National Junior College
2016 – 2017 H2 Mathematics
Complex Numbers (Lecture Notes)

Topic 13: Complex Numbers

Key questions to answer:

1. What is a complex number? How is it different from a real number?
   - *How is the complex number* i *defined*?
   - *How do we write a complex number in its Cartesian form?*
   - *What is the relationship between the set of real numbers and the set of complex numbers?*

<u>Complex Numbers in Cartesian Form</u>

2. When can we say that two complex numbers given in Cartesian form are equal?
   - *How do we apply this property to solve simple equations involving a complex variable?*

3. How do we carry out arithmetic operations (addition, subtraction, multiplication, division and taking square root) on complex numbers in Cartesian form?
   - *How do we relate the addition and subtraction of complex numbers to addition and subtraction of vectors?*

4. What is the conjugate of a complex number? What are its properties and applications?

5. What can we say about the roots of polynomial equations with real coefficients?
   - *How do we solve polynomial equations with real coefficients?*
   - *Understand that complex roots of a polynomial equation with real coefficients occur in conjugate pairs.*

6. How do we find the modulus and argument of a complex number given in Cartesian form?

7. How do we represent a complex number in Cartesian form by a point in the Argand diagram?
   - *How do we interpret geometrically, the terms 'real part', 'imaginary part', 'modulus', 'argument' and 'conjugate' of a complex number?*

<u>Complex Numbers in Polar & Exponential Form</u>

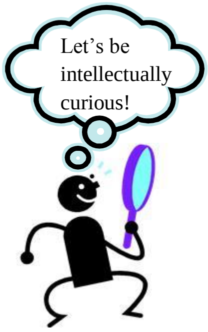
8. How do we convert a complex number from one of the following forms to another: (a) Cartesian form, (b) polar form and (c) exponential form?

9. How do we multiply and divide two complex numbers given in polar and exponential forms?

10. How do we represent a complex number in polar form by a point in the Argand diagram?

§1 Introduction

1.1 Roots of quadratic equations



How many roots do you expect to obtain when solving a quadratic equation?

Solve the following quadratic equations:

(a) $x^2+2x-3=0$ Two real and distinct roots

$x=\dfrac{-2\pm\sqrt{16}}{2}$ $\Rightarrow x=-3$ or $x=1$

(b) $x^2+2x+1=0$ Two real and repeated roots

$x=\dfrac{-2\pm\sqrt{0}}{2}$ $\Rightarrow x=-1$

(c) $x^2+2x+5=0$ No real roots

$x=\dfrac{-2\pm\sqrt{-16}}{2}$

Can we expand our notion of numbers to those that are non-real? How then can we express the roots of equation (c)?

1.2 Definition and Terminology

Definition 1.2.1 (Imaginary Unit)

The imaginary unit, i, is a number such that $\mathrm{i}^2=-1$. Hence, $\mathrm{i}=\sqrt{-1}$.

Thus, the square root of any negative real number can then be written in the form of *a*i, where *a* is a positive real number.

For example, $\sqrt{-3}=\sqrt{(3)(-1)}=\sqrt{3}\sqrt{-1}=\left(\sqrt{3}\right)\mathrm{i}$.

Definition 1.2.2 (Complex Number in Cartesian form)

A complex number, $z\in\mathbb{C}$, is a number of the form $x+\mathrm{i}y$, where $x, y\in\mathbb{R}$.

The symbol $\mathbb{C}$ is used to denote the set of complex numbers.

Hierarchy of the number system:

- Complex Numbers, $\mathbb{C}$
  - Real Numbers, $\mathbb{R}$
    - Rational Numbers, $\mathbb{Q}$
      - Integers, $\mathbb{Z}$
        - Negative Integers
        - Zero
        - Positive Integers
      - Fractions
    - Irrational Numbers
  - Purely Imaginary Numbers

Definition 1.2.3 (Real and Imaginary Parts of a Complex Number)

If $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, then

x is the **real** part of z and is denoted by $\mathrm{Re}(z)$. (i.e. $\mathrm{Re}(z) = x$)
y is the **imaginary** part of z is denoted by $\mathrm{Im}(z)$. (i.e. $\mathrm{Im}(z) = y$)

Note:

1. If $x = 0$, then $z = \mathrm{i}y$ is a **purely imaginary** number.
2. If $y = 0$, then $z = x$ is a **real** number.
3. $\mathrm{Im}(z) = y$ is a real number. $\mathrm{Im}(z)$ is **NOT** iy.

Definition 1.2.4 (Equality of Complex Numbers)

Two complex numbers are **equal** if and only if their real and imaginary parts are equal.

That is, given that $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, then

$$z_1 = z_2 \quad \Leftrightarrow \quad x_1 = x_2 \text{ and } y_1 = y_2 .$$

Note:

1. $z = 0 \quad \Leftrightarrow \quad x = 0$ and $y = 0$
2. Inequalities do not apply to complex numbers that are not real numbers. For example, we cannot say $2 + \mathrm{i}$ is larger or smaller than $2 - \mathrm{i}$.

Example 1.2.5

Find x and y, where $x, y \in \mathbb{R}$, if $x + 2y + \mathrm{i}(3x - y) = 4 + 2\mathrm{i}$.

Solution:

Comparing real part on both sides, we get $\quad x + 2y = 4 \quad ---(1)$

Comparing imaginary part on both sides, we get $\quad 3x - y = 2 \quad ---(2)$

Solving equations (1) and (2) simultaneously, we get $x = \dfrac{8}{7}$ and $y = \dfrac{10}{7}$.

Note: We must ensure that the terms on both sides of the simultaneous equations are real.

§2 Arithmetic Operations on Complex Numbers

2.1 'i' follows all arithmetic operations on real numbers

(i) Addition: 3i + 4i = 7i (ii) Subtraction: 10i – 3i = 7i

(iii) Multiplication:

$a \times \mathrm{i} = a\mathrm{i}$; $\mathrm{i}(a+b) = \mathrm{i}a + \mathrm{i}b$; $a\mathrm{i} \times b\mathrm{i} = ab\mathrm{i}^2 = -ab$, where $a, b \in \mathbb{R}$.

In particular,

$$\mathrm{i}^2 = -1 \text{ (by definition)}$$
$$\mathrm{i}^3 = \mathrm{i}\mathrm{i}^2 = -\mathrm{i}$$
$$\mathrm{i}^4 = \left(\mathrm{i}^2\right)^2 = 1$$
$$\mathrm{i}^5 = \mathrm{i}\mathrm{i}^4 = \mathrm{i}$$

Hence, i to any power can be reduced to one of i, -1, $-\mathrm{i}$ or 1.

