RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2018 Year 5

Chapter 7A: Complex Numbers I – Complex Numbers in Cartesian Form

SYLLABUS INCLUDES

H2 Mathematics:

- extension of the number system from real numbers to complex numbers
- complex roots of quadratic equations
- four operations of complex numbers expressed in the form $(x + \mathrm{i}y)$
- equating real parts and imaginary parts
- conjugate roots of a polynomial equation with real coefficients

PRE-REQUISITES

- Trigonometry
- Coordinate Geometry
- Vectors

CONTENT

1 Introduction to the Imaginary Number i

We know that the solution to the equation $x^2+1=0$ cannot be a real number, as the square of a real number cannot be negative. We say $x^2=-1$ has no real roots.

In order to solve the above equation, we need to find a "number" whose square is -1.

Let's suppose such a "number" exists.

Since we imagined it, let's call this number the **imaginary number** i.

We define i as

$$\boxed{\mathrm{i}=\sqrt{-1}}$$

Hence the solutions to $x^2+1=0$ are $x=\pm\sqrt{-1}=\pm\mathrm{i}$.

Example 1

(a) If $\mathrm{i}=\sqrt{-1}$, simplify i^2, i^3, i^4, i^{2009}, i^{2010}, i^{2011}, i^{2012}.

Solution

$\downarrow$ -1, $\downarrow$ $-i$, $\downarrow$ 1, $\downarrow$ i, $\downarrow$ -1, $\downarrow$ $-i$, $\downarrow$ 1

Let's generalize: If k is a positive integer, then $i^{4k}=1$, $i^{4k+1}=i$, $i^{4k+2}=-1$, $i^{4k+3}=-i$

(b) Perform the four basic operations on i:

(i)	$\mathrm{i}+\mathrm{i}=$ $2i$	**(ii)**	$5\mathrm{i}-\mathrm{i}=$ $4i$
(iii)	$5\mathrm{i}\times3\mathrm{i}=$ $15i^2=-15$	**(iv)**	$6\mathrm{i}\div3\mathrm{i}=$ 2

(c) Solve for x if $x^2-2x+2=0$.

Solution

$x^2-2x+2=0$

$x=\dfrac{2\pm\sqrt{4-8}}{2}$

$=\dfrac{2\pm\sqrt{-4}}{2}$

$=1\pm\dfrac{\sqrt{-4}}{2}$

$=1+1\sqrt{-1}$ or $1-1\sqrt{-1}$

$=$ $\cancel{\pm 2i}$ $1\pm i$

$(x-1)^2-1+2=0$

$(x-1)^2=-1$

$x-1=i$ or $x-1=-i$

$x=i+1$ or $x=x-i+1$

2 COMPLEX NUMBERS

Notice that the solution to Example 1(c) is a combination of real numbers and imaginary numbers. Such numbers are called complex numbers.

2.1 Definition of a Complex Number

A **complex number** is of the form $x+\mathrm{i}y$ where x and y are real numbers and $\mathrm{i}=\sqrt{-1}$.

The set of complex numbers is denoted by $\mathbb{C}=\{z:z=x+\mathrm{i}y,\ x,\ y\in\mathbb{R}\}$.

x is known as the real part of z, denoted by $\mathrm{Re}(z)$, and

y is known as the imaginary part of z, denoted by $\mathrm{Im}(z)$.

Note that $\mathrm{Im}(z)$ **does not** include i.

$x+\mathrm{i}y$ is known as the **cartesian form** of the complex number z.

Example 2

Write down the real and imaginary parts of the following complex numbers:

z	Re(z)	Im(z)
$-2+3\mathrm{i}$	-2	3
$1-\mathrm{i}$	1	-1
-4	-4	0
$5\mathrm{i}$	0	5

Remarks:

- $\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}\subseteq\mathbb{C}$
- If $y=0$, then $z=x$ is a real number.
- If $x=0$, then $z=\mathrm{i}y$ is a purely imaginary number.
- Complex numbers cannot be ordered, i.e. given any two complex numbers z_1 and z_2, we cannot compare whether $z_1<z_2$ unless they are real numbers.

Question: Is $\mathrm{i}<0$ or $\mathrm{i}>0$?

2.2 Operations on Complex Numbers

In this section, let $a,\ b,\ c,\ d \in \mathbb{R}$ and $z_1,\ z_2,\ z_3 \in \mathbb{C}$.

