1

$$\frac{4x^2+7x+1}{3x+1} \le x+2$$

$$\frac{4x^2+7x+1-(x+2)(3x+1)}{3x+1} \le 0$$

$$\frac{4x^2+7x+1-(3x^2+x+6x+2)}{3x+1} \le 0$$

$$\frac{x^2-1}{3x+1} \le 0$$

$$\frac{(x-1)(x+1)}{3x+1} \le 0$$

Sign diagram: $-$ for $x<-1$; $+$ for $-1<x<-\frac{1}{3}$; $-$ for $-\frac{1}{3}<x<1$; $+$ for $x>1$ (critical points -1, $-\frac{1}{3}$, 1)

$\therefore \; x \le -1$ or $-\frac{1}{3} < x \le 1$

$$\frac{4x+7\sqrt{x}+1}{3\sqrt{x}+1} \le \sqrt{x}+2$$

Replace x with $\sqrt{x}$,

$\therefore \; \sqrt{x} \le -1$ or $-\frac{1}{3} < \sqrt{x} \le 1$

(rejected as $\sqrt{x} \ge 0$)

Since $\sqrt{x} \ge 0$,

$-\frac{1}{3} < \sqrt{x} \le 1 \quad \Rightarrow \quad 0 \le \sqrt{x} \le 1$

$0 \le x \le 1$

2

(i)

$$\int n\cos^{-1}(nx)\,\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\int (nx)\left(-\frac{n}{\sqrt{1-(nx)^2}}\right)\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\int \left(-2n^2x\right)\left(1-n^2x^2\right)^{-1/2}\,\mathrm{d}x$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\times\frac{\left(1-n^2x^2\right)^{1/2}}{\frac{1}{2}}+C$$

$$=(nx)\cos^{-1}(nx)-\sqrt{\left(1-n^2x^2\right)}+C$$

(ii)

$$\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$$

$$=\Big[(nx)\cos^{-1}(nx)-\sqrt{(1-n^2x^2)}\ \Big]_0^{\frac{1}{2n}}$$

$$=\left[\frac{1}{2}\cos^{-1}\frac{1}{2}-\sqrt{1-\frac{1}{4}}\right]-(0-1)$$

$$=\frac{\pi}{6}-\frac{\sqrt{3}}{2}+1 \quad or \quad \frac{\pi}{6}+\frac{2-\sqrt{3}}{2}$$

---

3 (i)

$$\begin{aligned}(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) &= 4\mathbf{p}\times\mathbf{p}+10\mathbf{p}\times\mathbf{q}-10\mathbf{q}\times\mathbf{p}-25\mathbf{q}\times\mathbf{q}\\ &= 20\mathbf{p}\times\mathbf{q}\\ &= 20\begin{pmatrix}2\\1\\a\end{pmatrix}\times\begin{pmatrix}b\\1\\0\end{pmatrix}\\ &= 20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}\end{aligned}$$

Alternative:

$$\begin{aligned}(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) &= \left(2\begin{pmatrix}2\\1\\a\end{pmatrix}-5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\times\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}+5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\\ &= \begin{pmatrix}4-5b\\-3\\2a\end{pmatrix}\times\begin{pmatrix}4+5b\\7\\2a\end{pmatrix}\\ &= \begin{pmatrix}-6a-14a\\-(8a-10ab-8a-10ab)\\28-35b+12+15b\end{pmatrix}\\ &= \begin{pmatrix}-20a\\20ab\\40-20b\end{pmatrix}=20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}\end{aligned}$$

Given that the **i-** and **j-** components of the vector $20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ are equal,

$$-a=ab$$

$$ab+a=0$$

$$a(b+1)=0$$

Since $a\neq 0$, thus $b=-1$

(ii)

$$\left|(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})\right|=80$$

$$\left|20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}\right|=80$$

$$\left|\begin{pmatrix}-a\\ -a\\ 2+1\end{pmatrix}\right|=4$$

$$\sqrt{2a^2+9}=4$$

$$2a^2+9=16$$

$$a^2=\frac{7}{2}$$

$$a=\pm\sqrt{\frac{7}{2}}\quad or\quad \pm\frac{\sqrt{14}}{2}$$

(iii)