(iv) Division: $\mathrm{i}^{-1} = \frac{1}{\mathrm{i}} = \frac{1}{\mathrm{i}} \cdot \frac{\mathrm{i}}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = -\mathrm{i}$.

For Sections **2.2** to **2.3**, consider two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

2.2 Addition, Subtraction & Multiplication of Complex Numbers

(a) $z_1 + z_2 = \left(x_1 + \mathrm{i}\, y_1\right) + \left(x_2 + \mathrm{i}\, y_2\right) = (x_1 + x_2) + \mathrm{i}\,(y_1 + y_2)$

(b) $z_1 - z_2 = \left(x_1 + \mathrm{i}\, y_1\right) - (x_2 + \mathrm{i}\, y_2) = \left(x_1 - x_2\right) + \mathrm{i}\left(y_1 - y_2\right)$

(c) $kz_1 = k\left(x_1 + \mathrm{i}y_1\right) = kx_1 + \mathrm{i}ky_1$, $k \in \mathbb{R}$

(d) $$z_1 \times z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = x_1x_2 + \mathrm{i}y_1x_2 + \mathrm{i}x_1y_2 + \mathrm{i}^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1)$$

2.3 Division of Complex Numbers

This is done by **realising** the denominator, which is multiplying the complex conjugate (refer to page 6) of the denominator to the numerator and denominator.

$$\frac{z_1}{z_2} = \frac{x_1 + \mathrm{i}y_1}{x_2 + \mathrm{i}y_2} \cdot \frac{x_2 - \mathrm{i}y_2}{x_2 - \mathrm{i}y_2} = \frac{(x_1x_2 + y_1y_2) + \mathrm{i}(x_2y_1 - x_1y_2)}{{x_2}^2 + {y_2}^2} = \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2} + \mathrm{i}\left(\frac{x_2y_1 - x_1y_2}{{x_2}^2 + {y_2}^2}\right)$$

Example 2.3.1

Express the following complex numbers in the form $x+\mathrm{i}y$, where $x, y \in \mathbb{R}$.

(a) $(2 + 3\mathrm{i}) - (-1 + 2\mathrm{i})$ (b) $(2 + 4\mathrm{i})(1 - \mathrm{i})$ (c) $\dfrac{5+10\mathrm{i}}{4+3\mathrm{i}}$

Solution:

(a) $(2 + 3\mathrm{i}) - (-1 + 2\mathrm{i}) = 2 + 3\mathrm{i} + 1 - 2\mathrm{i} = 3 + \mathrm{i}$

(b) $(2 + 4\mathrm{i})(1 - \mathrm{i}) = 2 - 2\mathrm{i} + 4\mathrm{i} - 4\mathrm{i}^2 = 2 + 2\mathrm{i} + 4 = 6 + 2\mathrm{i}$

(c) $\dfrac{5+10\mathrm{i}}{4+3\mathrm{i}} = \dfrac{5+10\mathrm{i}}{4+3\mathrm{i}}.\dfrac{4-3\mathrm{i}}{4-3\mathrm{i}} = \dfrac{20-15\mathrm{i}+40\mathrm{i}+30}{16+9} = \dfrac{50+25\mathrm{i}}{25} = 2+\mathrm{i}$

Example 2.3.2

Solve the simultaneous equations $(1-\mathrm{i})z+2\mathrm{i}w=0$ and $3\mathrm{i}z+(1+\mathrm{i})w=\mathrm{i}$.

Solution:

$(1-\mathrm{i})z+2\mathrm{i}w=0$ ----- (1) ; $3\mathrm{i}z+(1+\mathrm{i})w=\mathrm{i}$ ----- (2)

From (1), we have $z=\dfrac{-2\mathrm{i}w}{1-\mathrm{i}}$. Substituting into (2), we get

$$3\mathrm{i}\left(\frac{-2\mathrm{i}w}{1-\mathrm{i}}\right)+(1+\mathrm{i})w=\mathrm{i}$$

$$\frac{6w}{1-\mathrm{i}}+(1+\mathrm{i})w=\mathrm{i}$$

$$6w+(1-\mathrm{i}^2)w=\mathrm{i}(1-\mathrm{i})$$

$$8w=\mathrm{i}+1$$

$$w=\frac{1}{8}+\frac{1}{8}\mathrm{i}$$

Therefore,

$$z=\frac{-2\mathrm{i}\left(\frac{1}{8}+\frac{1}{8}\mathrm{i}\right)}{1-\mathrm{i}}=\frac{\frac{1}{4}(1-\mathrm{i})}{1-\mathrm{i}}=\frac{1}{4}$$

Example 2.3.3

Find the square roots of $3+4\mathrm{i}$.

Solution:

The question requires us to evaluate $\sqrt{3+4\mathrm{i}}$. Let $\sqrt{3+4\mathrm{i}}=x+\mathrm{i}y$.

Then $(x+\mathrm{i}y)^2=3+4\mathrm{i} \Rightarrow x^2-y^2+2\mathrm{i}xy=3+4\mathrm{i}$.

Comparing the real and imaginary parts on both sides, we have:

$x^2-y^2=3$ ----- (1) ; $2xy=4$ ----- (2)

Solving simultaneously, we have $x=2, y=1$ or $x=-2, y=-1$.

The square roots of 3 + 4i are $2+\mathrm{i}$ and $-2-\mathrm{i}$.

§3 Complex Conjugates

Definition 3.1 (Complex Conjugate)

The **complex conjugate** of a complex number $z = x + \mathrm{i}y$ is defined to be the complex number $x - \mathrm{i}y$ and is denoted by $z*$.

For example, if $z = 3 - 5\mathrm{i}$, then $z* = 3 + 5\mathrm{i}$.

Properties 3.2

- $zz* = (x+\mathrm{i}y)(x-\mathrm{i}y) = x^2 - \mathrm{i}^2y^2 + \mathrm{i}yx - \mathrm{i}xy = x^2 + y^2$, i.e. the product of any complex number and its conjugate is real.

- $z + z* = 2\mathrm{Re}(z)$
- $z - z* = 2\mathrm{i}\mathrm{Im}(z)$
- $(z*)^* = z$
- $(kz)* = kz*$, where $k \in \mathbb{R}$
- $(z_1 \pm z_2)* = z_1* \pm z_2*$
- $(z_1z_2)* = z_1*\cdot z_2*$
- $(z^n)^* = (z*)^n$, where $n \in \mathbb{Z}$
- $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1*}{z_2*}$

Example 3.3

If $z_1 = 2 + \mathrm{i}$ and $z_2 = 4 - \mathrm{i}$, find $(2z_1 + z_2)*$ and $(z_1z_2)*$.