(a) Equality of Two Complex Numbers

2 complex numbers are equal if and only if their corresponding real and imaginary parts are equal, i.e. $a + \mathrm{i}b = c + \mathrm{i}d \iff a = c$ and $b = d$.

For example, if $x + \mathrm{i}y = 5 - 3\mathrm{i}$, we have $x =$ 5 and $y =$ -3.

(b) Addition of Complex Numbers

$$(a+\mathrm{i}b)+(c+\mathrm{i}d)=(a+c)+\mathrm{i}(b+d)$$

Addition of complex numbers is commutative: $z_1 + z_2 = z_2 + z_1$
Addition of complex numbers is associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

For example, $(3+4\mathrm{i})+(1-\mathrm{i}) =$ $4+3i$

(c) Subtraction of Complex Numbers

$$(a+\mathrm{i}b)-(c+\mathrm{i}d)=(a-c)+\mathrm{i}(b-d)$$

For example, $(3+4\mathrm{i})-(1-\mathrm{i}) =$ $2+5i$

(d) Multiplication of Complex Numbers

$$(a+\mathrm{i}b)(c+\mathrm{i}d) = ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2bd = ac + \mathrm{i}ad + \mathrm{i}bc + (-1)bd$$
$$= (ac - bd) + \mathrm{i}(ad + bc)$$

Multiplication of complex numbers is commutative: $z_1z_2 = z_2z_1$
Multiplication of complex numbers is associative: $(z_1z_2)z_3 = z_1(z_2z_3)$
Multiplication of complex numbers is distributive over addition: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

For example, $(3+4\mathrm{i})(1-\mathrm{i}) =$ $3-3i+4i-4i^2 - 7+i$ (note that $\mathrm{i}^2 = -1$)

$(3+4\mathrm{i})^2 = (3+4\mathrm{i})(3+4\mathrm{i}) =$ $9+12i+12i+16i^2 = -7+24i$ (note that $\mathrm{i}^2 = -1$)

Remarks: The above manipulation of complex numbers is the same as algebraic manipulation of expressions such as $(a+b)(c+d)$, $(a+b)\pm(c+d)$ and so on. The only additional consideration is $\mathrm{i}^2 = -1$.

We can use the GC to perform operations on complex numbers

Operation	Example	GC Screen
Multiplication of complex numbers (Press 2nd . to get i)	$(3+4\mathrm{i})(1-\mathrm{i})$	NORMAL FLOAT AUTO REAL RADIAN MP (3+4i)(1−i) 7+i
Square of a complex number	$(3+4\mathrm{i})^2$	NORMAL FLOAT AUTO REAL RADIAN MP (3+4i)² −7+24i
Division of complex numbers	$\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{3+4i}{1-i}$ $-\frac{1}{2}+\frac{7}{2}i$ (3+4i)/(1−i) −0.5+3.5i

How did the GC obtain $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}=-0.5+3.5\mathrm{i}$?

(e) Division of Complex Numbers

Recall when we tried to simplify $\dfrac{2-\sqrt{3}}{1+\sqrt{2}}$, we multiply it by $\dfrac{1-\sqrt{2}}{1-\sqrt{2}}$ so that we could rationalize the denominator.

For $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$, we will multiply it by $\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$, so that

$$\frac{3+4\mathrm{i}}{1-\mathrm{i}}=\frac{3+4\mathrm{i}}{1-\mathrm{i}}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}=\frac{3+3i+4i+4i^2}{1+1}=\frac{-1+7i}{2}=-\frac{1}{2}+\frac{7}{2}i$$

In general, $\dfrac{a+ib}{c+id}=\dfrac{a+ib}{c+id}\times\dfrac{c-id}{c-id}=\dfrac{(a+ib)(c-id)}{c^2-i^2d^2}=\dfrac{(a+ib)(c-id)}{c^2+d^2}$

Remark: Note that $c-id$ is chosen based on the denominator of $\dfrac{a+ib}{c+id}$.

We call $c-id$, the conjugate of $c+id$.

Example 3 [RJC Prelim 9233/2005/01/Q1(i)]

The complex numbers z and w are such that $z=-1+2\mathrm{i}$ and $w=1+b\mathrm{i}$, where $b\in\mathbb{R}$.

Given that the imaginary part of $\frac{w}{z}$ is $-\frac{3}{5}$, find the value of b.