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$\left(2\mathbf{p}-5\mathbf{q}\right)\cdot\left(2\mathbf{p}+5\mathbf{q}\right)=0$$

$$4|\mathbf{p}|^2-25|\mathbf{q}|^2=0$$

$$|\mathbf{p}|^2=\frac{25}{4}|\mathbf{q}|^2$$

$$=\frac{25}{4}\left((-1)^2+1^2\right)$$

$$=\frac{25}{2}$$

$$|\mathbf{p}|=\frac{5\sqrt{2}}{2}$$

Alternative:

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=\begin{pmatrix}4+5\\ -3\\ 2a\end{pmatrix}\cdot\begin{pmatrix}4-5\\ 7\\ 2a\end{pmatrix}$$

$$=16-25-21+4a^2$$

$$=4a^2-30$$

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4a^2-30=0$$

$$a^2=\frac{15}{2}$$

$$|\mathbf{p}|=\sqrt{2^2+1+a^2}=\sqrt{5+\frac{15}{2}}=\sqrt{\frac{25}{2}}=\frac{5\sqrt{2}}{2}$$

4	(a) **Method 1** Since the coefficients are real, $w=2+\mathrm{i}$ is another root of the equation. $(w-2+\mathrm{i})(w-2-\mathrm{i})=(w-2)^2-(\mathrm{i})^2$ $=w^2-4w+4+1$ $=w^2-4w+5$ $w^3+pw^2+qw+30=0$ $(w^2-4w+5)(w+6)=0$ (By inspection) Comparing coefficients of w^2, $p=6-4=2$ Comparing coefficients of w, $q=-24+5=-19$ **Method 2** Substitute $w=2-\mathrm{i}$ (or $w=2+\mathrm{i}$) into the given eqn, $(2-\mathrm{i})^3+p(2-\mathrm{i})^2+q(2-\mathrm{i})+30=0$ $(3-4\mathrm{i})(2-\mathrm{i})+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(6-3\mathrm{i}-8\mathrm{i}-4)+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(32+3p+2q)+(-11-4p-q)\mathrm{i}=0$ Comparing the real parts, $3p+2q=-32$ --- (1) Comparing the imaginary parts, $4p+q=-11$---- (2) (1) – (2) × 2: $3p-8p=-32+11\times 2$ $-5p=-10$ $p=2$ From (2): $q=-11-4\times 2=-19$ $\therefore\ p=2$, $q=-19$ (b) Substitute $z=3+u\mathrm{i}$ into the given eqn, $(3+u\mathrm{i})^2+(-5+2\mathrm{i})(3+u\mathrm{i})+(21-\mathrm{i})=0$ $9+6u\mathrm{i}-u^2-15-5u\mathrm{i}+6\mathrm{i}-2u+21-\mathrm{i}=0$ $(15-2u-u^2)+(u+5)\mathrm{i}=0$ Compare imaginary coefficient: $u+5=0$ $u=-5$ $\therefore z=3-5\mathrm{i}$ [Note: if using $15-2u-u^2=0$, need to reject $u=3$] **Method 1** Let the other root be *w*. $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=(z-3+5\mathrm{i})(z-w)$ Comparing coefficients of *z*, $-5+2\mathrm{i}=-w-3+5\mathrm{i}$ $w=2+3\mathrm{i}$

Method 2

Let the other solution be $a+b\mathrm{i}$,

$$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})$$
$$=(z-(3-5\mathrm{i}))(z-(a+b\mathrm{i}))$$
$$=z^2-(a+b\mathrm{i})z-(3-5\mathrm{i})z+(3-5\mathrm{i})(a+b\mathrm{i})$$
$$=z^2-\left[a+3+(b-5)\mathrm{i}\right]z+(3-5\mathrm{i})(a+b\mathrm{i})$$

Compare the z term: $-(a+3)=-5 \quad \Rightarrow a=2$

$-(b-5)=2 \quad \Rightarrow b=3$

$\therefore z=2+3\mathrm{i}$ is another root.