Solution:

$$(2z_1 + z_2)* = 2z_1* + z_2* = 2(2-\mathrm{i}) + (4+\mathrm{i}) = 4 - 2\mathrm{i} + 4 + \mathrm{i} = 8 - \mathrm{i}$$
$$(z_1z_2)* = z_1*z_2* = (2-\mathrm{i})(4+\mathrm{i}) = 8 + 2\mathrm{i} - 4\mathrm{i} - \mathrm{i}^2 = 9 - 2\mathrm{i}$$

Example 3.4

If $z = 4 + 9\mathrm{i}$, find $zz*$.

Solution:

$$zz* = (4+9\mathrm{i})(4-9\mathrm{i}) = 4^2 - (9\mathrm{i})^2 = 4^2 + 9^2 = 97$$

Example 3.5

Find $\dfrac{\left[(2+\mathrm{i})^5\right]^*}{(2-\mathrm{i})^3}$ in the form $x + \mathrm{i}y$.

Solution:

$$\frac{\left[(2+\mathrm{i})^5\right]^*}{(2-\mathrm{i})^3}=\frac{\left[(2+\mathrm{i})^*\right]^5}{(2-\mathrm{i})^3}=\frac{(2-\mathrm{i})^5}{(2-\mathrm{i})^3}=(2-\mathrm{i})^2=3-4\mathrm{i}$$

Example 3.6

Solve the simultaneous equations $(1-\mathrm{i})z+2\mathrm{i}w=0$ --- (1) and
$3\mathrm{i}z+(1+\mathrm{i})w^*=\mathrm{i}$ --- (2)

Solution:

From (1), $z=\dfrac{-2\mathrm{i}w}{1-\mathrm{i}}$ -- (3)

Substituting (3) into (2),

$$3\mathrm{i}\left(\frac{-2\mathrm{i}w}{1-\mathrm{i}}\right)+(1+\mathrm{i})w^*=\mathrm{i}$$
$$3\mathrm{i}(-2\mathrm{i}w)+(1+\mathrm{i})(1-\mathrm{i})w^*=(1-\mathrm{i})\mathrm{i}$$
$$6w+2w^*=1+\mathrm{i}$$

Let $w=x+\mathrm{i}y$, where x and y are real. Then

$$6(x+\mathrm{i}y)+2(x-\mathrm{i}y)=1+\mathrm{i}$$
$$8x+\mathrm{i}(4y)=1+\mathrm{i}$$

Comparing real and imaginary parts, $x=\dfrac{1}{8}, y=\dfrac{1}{4}$.

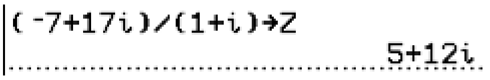
Hence $w=\dfrac{1}{8}+\dfrac{1}{4}\mathrm{i}$ and $z=\dfrac{-2\mathrm{i}w}{1-\mathrm{i}}=\dfrac{3}{8}+\dfrac{1}{8}\mathrm{i}$.

Example 3.7 (GC Practice. Refer to Appendix I for more GC information)

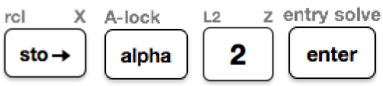
Given that $z=\dfrac{-7+17\mathrm{i}}{1+\mathrm{i}}$, find (i) $\sqrt{z}$ (ii) $z^2+\dfrac{1}{z^*}$

Solution:

(i)

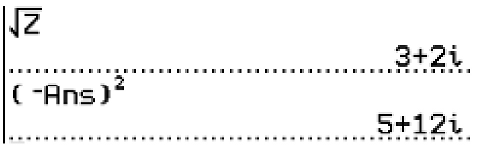


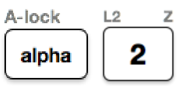
Step 1: Store z as $\dfrac{-7+17\mathrm{i}}{1+\mathrm{i}}$ in the GC.

Use 

to store the complex number as z. GC will simply express the number in Cartesian form. [Note that you should have changed the mode to "a + bi " form.]

Step 2: Find $\sqrt{z}$.



Using 

to call out z, we obtain 3 + 2i as a square root of z.

[To check your answer, you may want to square 3 + 2i to see if you get back z.]

In the above solution for Example 3.7(i), is 3 + 2i the only square root of z?

Check the answer using the algebraic method shown in Example 2.3.3.

(ii) Find $z^2 + \dfrac{1}{z^*}$.

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Z²+1/conj(Z)
 -118.9704142+120.0710059i
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Using conj(from MATH CMPLX menu to represent z^*, we see that $z^2 + \dfrac{1}{z^*} =$ $-118.97 + 120.07\text{i}$.

§4 Complex Roots of Equations

Theorem 4.1 (Fundamental Theorem of Algebra)

Every polynomial of **degree *n*** with coefficients in $\mathbb{C}$ has **exactly *n* roots** in $\mathbb{C}$, including repeated roots.

Note: Complex roots include real roots as the imaginary part can be zero.

Theorem 4.2 (Complex Conjugate Root Theorem)

All non-real roots of a polynomial with **real coefficients** must occur in **conjugate pairs**.

That is, if $z\ (= a + \text{i}b, b \neq 0)$ is a root of

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2 x^2 + a_1 x + a_0 = 0 \text{ where } a_i \in \mathbb{R}, \text{ for } i = 0, 1, 2, ..., n-1, n,$$

then $z^*\ (= a - \text{i}b, b \neq 0)$ is also a root of the equation.

Example 4.3

Solve $x^2 + 2x + 5 = 0$.

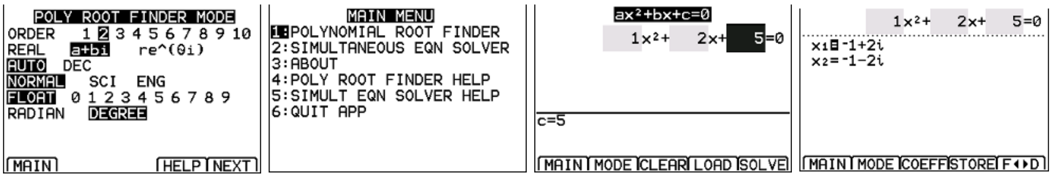
Solution:

By Fundamental Theorem of Algebra, we expect 2 roots.

$$x^2 + 2x + 5 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2}$$
$$= \frac{-2 \pm \sqrt{-16}}{2}$$
$$= -1 + 2\text{i} \text{ or } -1 - 2\text{i}$$

*We see that the Complex Conjugate Root Theorem holds true as the **<u>coefficients are real</u>**.*

Alternatively, using a GC,



$x=-1+2\text{i}$ or $x=-1-2\text{i}$

Example 4.4

Solve $z^2-3\text{i}z-2=0$.