Solution

$$\frac{w}{z}=\frac{1+bi}{-1+2i}\times\frac{-1-2i}{-1-2i}=\frac{-1-2i-bi-2bi^2}{(-1)^2+2^2}=\frac{-1+2b}{5}+i\frac{(-2-b)}{5}$$

Given $\text{Im}\left(\frac{w}{z}\right)=-\frac{3}{5} \Rightarrow \frac{-2-b}{5}=-\frac{3}{5} \Rightarrow b=1$

Example 4

Without using a calculator, find the square roots of $24-10i$.
By completing the square, or otherwise, solve $z^2-6z=15-10i$.

Solution:

Let the square roots of $24-10i$ be $x+yi$, where $x, y \in \mathbb{R}$.

Then
$$\begin{aligned}24-10i&=(x+yi)^2\\&=x^2+2xyi+y^2i^2\\&=x^2-y^2+2xyi\end{aligned}$$

Comparing real and imaginary parts:

$$24=x^2-y^2 \quad \text{---} (1)$$
$$-10=2xy \quad \text{---} (2)$$

From (2): $y=-\frac{5}{x}$ --- (3)

Sub (3) in (1): $24=x^2-\frac{25}{x^2}$

$$x^4-24x^2-25=0$$
$$(x^2-25)(x^2+1)=0$$

$x^2=25$ or $x^2=-1$ (NA since $x\in\mathbb{R}$)

$x=\pm5$

when $x=5$, $y=-1$ when $x=-5$, $y=1$

The square roots are $5-i$ or $-5+i$

Note that evaluating $\sqrt{24-10i}$ using GC only gives us $5-i$.

$$
\begin{aligned}
z^2 - 6z = 15 - 10\mathrm{i} \quad &\Rightarrow \quad (z-3)^2 - 3^2 = 15 - 10\mathrm{i} \\
&\Rightarrow \quad (z-3)^2 = 24 - 10\mathrm{i} \\
&\Rightarrow \quad z - 3 = \pm(5 - \mathrm{i}) \qquad \text{(from the above result)} \\
&\Rightarrow \quad z = 8 - \mathrm{i} \ \text{ or } \ z = -2 + \mathrm{i}
\end{aligned}
$$

Note:

We can use GC to check our answers.

Substitute $z = 8 - \mathrm{i}$ and $z = -2 + \mathrm{i}$ into $z^2 - 6z$, and check that both give $15 - 10\mathrm{i}$ as answers.

$24 - 10i = (a+ib)^2 = a^2 + 2abi - b^2 \Rightarrow a^2 - b^2 = 24,\ 2ab = -10$

$b = \cancel{\frac{-10}{}}\ -\frac{5}{a}$

$z^2 - 6z - 15 + 10i = 0$

$(z-3)^2 - 9 - 15 + 10i = 0$

$(z-3)^2 = 24 - 10i$

$z - 3 = 5 - i$ or $i - 5$

$z = 8 - i$ or $i - 2$

$a^2 - \left(\frac{-5}{a}\right)^2 = 24$

$a^2 - \frac{25}{a^2} = 24$

$a^4 - 24a^2 - 25 = 0$

$(a^2 - 25)(a^2 + 1) = 0$

$a = 5$ or $a = i$

$b = -1 \quad b = -\frac{5}{i} = 5i$

$\sqrt{24 - 10i} = (5 - i)$ or $(i - 5)$

2.3 Complex Conjugates

The **complex conjugate** of $z=x+\mathrm{i}y$, where $x, y\in\mathbb{R}$, is denoted by z^* and defined as

$$z^*=x-\mathrm{i}y$$

Observe that $\mathrm{Re}(z^*)=x=\mathrm{Re}(z)$ and $\mathrm{Im}(z^*)=-y=-\mathrm{Im}(z)$.
Note that $z=x+\mathrm{i}y$ and $z^*=x-\mathrm{i}y$ are conjugates of each other, and we call them a <u>conjugate pair</u>.

2.4 Some Properties of Complex Conjugates

The following properties can be proven by letting $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$ where $x, y, u, v\in\mathbb{R}$.