5

(i)

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$
$$=\sum_{n=2}^{N}\left[u_n-u_{n+1}\right]$$
$$=\begin{bmatrix}(u_2-u_3)\\+(u_3-u_4)\\+(u_4-u_5)\\ \dots\\ \dots\\+(u_{N-1}-u_N)\\+(u_N-u_{N+1})\end{bmatrix}$$
$$=u_2-u_{N+1}$$
$$=\frac{1}{2(2^2)(2-1)^2}-\frac{1}{2(N+1)^2((N-1)+1)^2}$$
$$=\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$$

(ii)

As $N\to\infty$, $\dfrac{1}{2N^2(N+1)^2}\to 0$

$\displaystyle\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}\to\frac{1}{8}$ which is a constant, hence it is a convergent series.

$$\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}=\frac{1}{8}-0$$
$$=\frac{1}{8}$$

	(iii) **Method 1** $\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2} = N\sum_{n=1}^{N}\frac{2}{(n+1)n^2(n+2)^2}$ $= N\sum_{n=2}^{N+1}\frac{2}{(n)(n-1)^2(n+1)^2}$ $= N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$ $= \frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$ **Method 2** By listing the terms $\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$ $= \frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{N(N-1)^2(N+1)^2}$ $\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}$ $= N\left[\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{(N+1)(N)^2(N+2)^2}\right]$ $= N\sum_{n=2}^{N+1}\frac{2}{n(n-1)^2(n+1)^2}$ $= N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$ $= \frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$
6	(i) $(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \quad \cdots(1)$ Differentiating (1) w.r.t. *x*: $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+k\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0 \quad \cdots(2)$

	Differentiating (2) w.r.t. x: $(x+y)\frac{d^3y}{dx^3}+\left(1+\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+(1+k)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$ $(x+y)\frac{d^3y}{dx^3}+\left(2+3\frac{dy}{dx}+k\right)\frac{d^2y}{dx^2}=0$ $x=0,\quad y=1:\ \frac{dy}{dx}=2-k$ $\frac{d^2y}{dx^2}=3k-6$ $\frac{d^3y}{dx^3}=6k^2-36k+48=6\left(k^2-6k+8\right)$ $\therefore y\ =1+(2-k)x+\left(\frac{3k-6}{2!}\right)x^2+\left(\frac{6\left(k^2-6k+8\right)}{3!}\right)x^3+...$ $=1+(2-k)x+\left(\frac{3k-6}{2}\right)x^2+\left(k^2-6k+8\right)x^3+...$ (ii) $\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$ $\frac{1}{\sin^2\left(x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$ $\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$ $=\left(1-2x^2\right)^{-2}$ $=1+4x^2+...$ $4=2\left(\frac{3k-6}{2}\right)$ $k=\frac{10}{3}$
7	(i) $\frac{dM}{dt}=k\left(100^2-M^2\right)\ ,\quad k>0$ Since $\frac{dM}{dt}>0$ and $M>0,\Rightarrow\left(100^2-M^2\right)>0$ and $0<M<100$ $\int\frac{1}{\left(100^2-M^2\right)}\,dM=\int k\,dt$ $\frac{1}{200}\ln\left(\frac{100+M}{100-M}\right)=kt+C$

$$\ln\left(\frac{100+M}{100-M}\right)=200kt+C'$$

$$\frac{100+M}{100-M}=Ae^{200kt}\ , \text{ where } A=e^{C'}$$

When $t=0, \quad M=5 \ \Rightarrow\ A=\frac{105}{95}=\frac{21}{19}$

When $t=5, \quad M=20 \ \Rightarrow\ \frac{3}{2}=\frac{21}{19}e^{1000k}$

$$e^{1000k}=\frac{19}{14} \quad or \quad 200k=\frac{1}{5}\ln\left(\frac{19}{14}\right)$$

