H2 Mathematics (9758)
Chapter 10 Integration Techniques
Extra Practice Solutions

Qn 1	**2018/ACJC Prelim/1/6**
(i)	Let $u=\sin^{-1}2x,\quad \frac{dv}{dx}=\frac{x}{\sqrt{1-4x^2}}=x\left(1-4x^2\right)^{-\frac{1}{2}}$ $\frac{du}{dx}=\frac{2}{\sqrt{1-4x^2}},\quad v=\int x\left(1-4x^2\right)^{-\frac{1}{2}}=-\frac{1}{8}\int -8x\left(1-4x^2\right)^{-\frac{1}{2}}dx$ $=-\frac{1}{4}\sqrt{1-4x^2}+C$ $\int \sin^{-1}2x\frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$ $=\left[\left(\sin^{-1}2x\right)\left(-\frac{1}{4}\sqrt{1-4x^2}\right)\right]-\int\left(-\frac{1}{4}\sqrt{1-4x^2}\right)\left(\frac{2}{\sqrt{1-4x^2}}\right)\mathrm{d}x$ $=\left[-\frac{1}{4}\left(\sin^{-1}2x\right)\sqrt{1-4x^2}\right]+\int\frac{1}{2}\,\mathrm{d}x$ $=\left[-\frac{1}{4}\left(\sin^{-1}2x\right)\sqrt{1-4x^2}\right]+\frac{1}{2}x+C$
(ii)	$\int\frac{x-1}{x^2+2x+6}\,\mathrm{d}x$ $=\frac{1}{2}\int\frac{2x+2}{x^2+2x+6}\,\mathrm{d}x-\int\frac{2}{\left(x+1\right)^2+5}\,\mathrm{d}x$ $=\frac{1}{2}\ln\left\|x^2+2x+6\right\|-\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$

Qn 2	**2011/CJC Prelim/2/2**
	$\frac{2+10x}{(1+3x)(1+3x^2)}=\frac{A}{1+3x}+\frac{Bx+C}{1+3x^2}$ By cover-up rule, when $x=-\frac{1}{3} \Rightarrow A=-1$ $2+10x=-1(1+3x^2)+(Bx+C)(1+3x)$ $\frac{2+10\mathrm{x}}{\left(1+3\mathrm{x}\right)\left(1+3\mathrm{x}^2\right)}=\frac{\mathrm{A}}{\left(1+3\mathrm{x}\right)}+\frac{\mathrm{Bx}+\mathrm{C}}{\left(1+3\mathrm{x}^2\right)}$ When $x = 0$, $C = 3$ When $x = 1$, $B = 1$ $\int_0^1 \frac{2+10x}{(1+3x)(1+3x^2)}\,\mathrm{d}x=\int_0^1 -\frac{1}{1+3x}+\frac{x+3}{1+3x^2}\,\mathrm{d}x$ $=\left[-\frac{1}{3}\ln\left\|1+3x\right\|+\frac{1}{6}\ln\left\|1+3x^2\right\|+\sqrt{3}\tan^{-1}\left(\sqrt{3}x\right)\right]_0^1$ $=-\frac{1}{3}\ln 4+\frac{1}{6}\ln 4+\sqrt{3}\tan^{-1}\left(\sqrt{3}\right)$ $=-\frac{1}{6}\ln 4+\frac{\sqrt{3}\pi}{3}$

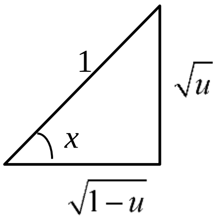
Qn3	**2011/DHS Prelim/1/8**
(a)	$\frac{x}{1-2x+x^2}=\frac{x}{\left(1-x\right)^2}$ $=\frac{-1}{1-x}+\frac{1}{\left(1-x\right)^2}$ Express $x=-\left(1-x\right)+1$ and split the fraction $\int \frac{x}{1-2x+x^2}\,\mathrm{d}x=\int \frac{-1}{1-x}+\frac{1}{\left(1-x\right)^2}\,\mathrm{d}x$ $=\ln\left\|1-x\right\|+\frac{1}{\left(1-x\right)}+C$
(b) **(i)**	$\int \sin^{-1}x\,\mathrm{d}x=x\sin^{-1}x-\int x.\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ $=x\sin^{-1}x+\frac{1}{2}\int\left(-2x\right)\left(1-x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $=x\sin^{-1}x+\sqrt{1-x^2}+C$ Let $u=\sin^{-1}x$ $v=1$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$ $\int v\,\mathrm{d}x=\int 1\,dx=x$
(b) **(ii)**	$\int\frac{x^2}{x^2-2x+3}\,\mathrm{d}x$ $=\int 1+\frac{2x-3}{x^2-2x+3}\,\mathrm{d}x$ Long division for improper fraction $=\int 1+\frac{2x-2}{x^2-2x+3}-\frac{1}{\left(x-1\right)^2+2}\,\mathrm{d}x$ $=x+\ln\left\|x^2-2x+3\right\|-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{2}}+C$