Properties	Proofs	Example
(a) $(z^*)^*=z$	$[(x+\mathrm{i}y)^*]^*=(x-\mathrm{i}y)^*=x+\mathrm{i}y$	$[(1-3\mathrm{i})^*]^*$ $=(1+3\mathrm{i})^*$ $=1-3\mathrm{i}$
(b) $z+z^*=2\mathrm{Re}(z)$	$(x+\mathrm{i}y)+(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $=2x$	$(1-3\mathrm{i})+(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})+(1+3\mathrm{i})$ $=2$
(c) $z-z^*=2\mathrm{i}\,\mathrm{Im}(z)$	$(x+\mathrm{i}y)-(x+\mathrm{i}y)^*=(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $=2\mathrm{i}y$	$(1-3\mathrm{i})-(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})-(1+3\mathrm{i})$ $=-6\mathrm{i}$
(d) $zz^*=x^2+y^2$	$(x+\mathrm{i}y)(x+\mathrm{i}y)^*$ $(a+\mathrm{i}b)(a-\mathrm{i}b)$ $\cancel{(a^2+b)(a+b)}$ $=(x+\mathrm{i}y)(x-\mathrm{i}y)$ $\cancel{=a^2+b^2}$ a^2-b^2 $=x^2-(\mathrm{i}y)^2$ (difference of 2 squares) $=x^2-\mathrm{i}^2y^2$ $(\mathrm{i}^2=-1)$ $=x^2+y^2$	$(1-3\mathrm{i})(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})(1+3\mathrm{i})$ $=1-(3\mathrm{i})^2$ $=1+9=10$
(e) $z=z^* \Leftrightarrow z\in\mathbb{R}$	$z=z^*\Leftrightarrow x+\mathrm{i}y=x-\mathrm{i}y$ $\Leftrightarrow 2\mathrm{i}y=0$ $\Leftrightarrow \mathrm{Im}(z)=0$ $\Leftrightarrow z$ is real	
(f) $(z+w)^*=z^*+w^*$	$(x+\mathrm{i}y+u+\mathrm{i}v)^*=(x+u+\mathrm{i}(y+v))^*$ $=x+u-\mathrm{i}(y+v)$ $=(x-\mathrm{i}y)+(u-\mathrm{i}v)$ $=z^*+w^*$ $(a+\mathrm{i}b+c+\mathrm{i}d)^*$ $=a+c-\mathrm{i}(b+d)$ $=\underbrace{a-\mathrm{i}b}+\underbrace{c-\mathrm{i}d}$	

Properties	Proofs	Example
(g) $(zw)^* = z^* w^*$	$((x+\text{i}y)(u+\text{i}v))^* = (xu - yv + \text{i}(xv+yu))$ $= xu - yv - \text{i}(xv+yu)$ $(x+\text{i}y)^*(u+\text{i}v)^* = (x-\text{i}y)(u-\text{i}v)$ $= xu - yv - \text{i}(xv+yu)$ $[(a+ib)(c+id)]^* = [ac + iad + ibc - bd]^*$ $= (ac-bd - i(ad+bc)]$ $(a+ib)^*(c+id)^* = (a-ib)(c-id)$ $= ac - iad - ibc - bd$ $= ac-bd - i(ad+bc)$	$[(1+3\text{i})(1-2\text{i})]^*$ $=(\ 7+i\ )^*$ $\cancel{=(\quad)^*}$ $= 7-1$ $(1+3\text{i})^*(1-2\text{i})^*$ $=(\ 1-3i\ )(1+2i\ )$ $= 7-i$

Note:

We can use **(g)** to show that $(z^2)^* = (z^*)^2$ by letting $w = z$.

We can also show that $(z^n)^* = (z^*)^n$, $n \in \mathbb{Z}^+$

Example 5

Let $z = 1 + \text{i}a$ and $w = 1 + \text{i}b$, where $a,\ b \in \mathbb{R}$ and $a > 0$. If $zw^* = 3 - 4\text{i}$, find the exact values of a and b.

Solution

$zw^* = 3 - 4\text{i} \Rightarrow (1+\text{i}a)(\ 1 - ib\ ) = 3 - 4\text{i}$

$\Rightarrow (1+ab) + i(a-b) = 3-4i$

Equating real and imaginary parts, $1+ab = 3$ —— (1)

$a - b = -4 \Rightarrow b = a+4$ —— (2)

Subst (2) into (1), $1 + a(a+4) = 3$

$a^2 + 4a - 2 = 0$

$a = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \sqrt{6}$

Since $a > 0$, $a = -2+\sqrt{6}$ and $b = a+4 = 2+\sqrt{6}$

$zw^* = (1+ia)(1-ib)$

$= 1 - ib + ia + ab$

$= 1 + ab + i(\cancel{b+a})^{a-b} = 3-4i$

$ab = 2,\ a-b = -4$

$a = b-4$

$b(b-4) = 2$

$b^2 - 4b - 2 = 0$

$b = \frac{4 \pm \sqrt{16+8}}{2} = 2 \pm \sqrt{6}$

$\cancel{= 2 \pm 3\sqrt{2}}$

$a = -2 \pm \sqrt{6}$ ✓

Example 6

Solve the simultaneous equations $z^*+w=-1$, $2z+(\mathrm{i}w)^*=-1$.

