H2 Mathematics (9758)
Chapter 9 Maclaurin Series
Assignment Suggested Solutions

1 **2015(9740)/I/6**

(i) Write down the first three non-zero terms in the Maclaurin series for $\ln(1+2x)$, where $-\frac{1}{2}<x\le\frac{1}{2}$, simplifying the coefficients. [2]

(ii) It is given that the three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^c$ for small x. Find the exact values of the constants a, b and c and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^c$, giving your answer as a simplified rational number. [6]

(i)	$\ln(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+\cdots$ $=2x-2x^2+\frac{8}{3}x^3+\cdots$ Replace x by $2x$ in $\ln(1+x)$ in MF26 $\boxed{\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{(-1)^{r+1}x^r}{r}+\ldots \quad (-1<x\le 1)}$
(ii)	$ax(1+bx)^c=ax\left(1+c(bx)+\frac{c(c-1)}{2}(bx)^2+\cdots\right)$ $=ax+abcx^2+\frac{c(c-1)}{2}ab^2x^3+\cdots$ Replace x by bx and n by c in $(1+x)^n$ in MF26 $\boxed{(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots \quad (\lvert x\rvert<1)}$ By comparing part **(i)** and **(ii)**, Coefficient of x: $a=2$ Coefficient of x^2 : $abc=-2\Rightarrow b=-\frac{1}{c}$ $(\because a=2)$ Coefficient of x^3 : $\frac{c(c-1)}{2}ab^2=\frac{8}{3}$ As $b=-\frac{1}{c}$, $\frac{c(c-1)}{2}\times 2\left(-\frac{1}{c}\right)^2=\frac{8}{3}$ $c(c-1)\left(\frac{1}{c^2}\right)=\frac{8}{3}$ $(c-1)\left(\frac{1}{c}\right)=\frac{8}{3}$

	$3c-3=8c$ $c=-\frac{3}{5}$ $\therefore b=\frac{5}{3}$ Coefficient of $x^4=a\left[\frac{c(c-1)(c-2)}{3!}\times b^3\right]=2\left[\frac{-\frac{3}{5}\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)}{3!}\left(\frac{5}{3}\right)^3\right]$ $=-\frac{104}{27}$

Question only wants coefficient, so don't give the whole term, i.e. you should not have the x^4

2 2019(9758)/ACJC Promo/Q7a & 2019(9758)/MI Promo/Q8

(a) Given that θ is sufficiently small, show that $\tan\left(\frac{\pi}{3}+\theta\right) \approx \sqrt{3}+4\theta+4\sqrt{3}\theta^2$. [3]

(b) It is given that $\mathrm{f}(x)=\ln(1+\sin x)$.

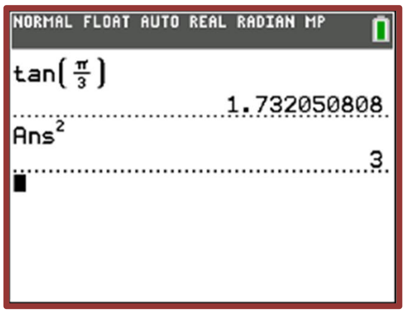
(i) Without the use of a calculator, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the first two non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [5]

(ii) By substituting $x=-\frac{\pi}{6}$, show that $\ln 2 \approx \frac{\pi^2}{72}+\frac{\pi}{6}$. [3]

(iii) John wants to use the Maclaurin series found in part **(i)** to estimate the value of $\mathrm{f}(x)$ when $x=a$, where a is a real number close to zero. Suggest one recommendation for John to improve the accuracy of his estimation. [1]

2	**ACJC JC1 Promo 9758/2019/Q7a ,**

(a)



$$\left[\tan\left(\frac{\pi}{3}\right)\right]^2=3$$

$$\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$\tan\left(\frac{\pi}{3}+\theta\right)=\frac{\tan\frac{\pi}{3}+\tan\theta}{1-\tan\frac{\pi}{3}\tan\theta}$$

Note that $\left(\frac{\pi}{3}+\theta\right)$ is not small, so you cannot do $\tan\left(\frac{\pi}{3}+\theta\right)\approx\frac{\pi}{3}+\theta$

We need to apply result from MF26.

$$\tan(A\pm B)\equiv\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$$

$$\approx\frac{\sqrt{3}+\theta}{1-\sqrt{3}\theta}$$

Recall that $\tan\theta\approx\theta$ when θ is sufficiently small.