Solution:

$$\begin{aligned} z^2-3\text{i}z-2=0 \Rightarrow z &= \frac{3\text{i}\pm\sqrt{(-3\text{i})^2-4(1)(-2)}}{2} \\ &= \frac{3\text{i}+\text{i}}{2} \text{ or } \frac{3\text{i}-\text{i}}{2} \\ &= 2\text{i} \text{ or } \text{i} \end{aligned}$$

We see that i *and* 2i *are not conjugate pairs since the Complex Conjugate Root Theorem would not hold when <u>the equation has a non-real coefficient</u>.*

Example 4.5

By completing the square, solve the equation $z^2+(4-2\text{i})z-8\text{i}=0$. Explain why the solutions are not a conjugate pair.

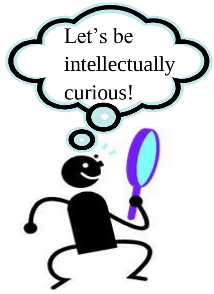
Solution:

$$\begin{aligned} z^2+(4-2\text{i})z-8\text{i} &= 0 \\ (z+2-\text{i})^2-8\text{i}-(2-\text{i})^2 &= 0 \\ (z+2-\text{i})^2 &= 8\text{i}+(2-\text{i})^2 \\ (z+2-\text{i})^2 &= 8\text{i}+4-4\text{i}-1 \\ &= 3+4\text{i} \end{aligned}$$

From **Example 2.3.3**, we have found that $\sqrt{3+4\text{i}}=\pm(2+\text{i})$.

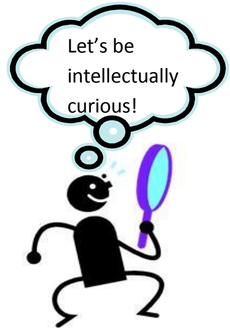
Hence, $z+2-\text{i}=\pm\sqrt{3+4\text{i}}=\pm(2+\text{i})\Rightarrow z=2\text{i}$ or $z=-4$.

Observe that since not all the coefficients are real, <u>the complex roots do not occur in conjugate pairs.</u>



Try to solve the equation in Examples 4.4 and 4.5 using the graphing calculator. Can the graphing calculator solve polynomial equations with non-real coefficients?

The graphing calculator is unable to solve polynomial equations with non-real coefficients.



How do we prove the Complex Conjugate Root Theorem? (Given that z is a complex root of a polynomial of degree n, prove that z^* is also a root of the polynomial)

Hint: You may use the following complex conjugate properties: $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$;

$(kz)^* = kz^*$, where $k \in \mathbb{R}$;

$\left(z^n\right)^* = (z^*)^n$, where $n \in \mathbb{Z}$

Example 4.6

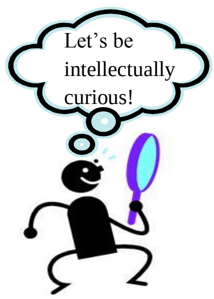
If $1-\mathrm{i}$ is a root of $x^3 - x^2 + 2 = 0$, find the other roots without using a graphing calculator.

Solution:

Since the coefficients are real, by Complex Conjugate Root Theorem, 1 + i is also a root of the equation. Also, the third root must be a real number, a.

By trial and error, we have
$\mathrm{f}(-1) = (-1)^3 - (-1)^2 + 2 = -1 - 1 + 2 = 0,$ where $\mathrm{f}(x) = x^3 - x^2 + 2$.
Therefore, $a = -1$ is the third root.

Hence the other roots are $x = -1$ and $x = 1 + \mathrm{i}$.



Can we solve this equation without using trial and error?

<u>Method 1: Long Division</u>
$\left(x - (1+\mathrm{i})\right)\left(x - (1-\mathrm{i})\right) = x^2 - 2x + 2$

Performing long division of $x^3 - x^2 + 2$ by $x^2 - 2x + 2$, we obtain $x + 1$ as the last factor.
Hence, $x = -1$ is a root.

Method 2: Comparing coefficients

The last root must be a real number, α. Thus we have

$$\left(x-(1+\mathrm{i})\right)\left(x-(1-\mathrm{i})\right)\left(x-\alpha\right)=x^3-x^2+2$$

$$\left(x^2-2x+2\right)\left(x-\alpha\right)=x^3-x^2+2$$

Comparing the constant terms on both sides of the equation, we get

$$-2\alpha=2$$

$$\alpha=-1$$

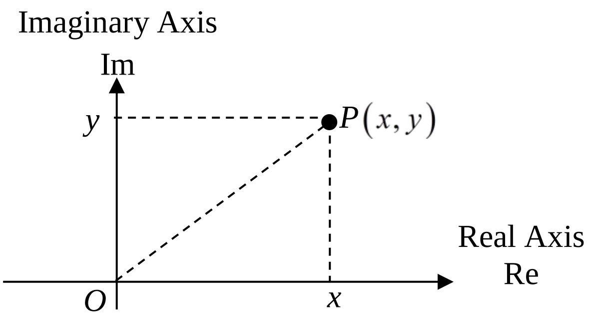
How many real roots can a quadratic/cubic equation with real coefficients have? How many real roots can it have if not all the coefficients are real?

§5 Geometrical Representation of Complex Numbers

5.1 The Argand Diagram

A complex number $z=x+\mathrm{i}y$ can be represented as a point P with Cartesian coordinates (x,y) on the x-y plane.

The x-y plane is called the ***Argand diagram*** where the horizontal axis is known as the ***real axis,*** denoted by **Re** and the vertical axis the ***imaginary axis,*** denoted by **Im**.

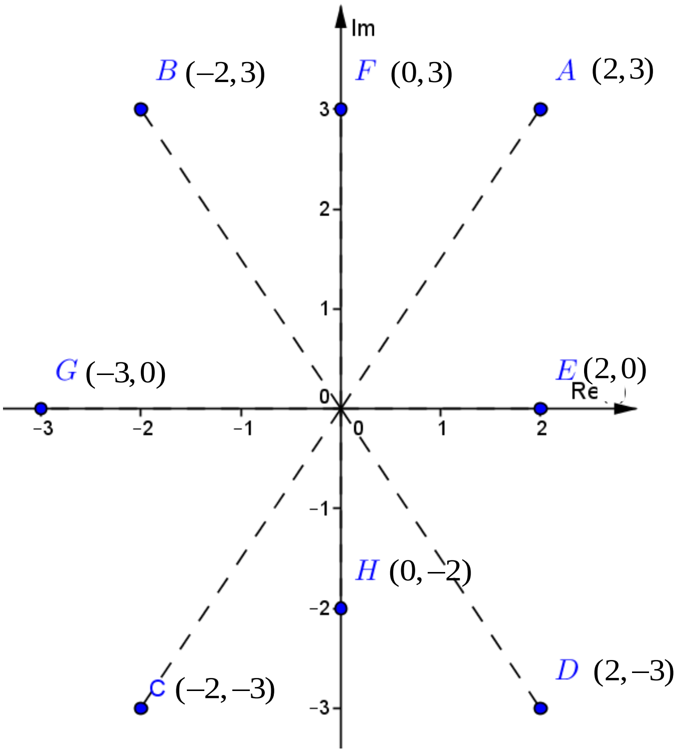


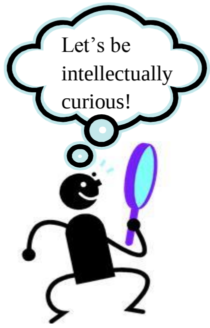
Example 5.1.1

Represent the following complex numbers in an Argand diagram:
$a=2+3\mathrm{i}$, $b=-2+3\mathrm{i}$, $c=-2-3\mathrm{i}$, $d=2-3\mathrm{i}$, $e=2$, $f=3\mathrm{i}$, $g=-3$, $h=-2\mathrm{i}$

