2018 P1 H2 (suggested solutions)

Q1(i)

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

(ii) From (i),

$$\int_1^e \frac{1-\ln x}{x^2}\,dx = \left[\frac{\ln x}{x}\right]_1^e = \frac{1}{e} - 0$$

$$= \frac{1}{e}$$

$$\int_1^e \frac{1}{x^2}\,dx - \int_1^e \frac{\ln x}{x^2}\,dx = \frac{1}{e}$$

$$\int_1^e \frac{\ln x}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^e - \frac{1}{e}$$

$$= \left(-\frac{1}{e} + 1\right) - \frac{1}{e}$$

$$= 1 - \frac{2}{e}$$

check using GC

Q2 (i)

$$\begin{cases} y = \frac{3}{x} \\ y = -2x + 7 \end{cases}$$

* Qn specifies : do not use calculator!!

→ must show working!

$$\frac{3}{x} = -2x + 7$$

$$3 = -2x^2 + 7x$$

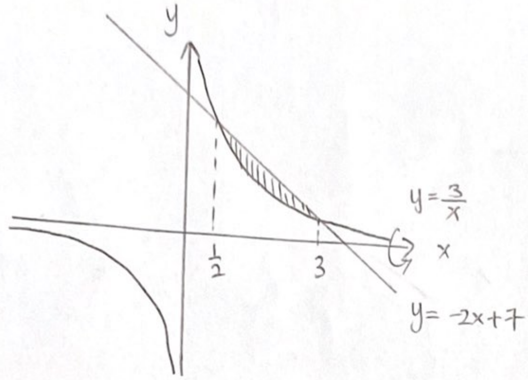
$$2x^2 - 7x + 3 = 0.$$

$$(2x-1)(x-3) = 0$$

$$x = \frac{1}{2} \text{ or } 3.$$

check using GC

(ii)



$$V = \pi\int_{\frac{1}{2}}^{3} (-2x+7)^2 - \left(\frac{3}{x}\right)^2 \, dx$$

$$= \pi\int_{\frac{1}{2}}^{3} (-2x+7)^2 - \frac{9}{x^2} \, dx$$

$$= \pi\left[\frac{(-2x+7)^3}{3(-2)} + \frac{9}{x}\right]_{\frac{1}{2}}^{3}$$

$$= \pi\left[-\frac{1}{6}+3 - (-36+18)\right]$$

$$= \frac{125}{6}\pi \text{ units}^3$$

check : using GC

Q3(i) $x\frac{dy}{dx} = 2y-6$. (given)

$y = ux^2$

Diff wrt x:

$$\frac{dy}{dx} = u(2x) + x^2\frac{du}{dx}$$

$$= 2ux + x^2\frac{du}{dx}$$

Sub into given DE :

$$\cancel{2ux^2} + x^3\frac{du}{dx} = \cancel{2ux^2} - 6$$

$$\frac{du}{dx} = -\frac{6}{x^3}$$

(ii) $\frac{du}{dx} = -\frac{6}{x^3}$

$u = \int -\frac{6}{x^3}\, dx$

$= -6\left(-\frac{1}{2x^2}\right) + c$

$= \frac{3}{x^2} + c$

Note:

$\int x^{-3}\, dx$

$= \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$\frac{y}{x^2} = \frac{3}{x^2} + c$

$y = 3 + cx^2$

When $x=1$, $y=2$: $2 = 3+c$

$\Rightarrow c = -1$

$\therefore y = 3 - x^2$ //

check:

$\frac{dy}{dx} = -2x$

$x\frac{dy}{dx} = -2x^2 = -2(3-y)$

$= -6+2y$

Q4(i) $|2x^2+3x-2| = 2-x$

$2x^2+3x-2 = 2-x$ or $2x^2+3x-2 = -(2-x) = -2+x$

$2x^2+4x-4=0$

$x = \frac{-4 \pm \sqrt{4^2-4(2)(-4)}}{2(2)}$

$= \frac{-4 \pm \sqrt{48}}{4}$

$= -1 \pm \frac{4\sqrt{3}}{4}$

$= -1 \pm \sqrt{3}$.