Solution

$(z^*+w)^*=(-1)^*$

$z^*+w=-1 \quad \Rightarrow z+w^*=-1$ ....(1)

$2z+(\mathrm{i}w)^*=-1 \quad \Rightarrow 2z-\mathrm{i}w^*=-1$..........(2)

(1) $\times$i : $\mathrm{i}z+\mathrm{i}w^*=-\mathrm{i}$ — (3)

(3)+(2): $(\mathrm{i}+2)z=-\mathrm{i}-1$

$z=\frac{-\mathrm{i}-1}{\mathrm{i}+2}=-\frac{3}{5}-\frac{1}{5}\mathrm{i}$

From (1),

$w^*=-1-z=-\frac{2}{5}+\frac{1}{5}\mathrm{i}$

$w=-\frac{2}{5}-\frac{1}{5}\mathrm{i}$

$z^*+w=-1$

$2z+(\mathrm{i}w)^*=-1$

$2z+\mathrm{i}^*w^*=-1$

$2z-\mathrm{i}w^*=-1$

$\mathrm{i}z^*+\mathrm{i}w=-\mathrm{i}$

$z^*+w=-1$

$2z-(\mathrm{i}w)^*=-1$

~~$z+z+z^*+w-w^*=-2$~~

$z=a+ib$, $w=c+id$

$a-ib+c+id=-1$

~~$2a+i2b-c+id=-1$~~

~~$3a+ib+id=-2$~~

~~$a=\frac{2}{3}$~~

~~$b+d=0$~~

~~$\frac{2}{3}-ib+c+id=\frac{4}{3}+i2b-c+id+ib$~~

~~$2c-ib=\frac{2}{3}$~~

~~$c=\frac{1}{3}$~~

~~$b=0$~~

$2a+i2b-i(c-id)=-1$

$2a+i2b-d-ic=-1$

$2a-d=-1$, $2b-c=0 \Rightarrow 2d-c=0$

$a+c=-1$, $d-b=0 \Rightarrow d=b$

~~$2a-b=-1=a+c$~~

$2d+a=2a-d=-1$

$3d=a=-1$

$d=-\frac{1}{3}$, $a=-1$, $c=0$

3 Complex Roots of Polynomial Equations

With complex numbers, we have the following theorem.

Fundamental Theorem of Algebra:

A polynomial equation of degree n has n roots (real or non-real).

Thus, taking non-real roots into account, a quadratic (degree 2) equation always has 2 roots, a cubic (degree 3) equation always has 3 roots, and so on.

Furthermore, if the coefficients of the polynomial equation are real, we have the following result:

Non-real roots of a polynomial equation with real coefficients occur in conjugate pairs.

In other words, if β is a non-real root of a polynomial equation with real coefficients, then β^* is also a non-real root of the equation.

Note that real coefficients include the constant term as well.

[Refer to Appendix for the proof of this result]

From **Example 1(c)** we obtained $x = 1 \pm \mathrm{i}$ as conjugate pair solutions to $x^2 - 2x + 2 = 0$.

Example 7 (Quadratic)
Find the roots of the equation $z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$.

Solution

$z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$

$$z = \frac{-(1-4i) \pm \sqrt{(-1+4i)^2 - 4(-5+i)}}{2}$$

$= 2-3i$ or $-1-i$

$$z = \frac{-(-1+4i) \pm \sqrt{(-1+4i)^2 - 4(-5+i)}}{2}$$
$$= \frac{1-4i \pm \sqrt{1-8i-16+20-4i}}{2}$$
$$= \frac{1}{2}(1-4i \pm \sqrt{5-12i})$$
$$= \frac{1}{2}(1-4i+3-2i) \text{ or } \frac{1}{2}(1-4i+2i-3)$$
$$= 2-3i \text{ or } -1-i \quad \checkmark$$

Note:
The expression under square root sign can be evaluated using GC.
$\sqrt{5-12\mathrm{i}}$ can be evaluated using GC

Question: Why are the roots not in conjugate pairs?