Solution:

Let A, B, …, H be the points representing a, b, …, h in the Argand diagram.





How are the following related in an Argand diagram?

(a) z and z^* (see A and D)
They are reflections about the real axis.

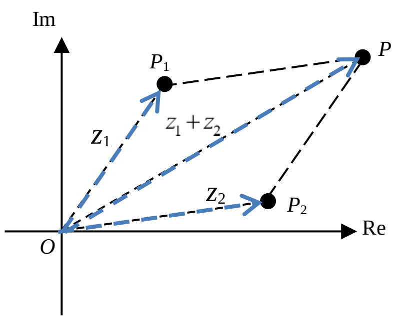
(b) z and $-z$ (see A and C)
They are reflections about the origin.

What about A and B? We see that $(-z)^* = -(z^*)$

Complex numbers can alternatively be represented by position vectors, i.e., a complex number $z = x + \mathrm{i}y$ can be represented as a position vector $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the x-y plane.

Techniques and operations used in coordinate geometry and vectors can be applied to complex numbers. Addition and subtraction of complex numbers correspond to the **parallelogram law of vector addition and subtraction**.

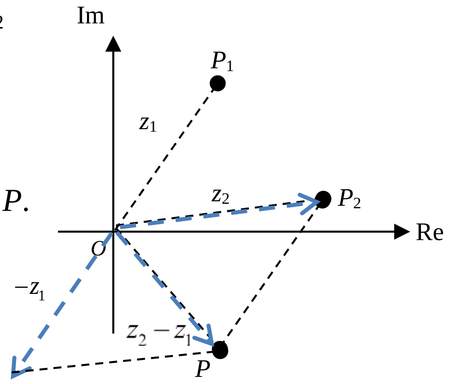
Let $z_1 = a + \mathrm{i}b$ be represented by P_1
$z_2 = c + \mathrm{i}d$ be represented by P_2.
$z = z_1 + z_2 = (a + c) + \mathrm{i}\,(b + d)$ be represented by P.



In terms of vectors,

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$$

The subtraction $z_2 - z_1$ is the same as the addition of z_2 and $-z_1$. Thus, let $\overrightarrow{P_2P}$ represents $-z_1$, then the $\overrightarrow{OP}$ represents $z_2 - z_1$.



$z = z_2 - z_1 = (c - a) + \mathrm{i}\,(d - b)$ be represented by P.

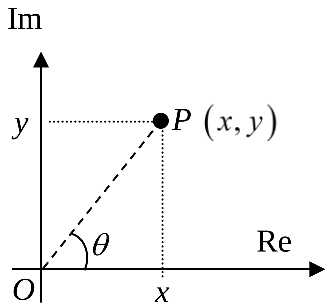
In terms of vectors,

$$\overrightarrow{OP} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

What are some possible limitations of considering complex numbers as vectors on the Argand Diagram?

Every complex number can only be represented by position vectors and not displacement vectors. So for example when finding the difference between two complex numbers (e.g. $z_2 - z_1$), the vector obtained must be translated to start from the origin first, before the real and imaginary parts of the complex number can be obtained. Finding differences of vectors on the other hand, has no such restriction.

5.2 Modulus and Argument of a Complex Number



Without loss of generality, P represents the complex number $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}^+$. We have:

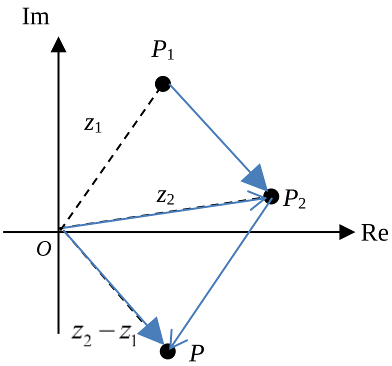
Definition 5.2.1 (Modulus)

The **modulus** of a complex number $z = x + \mathrm{i}y$ is defined as $\sqrt{x^2 + y^2}$, and is denoted by $|z|$.

Geometrically, it is the *distance* between the origin and the point P, i.e., the magnitude of $\overrightarrow{OP}$.

Note:

1. $|z|$ is always a non-negative real number, e.g., $|-3+4\text{i}|=\sqrt{(-3)^2+4^2}=5$.
2. From vector subtraction (diagram from parallelogram law of vector subtraction), we see that $|z_2-z_1|=|\overrightarrow{P_1P_2}|$ is the distance between P_1 and P_2.



Definition 5.2.2 (Argument)

The **argument** of *z* is defined as *θ*, the angle between OP and the positive real axis, and is denoted by arg(*z*).

Note:

1. *θ* is positive when measured anti-clockwise from the positive real axis and is negative when measured clockwise from the positive real axis.
2. It is important that you first check the quadrant in which *z* lies before computing its argument.

Example 5.2.3

Find the modulus and argument of each of the following complex numbers:

$$3,\ -5,\ -2\text{i},\ \sqrt{3}+\text{i} \text{ and } -\sqrt{3}-\text{i}.$$

Solution:

$$|3|=3, \qquad \arg(3)=0$$

$$|-5|=5, \qquad \arg(-5)=\pi$$

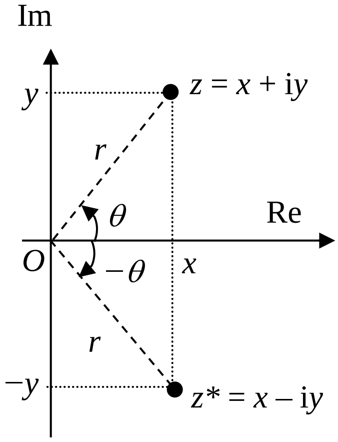
$$|-2\text{i}|=2, \qquad \arg(-2\text{i})=-\frac{\pi}{2}$$

$$|\sqrt{3}+\text{i}|=\sqrt{3+1}=2, \qquad \arg(\sqrt{3}+\text{i})=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

$$|-\sqrt{3}-\text{i}|=\sqrt{3+1}=2, \qquad \arg(-\sqrt{3}-\text{i})=-\left(\pi-\tan^{-1}\frac{1}{\sqrt{3}}\right)=-\frac{5\pi}{6}$$

Note: There can be multiple answers for arg(*z*), e.g., $\arg\left(\sqrt{3}+\text{i}\right)$ can be $\frac{\pi}{6}$ or other values differing by $2k\pi$, where $k\in\mathbb{Z}$. However, we will usually quote the **principal argument** as the answer. *θ* is the principal argument if it lies in the **principal range**, i.e., $-\pi<\theta\le\pi$.