check using GC

$2x^2+2x=0$

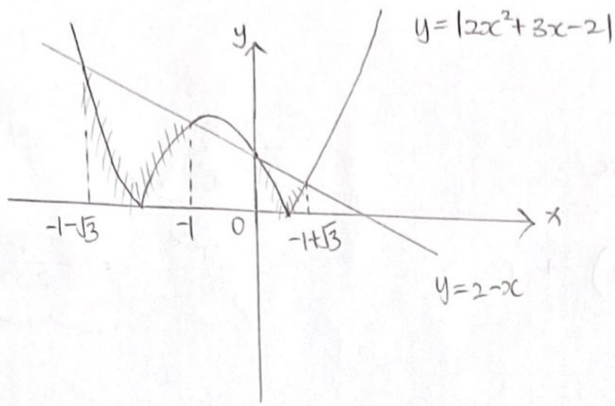
$x^2+x=0 \Rightarrow x(x+1)=0$

$x = 0$ or -1

* check using GC (graphs)

$\therefore x = -1 \pm \sqrt{3}, 0, -1$. //

(ii)



$|2x^2+3x-2| < 2-x$

$\therefore \quad -1-\sqrt{3} < x < -1 \quad \text{or} \quad 0 < x < -1+\sqrt{3}$ //

Q5 M1

$$ff(x) = f\left(\frac{x+a}{x+b}\right)$$

$$= \frac{\frac{x+a}{x+b} + a}{\frac{x+a}{x+b} + b}$$

$$= \frac{x+a+a(x+b)}{x+a+b(x+b)}$$

$$= \frac{(1+a)x + a+ab}{(1+b)x + a+b^2} = g(x) = x$$

$$(1+a)x + a+ab = (1+b)x^2 + (a+b^2)x$$

$$(1+b)x^2 + (\cancel{a}+b^2)x - (1+\cancel{a})x - (a+ab) = 0$$

$$(1+b)x^2 + (b^2-1)x - (a+ab) = 0.$$

$$(1+b)[x^2 + (b-1)x - a] = 0$$

$\therefore \ b = -1$ //

$ff(x) = x \Rightarrow f(x) = f^{-1}(x)$

$$\therefore f^{-1}(x) = \frac{x+a}{x-1}$$ //

M2 Since $ff(x) = x$

$\Rightarrow f(x) = f^{-1}(x)$.

Since $y=1$ is the asymptote for $y=f(x)$, thus $x=1$ is an asymptote for $y=f^{-1}(x)$.

$\therefore b = -1$

check:

$$ff(x) = f\left(\frac{x+a}{x-1}\right)$$

$$= \frac{\frac{x+a}{x-1} + a}{\frac{x+a}{x-1} - 1}$$

$$= \frac{x+\cancel{a}+a(x-\cancel{1})}{\cancel{x}+a-(\cancel{x}-1)} = \frac{x+ax}{a+1}$$

$$= \frac{x(a+1)}{a+1}$$

$$= x$$

Q6(i)

$\underline{a} \times 3\underline{b} = 2\underline{a} \times \underline{c}$

$\underline{a} \times 3\underline{b} - 2\underline{a} \times \underline{c} = \underline{0}$.

$\underline{a} \times (3\underline{b} - 2\underline{c}) = \underline{0}$

Since $\underline{a} \neq \underline{0}$, $\underline{a}$ and $3\underline{b} - 2\underline{c}$ are parallel.

$\therefore 3\underline{b} - 2\underline{c} = \lambda \underline{a}$.

(ii) given: $|\underline{a}| = 1$, $|\underline{c}| = 1$, $|\underline{b}| = 4$, $\measuredangle$ bet $\underline{b}$ & $\underline{c} = 60°$.

$(3\underline{b} - 2\underline{c}) \cdot (3\underline{b} - 2\underline{c}) = \lambda\underline{a} \cdot \lambda\underline{a}$

$9|\underline{b}|^2 - 12\underline{b} \cdot \underline{c} + 4|\underline{c}|^2 = \lambda^2 |\underline{a}|^2$

$9(4^2) - 12|\underline{b}||\underline{c}|\cos 60° + 4(1) = \lambda^2(1)$

$144 - 12(4)(\frac{1}{2}) + 4 = \lambda^2$

$\lambda^2 = 124$

$\lambda = \pm\sqrt{124} = \pm 2\sqrt{31}$ //

Q7(i) $\dfrac{x^2 - 4y^2}{x^2 + xy^2} = \dfrac{1}{2}$.