Answer: Because not all the coefficients of the quadratic equation are real.

Example 8 (Cubic) Do not use a calculator in answering this question.
Find the exact roots of the equation $z^3-2z^2+2z-1=0$.

Solution

$1-2+2-1=0$

Since the cubic equation has real coefficients, it has either 3 real roots or 1 real root and 1 conjugate pair of non-real roots.

① Sub a random value

$z=1$ is clearly a solution of $z^3-2z^2+2z-1=0$ since $1-2+2-1=0$. So $z-1$ is a factor of z^3-2z^2+2z-1.

z^2-z+1
$z-1 \,\overline{)\, z^3-2z^2+2z-1}$
$-(z^3-z^2)$
$-z^2+2z$
$-(-z^2+z)$
$z-1$

Observe that (or via long division) $z^3-2z^2+2z-1=(z-1)(z^2-z+1)$.

Hence $z^3-2z^2+2z-1=0 \Rightarrow (z-1)(z^2-z+1)=0$

$\Rightarrow z-1=0$ or $z^2-z+1=0$

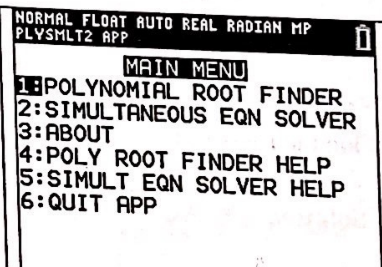
$\Rightarrow z=1$ or $z=\dfrac{1\pm\sqrt{1-4}}{2}$

i.e. $z=1$ or $z=\dfrac{1+\mathrm{i}\sqrt{3}}{2}$ or $z=\dfrac{1-\mathrm{i}\sqrt{3}}{2}$.

$z=\dfrac{1\pm\sqrt{1-4}}{2}$
$=\frac{1}{2}(1\pm\sqrt{-3})$
$=\frac{1}{2}(1\pm i\sqrt{3})$

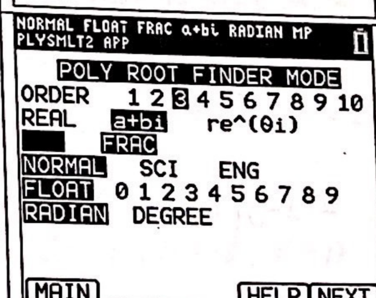
Note that the **Polynomial Root Finder** of the **PlySmlt2** app in the GC can be used to solve equations with real coefficients.

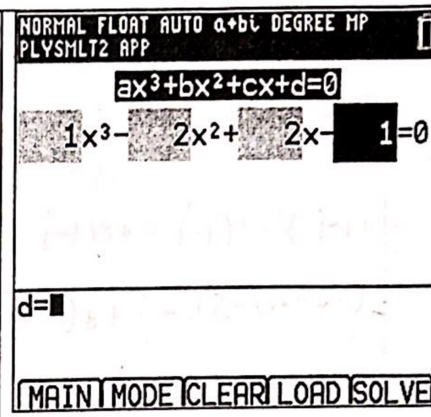
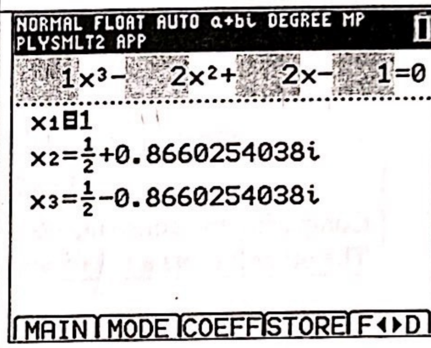
1. Press **[APPS]** and select **PlySmlt2.**
2. Select **1: Polynomial Root Finder.**



3. Adjust the settings as depicted on the screen to solve a cubic equation.

   Remember to select "a+bi" or "re^(θi)" for the GC to display all roots, not just the real roots.
4. Press **[GRAPH]** which is the button below **NEXT.**



5. Key in the values of the coefficients a_3, a_2, a_1 and a_0. 6. Press **[GRAPH]** to solve the system of equations.	
7. Read off the answers. x_1, x_2 and x_3 give the 3 roots of z. Note : The GC cannot display the roots in exact form.	

i -i

-i

Example 9 (Cubic) Do not use a calculator in answering this question.
Given that $1-\mathrm{i}$ is a root of the equation $z^3-5z^2+8z+p=0$, where $p\in\mathbb{R}$, find the other 2 roots and the value of p.