Note:



1. arg(0) is undefined.
2. z is **real** $\Leftrightarrow$ arg(z) = 0 or π.
3. z is **purely imaginary** $\Leftrightarrow$ $\arg(z) = \pm\frac{\pi}{2}$.
4. z, z^* and $-z$ have the same modulus, i.e., $|z| = |z^*| = |-z|$
5. $|z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$
6. $\arg(z^*) = -\arg(z)$
7. $\boxed{zz^* = |z|^2}$ (A very useful result.)

Example 5.2.4

Find the <u>reciprocal</u> of $z = 2 - 3\text{i}$, i.e., $\frac{1}{z}$.

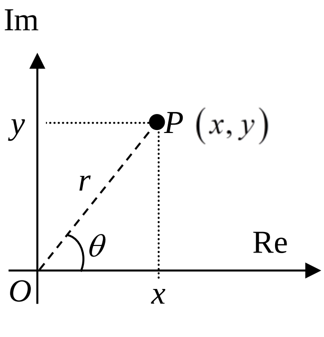
Solution:

$$\frac{1}{z} = \frac{1}{z} \times \frac{z^*}{z^*} = \frac{z^*}{|z|^2} = \frac{2+3\text{i}}{2^2 + (-3)^2} = \frac{2}{13} + \frac{3}{13}\text{i}.$$

(Here we are using the identity $zz^* = |z|^2$)

§6 Polar Form of a Complex Number

6.1 Cartesian Form $\rightleftharpoons$ Polar Form



Suppose $z = x + \text{i}y$ (in Cartesian form).
From the Argand diagram, we have

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \text{ for basic angle, } \theta$$

From the Argand diagram, we also have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore,

$$\begin{aligned} z &= x + \text{i}y \\ &= r\cos\theta + \text{i}r\sin\theta \\ &= r(\cos\theta + \text{i}\sin\theta) \end{aligned}$$

We have now expressed z in **polar form (also known as modulus–argument form or trigonometric form)**: $z = r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$.
The argument θ is usually expressed in the principal range (i.e. $-\pi < \theta \le \pi$) unless otherwise specified.

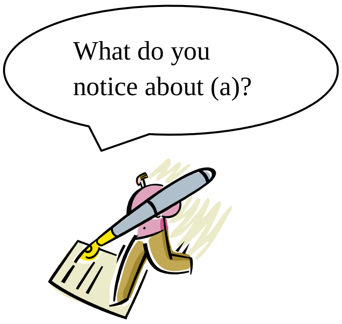
Note: The following expressions (where $r > 0$) are not in polar form. Can you explain why this is so? Next, try converting them to polar form.

(a) $r(\cos\theta - \mathrm{i}\sin\theta) = r[\cos(-\theta) + \mathrm{i}\sin(-\theta)]$

(b) $r(\sin\theta + \mathrm{i}\cos\theta) = r\left[\cos\left(\frac{\pi}{2} - \theta\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \theta\right)\right]$

(c) $-r(\cos\theta + \mathrm{i}\sin\theta) = r[\cos(\pi + \theta) + \mathrm{i}\sin(\pi + \theta)]$

(d) $r(\sin\theta - \mathrm{i}\cos\theta) = r\left[\cos\left(\theta - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \frac{\pi}{2}\right)\right]$



Example 6.1.1
Express (i) -3 and (ii) $-5 + 5\mathrm{i}$ in polar form.

Solution:
(i)

$|-3| = 3, \ \arg(-3) = \pi$

$\Rightarrow -3 = 3(\cos\pi + \mathrm{i}\sin\pi)$

(ii)

$|-5 + 5\mathrm{i}| = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}, \ \arg(-5 + 5\mathrm{i}) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$\Rightarrow -5 + 5\mathrm{i} = 5\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$$

§7 Exponential Form of a Complex Number

7.1 Euler's Formula

From the Maclaurin's expansion of $\cos\theta$, $\sin\theta$ and e^x for all real values of x and θ, we have

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots, \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots,$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

If we let $x = \mathrm{i}\theta$, we get

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \ldots \\ &= 1 + \mathrm{i}\theta - \frac{1}{2!}\theta^2 - \mathrm{i}\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \ldots \\ &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right) + \mathrm{i}\left(\theta - \frac{1}{3!}\theta^3 + \ldots\right) = \cos\theta + \mathrm{i}\sin\theta \end{aligned}$$

Euler's Formula: $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$

If we put $\theta = \pi$ in Euler's Formula, then we would obtain a remarkable identity which links together geometry, algebra, and five of the most essential symbols in math -- 0, 1, i, π and e -- that are essential tools in scientific work.

$$\mathrm{e}^{\mathrm{i}\pi} = \cos\pi + \mathrm{i}\sin\pi = -1 \Rightarrow \mathrm{e}^{\mathrm{i}\pi} + 1 = 0$$

7.2 Exponential Form

Since any complex number can be expressed in the polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, it follows from Euler's formula that we can write

$$z = r\left(\cos\theta + \mathrm{i}\sin\theta\right) = r\mathrm{e}^{\mathrm{i}\theta},$$

where $r = |z|$ and $\theta = \arg(z)$, usually expressed in principal range i.e. $-\pi < \theta \le \pi$.

$z = r\mathrm{e}^{\mathrm{i}\theta}$ is known as the e**xponential form** of z.

