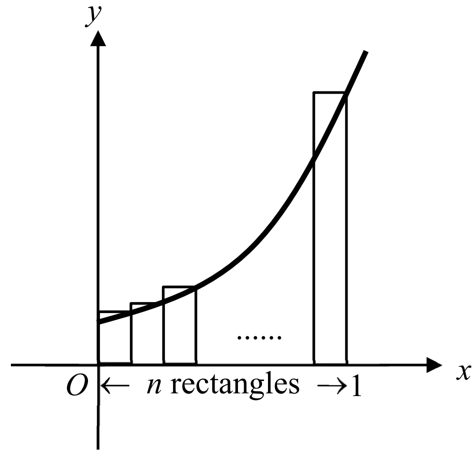
$$u_n = c\left(\frac{1+\sqrt{3}}{2}\right)^n + d\left(\frac{1-\sqrt{3}}{2}\right)^n,$$

where c and d are constants. **[2]**

(ii) Find an expression for u_n in terms of n in the case that $u_0 = 2$ and $u_1 = 1$. **[2]**

(iii) Find an expression for u_n in terms of n in the case that $u_0 = 2$ and u_n converges to zero. Hence find u_1. **[3]**

4 The diagram shows part of the graph of $y = (x+a)^2$, where a is a positive constant, with rectangles of equal width approximating the area under the curve between $x = 0$ and $x = 1$.



(i) Find the total area of the n rectangles, expressing your answer in the form
$$\frac{1}{n^3}\sum_{r=1}^{n} \mathrm{f}(r).$$ **[2]**

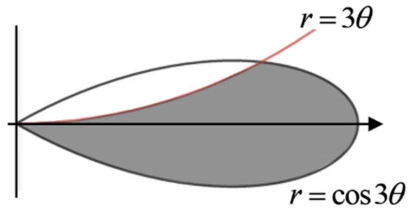
(ii) Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find a formula for $\frac{1}{n^3}\sum_{r=1}^{n} \mathrm{f}(r)$ in terms of n and a. Hence find the area under the curve between $x = 0$ and $x = 1$, in terms of a. **[6]**

5 The complex number z satisfies the equation

$$z^3(1+2\mathrm{i}) + z^2(1-2\mathrm{i}) + 4z + t = 0,$$

where t is a real number. It is given that one root is of the form $k + k\mathrm{i}$, where k is a negative real number. Find t and k, and the other roots of the equation. **[9]**

6 **(a)** The diagram below shows a section of two polar curves with equations $r = \cos 3\theta$ and $r = 3\theta$. Find the area of the shaded region. **[4]**



(b) The arc of a curve $y = 2\sqrt{x+1}$ from the point where $x = 0$ to the point where $x = 1$, is denoted by C. Show that the area of the surface generated when C is rotated through one revolution about the x-axis is $\frac{8\pi}{3}\left(a\sqrt{3} + b\sqrt{2}\right)$, where a and b are constants to be determined. **[5]**

7 A special curve, known as a cycloid, is defined parametrically as follows:

$$x = a(t - \sin t),\ y = a(-1 + \cos t),$$

where a is a positive constant.

The equations can be used to find the position of a point particle $P(x, y)$ travelling along a path, described by the curve, at time t seconds.

(i) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and sketch the cycloid for $0 \le t \le 4\pi$, indicating clearly the axial intercepts and stationary points. **[5]**

(ii) Find the exact distance travelled by P along the cycloid from $t = \pi$ to $t = 3\pi$, leaving your answer in terms of a. **[5]**

8 A medical research was done to test the effectiveness of a new antibiotic against the C-bacteria. In the beginning, there were 50 million C-bacteria. At the end of each day, the antibiotic eliminated 40% of the bacteria. In addition to the remaining bacteria, 10 million bacteria were added at the end of each day.