Thus $\dfrac{100+M}{100-M}=\dfrac{21}{19}\left(e^{1000k}\right)^{\frac{t}{5}}=\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{\frac{t}{5}}$

$$100+M=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}(100-M)$$

$$M\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1\right]=100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]$$

$$M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1} \quad OR \quad \frac{100\left[21\left(\frac{19}{14}\right)^{\frac{t}{5}}-19\right]}{21\left(\frac{19}{14}\right)^{\frac{t}{5}}+19} \quad OR \quad \frac{100\left[\left(\frac{19}{14}\right)^{\frac{t}{5}}-\frac{19}{21}\right]}{\left(\frac{19}{14}\right)^{\frac{t}{5}}+\frac{19}{21}}$$

(ii)

When t = 15, $M=\dfrac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{3}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{3}+1}=46.847$

$M\approx 47$ (nearest whole number)

(iii)

<u>Method 1:</u> Graphical Method

Sketch the graphs of M=f(t) and M=80

From the graph, when $t>34.336397, \quad M>80$

Least number of days required is 35.

<u>Method 2:</u> Use GC table

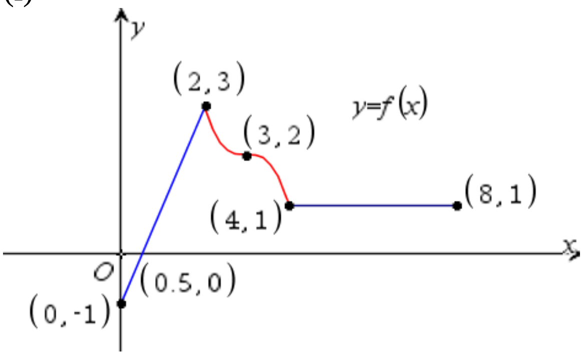
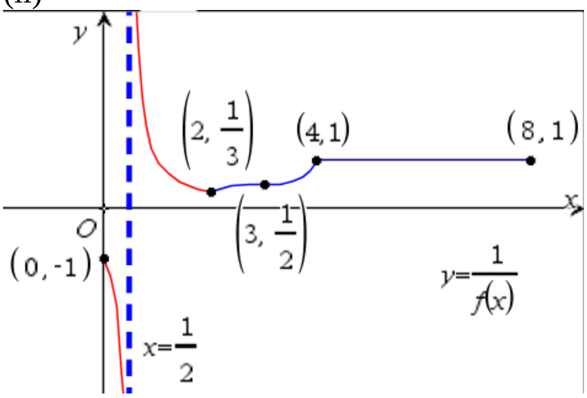
When $t=34,\ M=79.627<80$

When $t=35,\ M=80.718>80$

When $t=36,\ M=81.756>80$

$\Rightarrow\ t\geq 35$

Thus least number of days required is 35.

	Method 3: $$\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}>80$$ $$\frac{5}{4}\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]>\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1$$ $$\frac{1}{4}\cdot\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{9}{4}$$ $$\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{57}{7}$$ $$t>\frac{5\ln\left(\frac{57}{7}\right)}{\ln\left(\frac{19}{14}\right)}=34.336397$$ Least number of days required is 35.
8	(i)  Range of f is $[-1,3]$ or $R_f=[-1,3]$ or $R_f=\{y:-1\le y\le 3\}$ (ii) 

(iii)

$$\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x = \int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$$
$$= \text{area of rectangle}$$
$$= 2$$

9

$$\mathrm{f}(x) = \sin 2x + \cos 2x$$
$$R = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\tan\alpha = 1 \quad \Rightarrow \quad \alpha = \frac{\pi}{4}$$
$$\mathrm{f}(x) = \sin 2x + \cos 2x = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$$

(i)

Transforming $y = \sin x$ to $y = \sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$

Sequence of Transformation:

Either

A: A translation of $\frac{\pi}{4}$ units in the negative *x*-direction

B: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the *x*-axis.