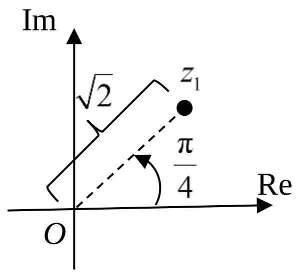
Note:

1. For $z = r\mathrm{e}^{\mathrm{i}\theta}$, θ must be in **radians** and usually expressed in the principal range, $-\pi < \theta \le \pi$.
2. $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = \left|\cos\theta + \mathrm{i}\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.
3. $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(2n\pi+\theta)}$, for $n \in \mathbb{Z}$.
4. $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta$, $\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = 2\mathrm{i}\sin\theta$
5. For $x, y \in \mathbb{R}$, $\left|\mathrm{e}^{x+\mathrm{i}y}\right| = \left|\mathrm{e}^{x}.\mathrm{e}^{\mathrm{i}y}\right| = \left|\mathrm{e}^{x}\right|\left|\mathrm{e}^{\mathrm{i}y}\right| = \mathrm{e}^{x}$

$$\arg\left(\mathrm{e}^{x+\mathrm{i}y}\right) = \arg\left(\mathrm{e}^{x}.\mathrm{e}^{\mathrm{i}y}\right) = \arg\left(\mathrm{e}^{x}\right) + \arg\left(\mathrm{e}^{\mathrm{i}y}\right) = 0 + y = y$$

Example 7.2.1

Express the following in exponential form:
(i) $z_1 = 1+\mathrm{i}$ (ii) $z_2 = \sqrt{3}-\mathrm{i}$



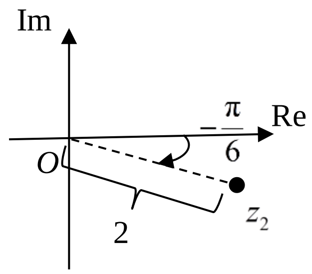
Solution:
(i)
$|z_1| = |1+\mathrm{i}| = \sqrt{2}$, $\arg(1+\mathrm{i}) = \frac{\pi}{4}$

$\Rightarrow z_1 = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

(ii)
$|z_2| = \left|\sqrt{3}-\mathrm{i}\right| = \sqrt{3+1} = 2,$

$\arg\left(\sqrt{3}-\mathrm{i}\right) = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

$\Rightarrow z_2 = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$



7.3 Multiplication/Quotient of Complex Numbers in Exponential Form

Let $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2) = r_2\mathrm{e}^{\mathrm{i}\theta_2}$. Then,

(1) $$z_1 \cdot z_2 = r_1\mathrm{e}^{\mathrm{i}\theta_1} \cdot r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1r_2\mathrm{e}^{\mathrm{i}\theta_1} \cdot \mathrm{e}^{\mathrm{i}\theta_2} = r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$$

Hence, $\lvert z_1 \cdot z_2\rvert = r_1r_2 = \lvert z_1\rvert\lvert z_2\rvert$ and $\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

We can extend the above result to get
(a) $|z_1z_2z_3...z_n| = |z_1||z_2||z_3|....|z_n|$
(b) $\arg(z_1z_2z_3....z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + ... + \arg(z_n) = \sum_{k=1}^{n}\arg(z_k)$

In particular, when $z_1 = z_2 = z_3 = ... = z_n = z$, we have
(c) $|z^n| = |z|^n$
(d) $\arg(z^n) = n\arg(z)$

(2) $$\frac{z_1}{z_2} = \frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}.$$

Hence, $\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{r_1}{r_2} = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$ and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

Example 7.3.1

Given that $z_1 = 1+\sqrt{3}\text{i}$ and $z_2 = -\sqrt{3}+\text{i}$. Find $\arg\left(z_1 z_2\right)$.

Solution:

$$\arg\left(z_1\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \text{ and } \arg\left(z_2\right) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{5\pi}{6}$$

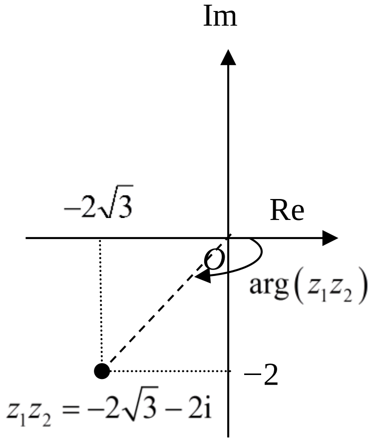
$$\arg\left(z_1 z_2\right) = \arg\left(z_1\right) + \arg\left(z_2\right) = \frac{\pi}{3} + \frac{5\pi}{6} = \frac{7\pi}{6} \equiv -\frac{5\pi}{6}.$$

Note: The argument is usually expressed in the principal range (i.e. $-\pi < \theta \le \pi$) unless otherwise specified.

<u>Alternative Method</u>

$$\begin{aligned} z_1 z_2 &= \left(1+\sqrt{3}\text{i}\right)\left(-\sqrt{3}+\text{i}\right) \\ &= -\sqrt{3}+\text{i}-3\text{i}-\sqrt{3} \\ &= -2\sqrt{3}-2\text{i} \end{aligned}$$

$$\begin{aligned} \arg\left(z_1 z_2\right) &= -\pi + \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) \\ &= -\pi + \frac{\pi}{6} = -\frac{5\pi}{6} \end{aligned}$$



What are the advantages/disadvantages compared to the previous method?

Example 7.3.2 (Simplifying using laws of indices)

Express the following in exponential form:

(i) $\dfrac{1+\text{i}}{\sqrt{3}-\text{i}}$ (ii) $(1+\text{i})^3(\sqrt{3}-\text{i})^4$

Solution:

(i) $$\frac{1+\text{i}}{\sqrt{3}-\text{i}} = \frac{\sqrt{2}\text{e}^{\text{i}\frac{\pi}{4}}}{2\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}} = \frac{1}{\sqrt{2}}\text{e}^{\text{i}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)} = \frac{1}{\sqrt{2}}\text{e}^{\text{i}\left(\frac{5\pi}{12}\right)}$$

(ii) $$(1+\text{i})^3(\sqrt{3}-\text{i})^4 = {z_1}^3{z_2}^4 = \left(\sqrt{2}\text{e}^{\text{i}\frac{\pi}{4}}\right)^3\left(2\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}\right)^4 = 32\sqrt{2}\text{e}^{\text{i}\left(\frac{3\pi}{4}-\frac{2\pi}{3}\right)} = 32\sqrt{2}\text{e}^{\text{i}\left(\frac{\pi}{12}\right)}$$

* *You may use the GC if exact form is not required.*

Note that by expressing a complex number in exponential form first can aid in the subsequent manipulations/simplifications.

Example 7.3.3 (Simplifying using properties of modulus and argument)

Express the following in polar and exponential form.