(i) Let C_n ($n \geq 0$) denote the number of bacteria at the end of n days after the research had started. Write down a first-order recurrence relation for C_{n+1} and solve it. **[5]**

(ii) For this antibiotic to be considered effective, it must reduce the number of bacteria to 15 million or less eventually. Using your answer to part **(i)**, comment on the effectiveness of this antibiotic. **[2]**

(iii) Show that the antibiotic needs to eliminate 67% of the bacteria every day in order to reduce the number of bacteria to 15 million eventually. **[3]**

(iv) State one assumption for this modelling process to be considered accurate. **[1]**

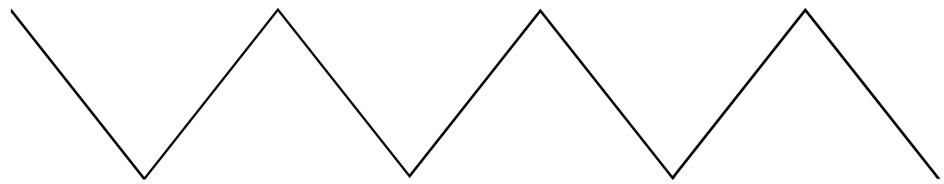
9 **(i)** Given that $y = \tan^{-1}(\mathrm{e}^x)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x} - 2\mathrm{e}^x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$. By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3. **[6]**

(ii) Use your series from part **(i)** to estimate $\int_0^{0.05} \tan^{-1}(\mathrm{e}^x)\,\mathrm{d}x$, correct to 5 decimal places. **[2]**

(iii) Use your calculator to find $\int_0^{0.05} \tan^{-1}(\mathrm{e}^x)\,\mathrm{d}x$, correct to 5 decimal places. **[1]**

(iv) Comparing your answers to parts **(ii)** and **(iii)**, and with reference to the value of x, comment on the accuracy of your approximations. **[2]**

10



An engineer is trying to model the above *triangular-wave curve* mathematically. His co-worker tells him this can be done using an 'infinite sum of cosine waves'. The engineer investigates possible models.

(a) The first model he uses is

$$y_1 = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \ldots + \frac{1}{n}\cos nx.$$

Sketch the graph of y_1, for $0 \le x \le 4\pi$, in the case where $n = 5$. **[2]**

(b) The engineer decides that y_1 will not give the *triangular -wave curve* when $n \to \infty$. He then uses the following refinement to the above model.

$$y_2 = \cos x + \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x + \ldots + \frac{1}{2n-1}\cos(2n-1)x.$$

Sketch the graph of y_2, for $0 \le x \le 4\pi$, in the case where $n = 5$. **[2]**

(c) The engineer observes that y_2 does not converge rapidly enough as $n \to \infty$ and so instead considers the infinite series

$$y_3 = \cos x + \frac{1}{9}\cos 3x + \frac{1}{81}\cos 5x + \ldots + \frac{1}{3^{2n-2}}\cos(2n-1)x + \ldots\ .$$

By expressing y_3 as the real part of a convergent geometric series involving $\cos x + \mathrm{i}\sin x$, show that $y_3 = \dfrac{36\cos x}{41 - 9\cos 2x}$. **[8]**

~~**11** A conic section has polar equation $r = \dfrac{2}{a - \sin\theta}$, $0 \le \theta < 2\pi$ where a is a constant.~~

~~**(i)** Given that $a = 3$, find the equation of this curve in a standard cartesian form and show that it is an ellipse. **[5]**~~

~~**(ii)** Determine the eccentricity of the curve and the coordinates of the two foci. **[4]**~~

~~Given instead that the curve is a parabola with focus F. The point P lies on the parabola and the foot of perpendicular from P to the directrix of the parabola is A.~~

~~**(iii)** State the value of a. **[1]**~~

~~**(iv)** Show that the foot of the perpendicular from the focus to the tangent at P lies on the tangent at the vertex. **[3]**~~

~The End~

BLANK PAGE