1+i

Solution
Method 1:
Since the cubic equation has real coefficients, 1+i is also a root.

The third root must be a real no k

smart method

$z^3-5z^2+8z+p=[z-(1+i)][z-(1-i)](z-k)$

$=[(z-1)-i][(z-1)+i](z-k)$

$=[(z-1)^2-i^2](z-k)$

$=(z^2-2z+2)(z-k)$

$z^3-5z^2+8z+p=(z-1+i)(z-1-i)(z-\alpha)$
$=(z^2-z-iz-z+1+i+iz-1+i)(z-\alpha)$
$=(z^2-2z+1+1)(z-\alpha)$
$=(z^3-2z^2+z+iz-\alpha z^2+2\alpha z-\alpha-i\alpha)$
$=z^3+(-2-\alpha)z^2+(1+2\alpha+i)z-\alpha-i\alpha$

$p=-\alpha-i\alpha$
$1+2\alpha+i=8$
$-2-\alpha=-5$
$\alpha=3$
$p=-3-3i$

Comparing coefficient of z, $8=2k+2 \Rightarrow k=3$ Comparing the constant, $p=-2k=-6$

The other 2 roots are 1+i and 3, and $p=$ -6.

$(1-i)^3-5(1-i)^2+8(1-i)+p=0$

$(1-i)^2 = 1-2i-1=-2i$

$(1-i)^3=(1-i)(-2i)=-2i-2$

Method 2:
Since $1-\mathrm{i}$ is a root of the given equation, an alternative way of finding p is by **<u>substituting the root into the equation.</u>**

$$z^3-5z^2+8z+p=0$$

$(1-i)^3-5(1-i)^2+8(1-i)+p=0$

$(-2-2i)-5(-2i)+8(1-i)+p=0$

$p=-6$

Since ~~$(1-i)^2 = 1-2i+i^2-2i$~~

~~$(1-i)^3 = (1-i)^2(1-i)$~~

~~$= -2-2i$~~

Since the cubic equation has real coefficients, $1+\mathrm{i}$ is also a root.
The third root must be a real number k.

$z^3-5z^2+8z-6=[z-(1+\mathrm{i})]\,[z-(1-\mathrm{i})](z-k)$
$\quad =(z^2-2z+2)(z-k)$

Comparing the constant, $-6=-2k \Rightarrow k=3$
The other 2 roots are $1+\mathrm{i}$ and 3.

Remark: The Fundamental Theorem of Algebra tells us that an equation like $z^{10}=1024$ has 10 roots (real or non-real). How do we find them?

We will deal with it in Chapter 7C.

APPENDIX

PROOF OF RESULT THAT NON-REAL ROOTS OF A POLYNOMIAL EQUATION WITH REAL COEFFICIENTS OCCUR IN CONJUGATE PAIRS

Consider the equation

$$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0 = 0,$$

where $a_n,\ a_{n-1},\ a_{n-2},\ldots,\ a_1,\ a_0 \in \mathbb{R},\ a_n \neq 0,\ n \in \mathbb{Z}^+$.

Suppose β is a non-real root of the equation,
i.e.
$$a_n\beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + \ldots + a_1\beta + a_0 = 0.$$

Taking conjugates on both sides of the equation,
$$(a_n\beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + \ldots + a_1\beta + a_0)^* = 0^*$$
$$(a_n\beta^n)^* + (a_{n-1}\beta^{n-1})^* + (a_{n-2}\beta^{n-2})^* + \ldots + (a_1\beta)^* + a_0^* = 0^*$$

Now $(\beta^k)^* = (\beta^*)^k$ and $(a_k)^* = a_k$ since $a_k \in \mathbb{R}$.
Clearly $0^* = 0$ since $0 \in \mathbb{R}$.

Thus we have
$$a_n(\beta^*)^n + a_{n-1}(\beta^*)^{n-1} + a_{n-2}(\beta^*)^{n-2} + \ldots + a_1(\beta^*) + a_0 = 0,$$
i.e. β^* is also a non-real root of the given equation.

SUMMARY