Always remember to rewrite the term in the denominator as negative power before expanding.

$$=\left(\sqrt{3}+\theta\right)\left(1-\sqrt{3}\theta\right)^{-1}$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots \qquad (|x|<1)$$

Replace x by $-\sqrt{3}\theta$ in $(1+x)^n$ in MF26

$$=\left(\sqrt{3}+\theta\right)\left(1+(-1)\left(-\sqrt{3}\theta\right)+\frac{(-1)(-2)}{2!}\left(-\sqrt{3}\theta\right)^2+\ldots\right)$$

$$=\left(\sqrt{3}+\theta\right)\left(1+\sqrt{3}\theta+3\theta^2+\ldots\right)$$

$$=\sqrt{3}+3\theta+3\sqrt{3}\theta^2+\theta+\sqrt{3}\theta^2+\ldots$$

$$=\sqrt{3}+4\theta+4\sqrt{3}\theta^2+\ldots$$

(bi)

Method 1 (Implicit differentiation)

$\mathrm{f}(x)=\ln(1+\sin x)$

Differentiate wrt x,

$\mathrm{f}'(x)=\dfrac{\cos x}{1+\sin x}$

$\Rightarrow (1+\sin x)\mathrm{f}'(x)=\cos x$

Cross-multiply to simplify before differentiate

Differentiate wrt x,

$(1+\sin x)\mathrm{f}''(x)+(\cos x)\mathrm{f}'(x)=-\sin x ------(1)$

$\mathrm{f}(0)=\ln(1+\sin 0)=0$

$\mathrm{f}'(0)=\dfrac{\cos 0}{1+\sin 0}=1$

Notice the question asks to find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. And only the first two non-zero terms are required, implying one of the values are likely to be 0.

Sub $x=0$ into (1):

$(1+\sin 0)\mathrm{f}''(0)+(\cos 0)\mathrm{f}'(0)=-\sin 0$

$\Rightarrow \mathrm{f}''(0)+1=0$

$\Rightarrow \mathrm{f}''(0)=-1$

Concept: Use Maclaurin Theorem to find the Maclaurin Series required

➔ $\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\dfrac{x^2}{2!}\mathrm{f}''(0)+\cdots+\dfrac{x^n}{n!}\mathrm{f}^{(n)}(0)+\cdots$ (in MF 26)

$$\mathrm{f}(x)=0+(1)x+(-1)\left(\frac{x^2}{2!}\right)+\ldots$$

$$=x-\frac{1}{2}x^2+\ldots$$

Method 2 (using quotient rule – not recommended)

$\mathrm{f}(x)=\ln(1+\sin x)$

$\mathrm{f}'(x)=\dfrac{\cos x}{1+\sin x}$

$$\mathrm{f}''(x)=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$\mathrm{f}(0)=\ln(1+\sin 0)=0$

$\mathrm{f}'(0)=\dfrac{\cos 0}{1+\sin 0}=1$

$$\mathrm{f}''(0)=\frac{-\sin 0-\sin^2 0-\cos^2 0}{(1+\sin 0)^2}=-1$$

	$\mathrm{f}(x) = 0 + (1)x + (-1)\left(\frac{x^2}{2!}\right) + \ldots$ $= x - \frac{1}{2}x^2 + \ldots$
(ii)	When $x = -\frac{\pi}{6}$, $\ln\left[1+\sin\left(-\frac{\pi}{6}\right)\right] = -\frac{\pi}{6} - \frac{1}{2}\left(-\frac{\pi}{6}\right)^2 + \ldots$ $\ln\left(1-\frac{1}{2}\right) = -\frac{\pi}{6} - \frac{\pi^2}{72} + \ldots$ $\ln\left(\frac{1}{2}\right) = -\frac{\pi}{6} - \frac{\pi^2}{72} + \ldots$ $\ln 2^{-1} = -\frac{\pi}{6} - \frac{\pi^2}{72} + \ldots$ $-\ln 2 = -\frac{\pi}{6} - \frac{\pi^2}{72} + \ldots$ $\ln 2 \approx \frac{\pi^2}{72} + \frac{\pi}{6}$ (shown) Substitute $x = -\frac{\pi}{6}$ into **both the LHS and the RHS** of the Maclaurin series expansion in part **(i)**
(iii)	Use more terms in the Maclaurin series of $y = \ln(1+\sin x)$ for the estimation.