$2(x^2 - 4y^2) = x^2 + xy^2 \Rightarrow 2x^2 - 8y^2 = x^2 + xy^2 \Rightarrow x^2 - 8y^2 = xy^2$

Diff wrt x:

$2x - 16y\frac{dy}{dx} = x(2y)\frac{dy}{dx} + y^2$

$2x - y^2 = \frac{dy}{dx}(2xy + 16y)$

$\therefore \frac{dy}{dx} = \dfrac{2x - y^2}{2xy + 16y}$ (shown).

* Avoid making y the subject, rather use implicit differentiation here.

(ii) when $x=1$:

$$\frac{1-4y^2}{1+y^2} = \frac{1}{2}$$

$$2-8y^2 = 1+y^2$$

$$1 = 9y^2$$

$$y^2 = \frac{1}{9}.$$

$$y = \pm\frac{1}{3}.$$

$$\frac{dy}{dx} = \frac{2(1)-\left(\frac{1}{3}\right)^2}{2\left(\frac{1}{3}\right)+16\left(\frac{1}{3}\right)} \quad \text{or} \quad \frac{2-\left(-\frac{1}{3}\right)^2}{2\left(-\frac{1}{3}\right)+16\left(-\frac{1}{3}\right)}$$

$$= \frac{17}{54} \quad \text{or} \quad -\frac{17}{54}$$

tan at P: $y-\frac{1}{3} = \frac{17}{54}(x-1)$

tan at Q: $y-\left(-\frac{1}{3}\right) = -\frac{17}{54}(x-1)$

$$\Rightarrow y+\frac{1}{3} = -\frac{17}{54}(x-1)$$

$$\Rightarrow -\left(y+\frac{1}{3}\right) = \frac{17}{54}(x-1)$$

$$y-\frac{1}{3} = -\left(y+\frac{1}{3}\right)$$

$$= -y-\frac{1}{3}$$

$$2y = 0.$$

$$y=0$$

$$-\frac{1}{3} = \frac{17}{54}(x-1) \Rightarrow x = -\frac{1}{17}.$$

$\therefore N\left(-\frac{1}{17}, 0\right).$

check: intersecn of 2 tangents on graph (GC)

8 (i) $u_{n+1} = 2u_n + An$

$u_1 = 5$

$u_2 = 15$

$u_2 = 2u_1 + A = 2(5) + A$

$= 10 + A = 15$

$A = 5.$

$u_3 = 2u_2 + A(2)$

$= 2(15) + 5(2) = 40.$ //

(ii) $u_n = a(2^n) + bn + c$

$u_1 = a(2) + b(1) + c = 5$ —— (1)

$u_2 = a(4) + b(2) + c = 15$ —— (2)

$u_3 = a(8) + b(3) + c = 40$ —— (3)

using GC,

$a = \frac{15}{2}$, $b = -5$, $c = -5.$ //

(iii) $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (a(2^r) + br + c)$

$= a \sum_{r=1}^{n} 2^r + b \sum_{r=1}^{n} r + c \sum_{r=1}^{n} 1$

$= a \frac{2(1-2^n)}{1-2} + b \frac{n}{2}(1+n) + cn.$

$= -15(1-2^n) - \frac{5}{2}n(1+n) - 5n$

$= 15(2^n - 1) - \frac{5}{2}n(1+n) - 5n.$ //

Q9 (i) $x = 2\theta - \sin 2\theta$, $y = 2\sin^2\theta$

$\frac{dx}{d\theta} = 2 - 2\cos 2\theta$ $\frac{dy}{d\theta} = 4\sin\theta\cos\theta$

$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{2-2\cos 2\theta}$

$= \frac{2\sin\theta\cos\theta}{1-\cos 2\theta}$ $\cos 2\theta = 1 - 2\sin^2\theta$

$= \frac{\cancel{2}\cancel{\sin\theta}\cos\theta}{\cancel{2}\sin^{\cancel{2}}\theta}$

$= \cot\theta$ (shown).

(ii) When $\theta = \alpha$,

$x = 2\alpha - \sin 2\alpha$, $y = 2\sin^2\alpha$

$\frac{dy}{dx} = \cot\alpha$

grad of normal $= -\frac{1}{\cot\alpha} = -\tan\alpha$.