(i) $\left(1-\mathrm{i}\sqrt{3}\right)^3\left(-1+\mathrm{i}\right)^2$ (ii) $\dfrac{\left(1-\mathrm{i}\sqrt{3}\right)^6}{\left(-1+\mathrm{i}\right)^3}$

Solution:

(i)

$$\left|\left(1-\mathrm{i}\sqrt{3}\right)^3\left(-1+\mathrm{i}\right)^2\right| = \left|1-\mathrm{i}\sqrt{3}\right|^3\left|-1+\mathrm{i}\right|^2$$
$$=\left(\sqrt{1+3}\right)^3\left(\sqrt{1+1}\right)^2 = 16$$

$$\arg\left[\left(1-\mathrm{i}\sqrt{3}\right)^3\left(-1+\mathrm{i}\right)^2\right] = \arg\left(1-\mathrm{i}\sqrt{3}\right)^3 + \arg\left(-1+\mathrm{i}\right)^2$$
$$= 3\arg\left(1-\mathrm{i}\sqrt{3}\right) + 2\arg\left(-1+\mathrm{i}\right)$$
$$= 3\left(-\tan^{-1}\sqrt{3}\right) + 2\left(\pi - \frac{\pi}{4}\right) = \frac{\pi}{2}$$

Thus,

$$\left(1-\mathrm{i}\sqrt{3}\right)^3\left(-1+\mathrm{i}\right)^2 = 16\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}$$
$$= 16\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right).$$

(ii)

$$\left|\frac{\left(1-\mathrm{i}\sqrt{3}\right)^6}{\left(-1+\mathrm{i}\right)^3}\right| = \frac{\left|1-\mathrm{i}\sqrt{3}\right|^6}{\left|-1+\mathrm{i}\right|^3} = \frac{\left(\sqrt{1+3}\right)^6}{\left(\sqrt{1+1}\right)^3} = \frac{2^6}{\left(\sqrt{2}\right)^3} = 16\sqrt{2}$$

$$\arg\left[\frac{\left(1-\mathrm{i}\sqrt{3}\right)^6}{\left(-1+\mathrm{i}\right)^3}\right] = 6\arg\left(1-\mathrm{i}\sqrt{3}\right) - 3\arg\left(-1+\mathrm{i}\right) = 6\left(-\tan^{-1}\sqrt{3}\right) - 3\left(\pi - \frac{\pi}{4}\right) = -\frac{17}{4}\pi$$

$$\text{principal arg}\left[\frac{\left(1-\mathrm{i}\sqrt{3}\right)^6}{\left(-1+\mathrm{i}\right)^3}\right] = -\frac{17}{4}\pi + 2\left(2\pi\right) = -\frac{\pi}{4}$$

Thus, $\dfrac{\left(1-\mathrm{i}\sqrt{3}\right)^6}{\left(-1+\mathrm{i}\right)^3} = 16\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)} = 16\sqrt{2}\left[\cos\left(-\dfrac{\pi}{4}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right].$

** You may use the GC if exact form is not required.*

Example 7.3.4 ("Half-Argument" approach)

The complex number q is given by $q=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}$, where $0<\theta<2\pi$. In either order,

(i) find the real part of q (ii) show that the imaginary part of q is $\dfrac{1}{2}\cot\dfrac{\theta}{2}$.

Solution:

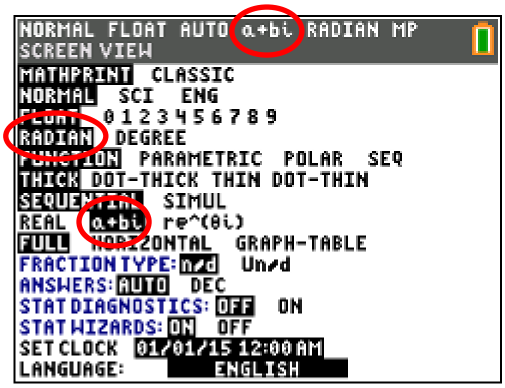
$$\begin{aligned}
q &= \frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} \\
&= \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta/2}\left(\mathrm{e}^{-\mathrm{i}\theta/2}-\mathrm{e}^{\mathrm{i}\theta/2}\right)} \\
&= \frac{\mathrm{e}^{\mathrm{i}\theta/2}}{\mathrm{e}^{-\mathrm{i}\theta/2}-\mathrm{e}^{\mathrm{i}\theta/2}} \\
&= \frac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{\left(\cos\frac{-\theta}{2}+\mathrm{i}\sin\frac{-\theta}{2}\right)-\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)} \\
&= \frac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{-2\mathrm{i}\sin\frac{\theta}{2}} = \frac{\mathrm{i}\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\
&= -\frac{1}{2}+\mathrm{i}\,\frac{1}{2}\cot\frac{\theta}{2}
\end{aligned}$$

Thus, $\mathrm{Re}(q)=-\dfrac{1}{2}$, and $\mathrm{Im}(q)=\dfrac{1}{2}\cot\dfrac{\theta}{2}$.

Appendix I: Complex Numbers in GC

This section aims to equip you with some basics of using GC in complex numbers. However, you are also strongly encouraged to read up any GC guidebook to acquire the basic skills required to utilise a GC and to explore its various functions.

Note that you may use the GC to help you in your calculations involving complex numbers, unless it is stated that the problem given needs to be solved without the use a calculator, or if an exact solution is required.

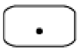


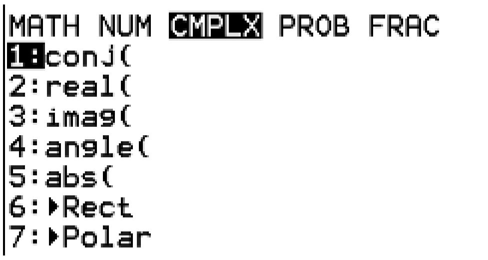
Getting Started

1. Press 

to display mode settings. Scroll down to select "*a* + *b i* " so that you can obtain complex number solutions in Cartesian form.
   It is also recommended that you use radian mode for calculations involving complex numbers.

2. To enter the complex number *i*, press 

.



3. Other operations or functions for complex numbers can be found in the Math CPX menu, which contains standard operations involving complex numbers.

   Press 

  to get to the CMPLX menu.

A brief description of the 7 operations or functions in the menu:

Operation:	**Form :**	**Explanation:**
1: conj(	conj(*complex number z*)	Returns the complex conjugate z^* of z.
2: real(	real(*complex number z*)	Returns the real part, x, of $z = x + y\mathrm{i} = re^{\mathrm{i}\theta}$.
3: imag(	imag(*complex number z*)	Returns the complex part, y of $z = x + y\mathrm{i} = re^{\mathrm{i}\theta}$.
4. angle(	angle(*complex number z*)	Returns the principal argument, θ, of $z = x + y\mathrm{i} = re^{\mathrm{i}\theta}$.
5: abs(	abs(*complex number z*)	Returns the modulus, r, of $z = x + y\mathrm{i} = re^{\mathrm{i}\theta}$.
6: ►Rect	*Complex number z* ►Rect	Displays z in Cartesian form, $z = x + y\mathrm{i}$.
7: ►Polar	*Complex number z* ►Polar	Displays z in Exponential form, $z = re^{\mathrm{i}\theta}$.