C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the *y*-axis.

Acceptable sequence: ABC, ACB, CAB.

OR $y = \sqrt{2}\sin\left[2\left(x + \frac{\pi}{8}\right)\right]$

D: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the *x*-axis.

E: A translation of $\frac{\pi}{8}$ units in the negative *x*-direction.

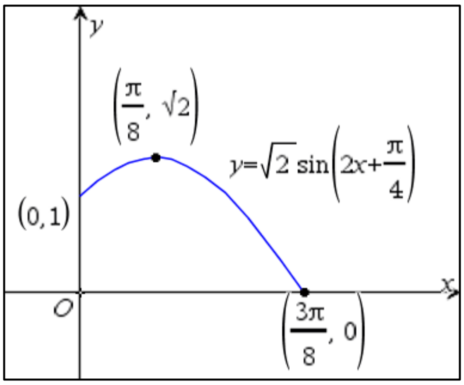
F: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the *y*-axis.

Acceptable sequence: DEF, DFE, FDE

(ii)

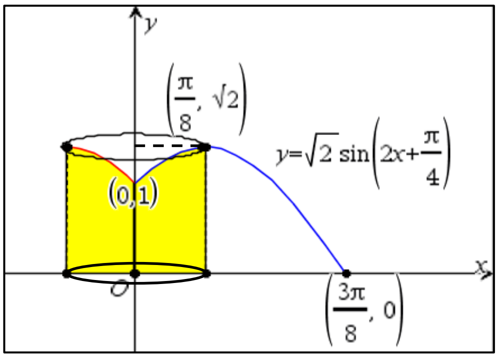
Max point occurs when $\sin\left(2x + \frac{\pi}{4}\right) = 1$

$$\Rightarrow \quad \left(2x + \frac{\pi}{4}\right) = \frac{\pi}{2}$$
$$\Rightarrow \quad x = \frac{\pi}{8},\ y = \sqrt{2}$$



(iii)

$y=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$



The curve is one-one for $0\le x\le\frac{\pi}{8}$,

thus inverse function exists.

$\sin\left(2x+\frac{\pi}{4}\right)=\frac{y}{\sqrt{2}}$

$2x+\frac{\pi}{4}=\sin^{-1}\frac{y}{\sqrt{2}}$

$x=\frac{1}{2}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]$

Volume = Volume of cylinder - $\pi\int_1^{\sqrt{2}} x^2 \ \mathrm{d}y$

$=\pi\left(\frac{\pi}{8}\right)^2\sqrt{2}-\pi\int_1^{\sqrt{2}}\frac{1}{4}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]^2 \ \mathrm{d}y$

$=0.6506458$

$\approx 0.6506 \ (4 \ \text{d.p.})$

10 (i)

$\overrightarrow{AB}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$

$l_{AB}:\underset{\sim}{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$ *or* $\underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$ or equivalent

(ii)

$\sin\theta=\frac{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right|}{\sqrt{5}\sqrt{2}}=\frac{1}{\sqrt{10}}$

$\theta=18.4^\circ$

(iii)

Let $\underset{\sim}{m}$ be a vector perpendicular to the plane containing the light ray and $\underset{\sim}{n}$.