Eqn of normal : $y - 2\sin^2\alpha = -\tan\alpha\,(x - 2\alpha + \sin 2\alpha)$

when $y = 0$:

$-2\sin^2\alpha = -\tan\alpha\,(x - 2\alpha + \sin 2\alpha)$

$= -\frac{\sin\alpha}{\cos\alpha}(x - 2\alpha + 2\sin\alpha\cos\alpha)$

$\cancel{2\sin^2\alpha\cos\alpha} = \sin\alpha\,(x - 2\alpha + 2\sin\alpha\cos\alpha)$

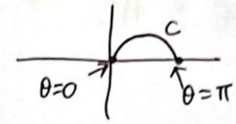
$= \sin\alpha\,(x - 2\alpha) + \cancel{2\sin^2\alpha\cos\alpha}$

$\sin\alpha\,(x - 2\alpha) = 0$

Since $\sin\alpha \neq 0$, $x = 2\alpha$

$\therefore k = 2$ //

(iii)



$$\int_0^\pi \sqrt{(2-2\cos 2\theta)^2 + (4\sin\theta\cos\theta)^2}\; d\theta$$

$$= \int_0^\pi \sqrt{(4 - 8\cos 2\theta + 4\cos^2 2\theta) + 4\sin^2 2\theta}\; d\theta$$

$$= \int_0^\pi \sqrt{4 - 8\cos 2\theta + 4}\; d\theta$$

$$= \int_0^\pi \sqrt{8(1-\cos 2\theta)}\; d\theta$$

$$= \int_0^\pi \sqrt{8(2\sin^2\theta)}\; d\theta$$

$$= \int_0^\pi \sqrt{16\sin^2\theta}\; d\theta$$

$$= \int_0^\pi 4\sin\theta\; d\theta$$

$$= \Big[-4\cos\theta\Big]_0^\pi$$

$$= 4 - (-4) = 8 \text{ units}$$

Q10 (i) $L\frac{dI}{dt} + RI + \frac{q}{C} = V$

Diff wrt t:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{dV}{dt}$$

$$\therefore\; L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt} = 0$$ if V is a constant.

constant   constant

(ii) $I = Ate^{-\frac{Rt}{2L}}$

$$\frac{dI}{dt} = At\left(-\frac{R}{2L}e^{-\frac{Rt}{2L}}\right) + Ae^{-\frac{Rt}{2L}}$$

$$\frac{dI}{dt} = Ae^{\frac{-Rt}{2L}}\left(-\frac{R}{2L}t+1\right)$$

$$\frac{d^2I}{dt^2} = Ae^{-\frac{Rt}{2L}}\left(-\frac{R}{2L}\right) + A\left(-\frac{R}{2L}t+1\right)\left(-\frac{R}{2L}\right)e^{\frac{-Rt}{2L}}.$$

$$= -\frac{AR}{2L}e^{-\frac{Rt}{2L}}\left(1-\frac{R}{2L}t+1\right)$$

$$= -\frac{AR}{2L}e^{\frac{-Rt}{2L}}\left(2-\frac{R}{2L}t\right)$$

Sub into DE in (i):

$$L\left(-\frac{AR}{2L}\right)\cancel{e^{\frac{-Rt}{2L}}}\left(2-\frac{R}{2L}t\right) + R\cancel{Ae^{\frac{-Rt}{2L}}}\left(-\frac{R}{2L}t+1\right) + \frac{\cancel{A}te^{\cancel{\frac{-Rt}{2L}}}}{C} = 0$$

$$-\frac{R}{2}\left(2-\frac{R}{2L}t\right) + R\left(-\frac{R}{2L}t+1\right) + \frac{t}{C} = 0.$$

$$\frac{R^2}{4L}t - \frac{R^2}{2L}t + \frac{t}{C} = 0$$

$$t\left(-\frac{1}{4}\frac{R^2}{L}+\frac{1}{C}\right) = 0.$$

$$\frac{1}{C} = \frac{1}{4}\frac{R^2}{L} = \frac{R^2}{4L}$$ since $t>0$. $\therefore C = \frac{4L}{R^2}$ (shown) //