3 2018(9758)/CJC Prelim/II/1

Given that $\mathrm{f}(x)=\mathrm{e}^{\sin x}$, use the standard series to find the series expansion for $\mathrm{f}(x)$ in the form $a+bx+cx^2+dx^3$, where a, b, c and d are constants to be determined.

Hence show that the first three non-zero terms for the expansion of $\dfrac{1}{\left(\mathrm{e}^{\sin x}\right)^2}$ in ascending powers of x is $1-2x+2x^2$. [4]

The function $y=\mathrm{g}(x)$ satisfies $4\dfrac{\mathrm{d}y}{\mathrm{d}x}=(y+1)^2$ and $y=1$ at $x=0$.

(i) By further differentiation, find the series expansion for $\mathrm{g}(x)$, up to and including the term in x^3.

Hence show that when x is small, $\mathrm{g}(x)-\mathrm{f}(x)\approx\dfrac{1}{4}x^3$. [5]

(ii) By using the result in **(i)**, justify whether $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of x close to zero. [1]

Q3	**2018(9758)/CJC Prelim/II/1**
	$\mathrm{f}(x)=\mathrm{e}^{\sin x}$ $=\mathrm{e}^{x-\frac{x^3}{3!}+\cdots}$ $\boxed{\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots+\dfrac{(-1)^r x^{2r+1}}{(2r+1)!}+\ldots}$ Standard series for $\sin x$ Replace x by $x-\dfrac{x^3}{3!}$ in e^x in MF26 $\boxed{\mathrm{e}^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^r}{r!}+\ldots}$ Standard series for e^x $=1+\left(x-\dfrac{x^3}{3!}\right)+\dfrac{\left(x-\dfrac{x^3}{3!}\right)^2}{2!}+\dfrac{\left(x-\dfrac{x^3}{3!}\right)^3}{3!}+\cdots$ $=1+x-\dfrac{x^3}{6}+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\cdots$ $=1+x+\dfrac{x^2}{2}+\cdots$ $\therefore\ a=1,\ b=1,\ c=\dfrac{1}{2}$ and $d=0$

From previous result, $e^{\sin x}=1+x+\frac{x^2}{2}+\cdots$,

$$\frac{1}{\left(e^{\sin x}\right)^2}=\frac{1}{\left(1+x+\frac{x^2}{2}+\cdots\right)^2}$$

$$\approx\left(1+x+\frac{x^2}{2}\right)^{-2}$$

Always remember to rewrite the term in the denominator as negative power before expanding.

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots \qquad (|x|<1)$$

$$=1+(-2)\left(x+\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2!}\left(x+\frac{x^2}{2}\right)^2+\cdots$$

$$\approx 1-2x+2x^2$$

Replace x by $\left(x+\frac{x^2}{2}\right)$ and n by -2

(i)

$$4\frac{dy}{dx}=(y+1)^2$$

Differentiating with respect to *x*,

$$4\frac{d^2y}{dx^2}=2(y+1)\frac{dy}{dx}$$

Differentiating with respect to *x*,

$$4\frac{d^3y}{dx^3}=2(y+1)\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

Note that: $\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)=\left(\frac{dy}{dx}\right)^2$, **NOT** $\frac{d^2y}{dx^2}$

$$4\frac{d^3y}{dx^3}=2(y+1)\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{dy}{dx}\right)^2$$

Sub $x=0$, $y=1$, $\frac{dy}{dx}=1$, $\frac{d^2y}{dx^2}=1$, $\frac{d^3y}{dx^3}=\frac{3}{2}$