$\underset{\sim}{m}=\underset{\sim}{n}\times\overrightarrow{AB}=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ or $\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} \Rightarrow \frac{2}{3} - q = 1$$

$$q = -\frac{1}{3}$$

$$\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \perp \underset{\sim}{m} \Rightarrow \begin{pmatrix} -\frac{2}{3} \\ p \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-\frac{4}{3} - p + \frac{2}{3} = 0 \Rightarrow p = -\frac{2}{3}$$

(iv)

Glass upper surface is $x + z = 2$

Glass bottom surface is $3x + 3z = -4 \Rightarrow x + z = -\frac{4}{3}$

Distance between two planes $= \frac{\left|2 - \left(-\frac{4}{3}\right)\right|}{\sqrt{2}} = \frac{10}{3\sqrt{2}} = \frac{5\sqrt{2}}{3}$

Thickness of the glass object is $\frac{5\sqrt{2}}{3}$ cm

(v)

Let the point at which the light ray leaves the glass object be *F*.

Method 1:

$$l_{BF} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad or \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

At *F*,

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4$$

$$6 + \mu(6+3) = -4$$

$$\mu = -\frac{10}{9}$$

OR

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4$$

$$6 + \mu(-2-1) = -4$$

$$\mu = \frac{10}{3}$$

The coordinates of *F* are

$$\left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9}\right)$$

Method 2:

$$\cos 45^\circ = \frac{\frac{5\sqrt{2}}{3}}{BF} \Rightarrow \left|\overrightarrow{BF}\right| = \frac{5\sqrt{2}}{3} \times \sqrt{2} = \frac{10}{3}$$

(or using Pythagoras' theorem)

	$\vec{BF} = \frac{10}{3}\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = -\frac{10}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\vec{OF} = -\frac{10}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -20 \\ -20 \\ 8 \end{pmatrix}$ The coordinates of F are $\left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9}\right)$
11	(i) Let l be the slant height of the cone. $l^2 = h^2 + r^2 \quad ----(1)$ Using similar triangles, $\frac{h-3}{l} = \frac{3}{r}$ $l = \frac{rh-3r}{3} \quad ----(2)$ Equating (1) and (2), $\left(\frac{rh-3r}{3}\right)^2 = h^2 + r^2 \quad ----(*)$ $r^2h^2 - 6r^2h + 9r^2 = 9h^2 + 9r^2$ $r^2(h^2 - 6h) = 9h^2$ $\therefore r = \frac{3h}{\sqrt{h^2 - 6h}} \qquad (\text{Since } r > 0)$ (ii) Volume of cone, $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$ $= \frac{3\pi h^3}{h^2 - 6h}$ $= \frac{3\pi h^2}{h-6}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{6\pi h(h-6) - 3\pi h^2}{(h-6)^2}$ $= \frac{3\pi h^2 - 36\pi h}{(h-6)^2}$

$\frac{dV}{dh} = 0 \quad \Rightarrow \quad 3\pi h^2 - 36\pi h = 0$

$h(h-12) = 0$

$h = 12$ or $h = 0$ (reject $\because h > 0$)

h	12^-	12	12^+
Sign of $\frac{dV}{dh}$	– ve	0	+ ve
Tangent	\	—	/

Thus, V is a minimum when $h = 12$

When $h = 12$,

$$r = \frac{3(12)}{\sqrt{(12)^2 - 6(12)}} = \frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

$$V = \frac{3\pi(12)^2}{12-6} = 72\pi \qquad (\approx 226.195)$$

(iii)

Let R be the radius of the snowball

$$S = 4\pi R^2 \quad \Rightarrow \quad \frac{dS}{dt} = 8\pi R\frac{dR}{dt}$$

$$V = \frac{4}{3}\pi R^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi R^2\frac{dR}{dt}$$

When $R = 2.5$, $\frac{dS}{dt} = -0.75 \quad \Rightarrow \quad 8\pi(2.5)\frac{dR}{dt} = -0.75$

$$\frac{dR}{dt} = -\frac{3}{80\pi} \quad or \quad -\frac{0.0375}{\pi} \quad or \quad -0.0119366$$

$$\frac{dV}{dt} = 4\pi(2.5)^2\left(-\frac{3}{80\pi}\right) = -\frac{15}{16} \quad or \quad -0.9375$$

At the instant when $R = 2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.