(iii) Since $C = 0.75 = \frac{4L}{R^2}$, $I = Ate^{\frac{-Rt}{2L}}$

Thus, $\frac{dI}{dt} = 0 \Rightarrow Ae^{\frac{-Rt}{2L}}\left(-\frac{R}{2L}t+1\right) = 0$

$$-\frac{R}{2L}t+1 = 0$$

$$-\frac{4}{2(3)}t+1 = 0$$

$$\frac{2}{3}t = 1$$

$$t = \frac{3}{2}.$$

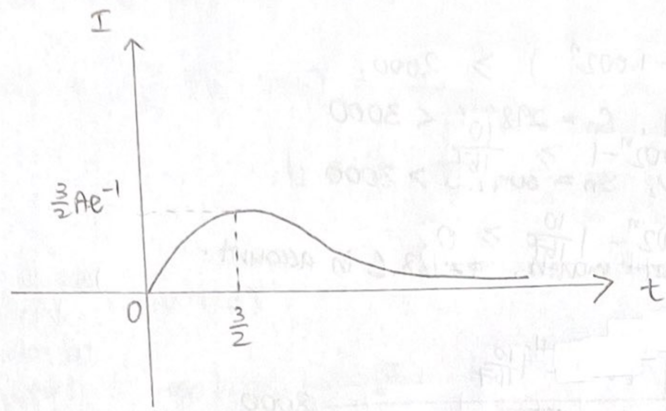
Sub $t=\frac{3}{2}$ into DE in (i) (with $R=4$, $L=3$, $C=0.75$, $I = Ate^{-\frac{Rt}{2L}}$)

$$3\frac{d^2I}{dt^2} + 4(0) + \frac{A\left(\frac{3}{2}\right)e^{-\frac{2}{3}\left(\frac{3}{2}\right)}}{0.75} = 0$$

$$3\frac{d^2I}{dt^2} + 2Ae^{-1} \Rightarrow \frac{d^2I}{dt^2} = -\frac{2}{3}Ae^{-1} < 0 \quad \therefore I \text{ max.}$$

$$\therefore I = A\left(\frac{3}{2}\right)e^{-\frac{2}{3}\left(\frac{3}{2}\right)} = \frac{3}{2}Ae^{-1} \quad //$$

) $I = Ate^{-\frac{2}{3}t}$.



Q11 (i) $a = 0.2$.

end Jan $100(1+\frac{0.2}{100})$

Feb $100(1+\frac{0.2}{100})^2$

Mar $100(1+\frac{0.2}{100})^3$

Dec $100(1+\frac{0.2}{100})^{12}$

(a) $100(1+\frac{0.2}{100})^{12} = \$102.43$

(b)

	beginning	end
$n=1$	100	$100(1.002)$
$n=2$	$100(1.002)+100$	$100(1.002^2)+100(1.002)$
	$\vdots$	$\vdots$
$n=12$		$100(1.002^{12})+100(1.002)^{11}+\dots+100(1.002)$

$$= 100(1.002+1.002^2+\dots+1.002^{12})$$

$$= \frac{100(1.002)(1-1.002^{12})}{1-1.002}$$

$$= \$1215.71.$$

* (c) $$S_n = \frac{100(1.002)(1-1.002^n)}{1-1.002} > 3000.$$

using GC,

when $n=29$, $S_n = 2988.6 < 3000$

when $n=30$, $S_n = 3094.8 > 3000$

$\Rightarrow$ On the last day of 29th month, $\$2988.6$ in account.

At the 1st day of 30^{th} month,

there will be $\$2988.6 + 100 = \$3088.6 > 3000$

$\therefore$ This occurs on the 1st day of the 30^{th} month i.e.
1st day of June 2018.

Jan 2016
$\downarrow 12$
Dec 2016
$\downarrow 12$
Dec 2017
$\downarrow 6$
Jun 2018

(ii) (a) $\$(100 + 12b)$

Note:

end Jan 2016	$100+b$
Feb	$100+2b$
Mar	$100+3b$
⋮	
Dec 2016	$100+12b$

(b)

	beginning	end
$n=1$	100	$100+b$
$n=2$	$(100+b)+100$	$(100+2b)+(100+b) = 2(100)+b+2b$
$n=3$	$(100+2b)+(100+b)+100$	$(100+3b)+(100+2b)+(100+b) = 3(100)+b+2b+3b.$
	⋮	⋮

when
$n = 24$

$24(100) + \overbrace{b + 2b + \cdots + 24b}^{AP} = 2800.$

$2400 + \frac{24}{2}(b + 24b) = 2800$

$b = \frac{4}{3} \approx \$1.33$ //

* (iii) $a = 1$, $n = 60$ (end)

(Plan P) : $\frac{100(1.01)(1-1.01^{60})}{1-1.01} = 60(100) + b + 2b + 3b + \cdots + 60b$ (Plan Q)

$= 6000 + \frac{60}{2}(b + 60b)$

$\therefore b = \$1.23$ //