Concept: Use Maclaurin Theorem to find the Maclaurin Series required

➔ $f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^{(n)}(0)+\cdots$ (in MF 26)

Using Maclaurin's formula, $g(x)=1+x+\frac{x^2}{2!}+\left(\frac{3}{2}\right)\frac{x^3}{3!}+\cdots$

$$g(x)=1+x+\frac{x^2}{2}+\frac{x^3}{4}+\cdots$$

$$g(x)-f(x)=\left(1+x+\frac{x^2}{2}+\frac{x^3}{4}+\cdots\right)-\left(1+x+\frac{1}{2}x^2+\cdots\right)$$

$$\approx\frac{x^3}{4} \text{ (shown)}$$

(ii)	For values of x close to zero, $\mathrm{g}(x)-\mathrm{f}(x)\approx\frac{1}{4}x^3\to 0$ Therefore, $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of x close to zero.

Note that: if $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$, that means that the difference, $\mathrm{g}(x)-\mathrm{f}(x)$, would be very small. And notice that when $x\to 0$, $\frac{1}{4}x^3\to 0$ (This explanation must be given before you conclude that the estimation is good.)

Also note that question wants you to use the result in part (i), so graphical method is not accepted here.

4 2012/RVHS Prelim/I/Q4

It is given that $y=\left(\cos^{-1}x\right)^2$. Show that $(1-x^2)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y$. [2]

By further differentiation of this result, find the Maclaurin's series for y up to and including the term in x^2. [4]

Deduce

(i) the equation of the tangent to the curve $y=\left(\cos^{-1}x\right)^2$ at the point where $x = 0$, [1]

(ii) the first two non-zero terms in the series expansion of $\dfrac{2\cos^{-1}x}{x^2-1}$ by expressing $1-x^2$ as $\left(\sqrt{1-x^2}\right)^2$. [2]

Q4	2012/RVHS Prelim/I/Q4
	$y=\left(\cos^{-1}x\right)^2$ --- (1) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(\cos^{-1}x\right)\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$ --- (2) Squaring both sides, we get $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{4\left(\cos^{-1}x\right)^2}{1-x^2}$ $\left(1-x^2\right)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y$ (shown)
	$\left(1-x^2\right)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y$ Differentiate with respect to x: $\left(1-x^2\right)(2)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(-2x)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4\dfrac{\mathrm{d}y}{\mathrm{d}x}$ --- (3) Substitute $x = 0$ into (1), (2) and (3): $y=\left(\cos^{-1}0\right)^2=\left(\dfrac{\pi}{2}\right)^2=\dfrac{\pi^2}{4}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(\cos^{-1}0\right)\left(-\dfrac{1}{\sqrt{1-0^2}}\right)=2\left(\dfrac{\pi}{2}\right)(-1)=-\pi$ $\left(1-0^2\right)(2)(-\pi)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4\pi \Rightarrow \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2$ $\therefore\ y=\dfrac{\pi^2}{4}+(-\pi)x+\dfrac{2}{2!}x^2+\ldots$ $y=\dfrac{\pi^2}{4}-\pi x+x^2+\ldots$

(i)	Equation of tangent: $y=-\pi x+\dfrac{\pi^2}{4}$. Tangent to graph at $x=0$ is the equation of straight line $y=mx+c$ obtained from the series.
(ii)	$\dfrac{2\cos^{-1}x}{x^2-1}$ $=-\dfrac{2\cos^{-1}x}{1-x^2}$ $=-\dfrac{2\cos^{-1}x}{\sqrt{1-x^2}\,\sqrt{1-x^2}}$ Observe that $\dfrac{dy}{dx}=\dfrac{-2\cos^{-1}x}{\sqrt{1-x^2}}$ $=\dfrac{dy}{dx}\left(1-x^2\right)^{\frac{-1}{2}}$ $=(-\pi+2x+...)\left[1+\dfrac{1}{2}x^2+...\right]$ Find $\dfrac{dy}{dx}$ from Maclaurin's expansion above and the expansion of $\left(1-x^2\right)^{-\frac{1}{2}}$ using binomial expansion $\approx -\pi+2x$