Qn 4	**2011/IJC Prelim/1/3**
	$\int \frac{\left[\ln\left(2x\right)\right]^2}{x\left[25-2\left[\ln\left(2x\right)\right]^2\right]}\mathrm{d}x$ $=\int \frac{2u^2}{\mathrm{e}^u\left(25-2u^2\right)}\left(\frac{1}{2}\mathrm{e}^u\right)\mathrm{d}u$ $x=\frac{1}{2}\mathrm{e}^u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{2}\mathrm{e}^u$ Visualise $\mathrm{d}x=\left(\frac{1}{2}\mathrm{e}^u\right)\mathrm{d}u$ $=\int \frac{u^2}{25-2u^2}\mathrm{d}u$ $=-\frac{1}{2}\int \frac{-2u^2+25-25}{25-2u^2}\mathrm{d}u$ $=-\frac{1}{2}\int 1-\frac{25}{25-2u^2}\mathrm{d}u$ $=-\frac{1}{2}\left[u-\left(25\right)\left(\frac{1}{\sqrt{2}\left(2\right)\left(5\right)}\ln\left\|\frac{5+u\sqrt{2}}{5-u\sqrt{2}}\right\|\right)\right]+c$ $=-\frac{1}{2}\left[u-\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+u\sqrt{2}}{5-u\sqrt{2}}\right\|\right]+c$ $=-\frac{1}{2}\left[\ln\left(2x\right)-\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+\sqrt{2}\ln\left(2x\right)}{5-\sqrt{2}\ln\left(2x\right)}\right\|\right]+c$

Qn 5	**2015/MI Prelim/1/2**
(i)	$\int \frac{\sin x}{1+2\cos x}\ \mathrm{d}x$ $= -\frac{1}{2}\int \frac{-2\sin x}{1+2\cos x}\ \mathrm{d}x$ $= -\frac{1}{2}\ln\lvert 1+2\cos x\rvert + C$
(ii)	$\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x$ $= \left[\mathrm{e}^x \cos 2x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2\mathrm{e}^x \sin 2x\ \mathrm{d}x$ $= \left[\mathrm{e}^x \cos 2x + 2\mathrm{e}^x \sin 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 4\mathrm{e}^x \cos 2x\ \mathrm{d}x$ $5\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x = -\mathrm{e}^{\frac{\pi}{2}} - 1$ $\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x = -\frac{1}{5}\left(\mathrm{e}^{\frac{\pi}{2}} + 1\right)$ Combine $\int_0^{\frac{\pi}{2}} 4e^x \cos 2x\ dx$ with LHS to become $5\int_0^{\frac{\pi}{2}} e^x \cos 2x\ dx$ **<u>Alternatively,</u>** $\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x$ $= \left[\frac{1}{2}\mathrm{e}^x \sin 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{2}\mathrm{e}^x \sin 2x\ \mathrm{d}x$ $= \left[\frac{1}{2}\mathrm{e}^x \sin 2x + \frac{1}{4}\mathrm{e}^x \cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{4}\mathrm{e}^x \cos 2x\ \mathrm{d}x$ $\frac{5}{4}\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x = -\frac{1}{4}\left(\mathrm{e}^{\frac{\pi}{2}} + 1\right)$ $\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\ \mathrm{d}x = -\frac{1}{5}\left(\mathrm{e}^{\frac{\pi}{2}} + 1\right)$

Qn 6	**2015/ACJC Prelim/1/1**
	$u = 3 - x^2, \quad x^2 = 3 - u, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -2x.$ $\int x^3\sqrt{3-x^2}\ \mathrm{d}x = -\frac{1}{2}\int (3-u)u^{\frac{1}{2}}\ \mathrm{d}u$ $= -\frac{1}{2}\int 3u^{\frac{1}{2}} - u^{\frac{3}{2}}\ \mathrm{d}u$ $= \frac{1}{5}u^{\frac{5}{2}} - u^{\frac{3}{2}} + c$ $= \frac{1}{5}\left(3-x^2\right)^{\frac{5}{2}} - \left(3-x^2\right)^{\frac{3}{2}} + c.$

Qn 7	**2015/NJC Prelim/2/1**
(a)	$x = 3\tan\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec^2\theta$ When $x = 3,\ \tan\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$ When $x = \sqrt{3}$, $\tan\theta = \frac{\sqrt{3}}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$ Change limits to θ $\int_{\sqrt{3}}^{3} \frac{1}{x^2\sqrt{x^2+9}}\,\mathrm{d}x$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}\left(3\sec^2\theta\right)\mathrm{d}\theta$ $= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\left(\tan^2\theta\right)\left(3\sec\theta\right)}\left(3\sec^2\theta\right)\mathrm{d}\theta$ $= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta$ $= \frac{1}{9}\left[\frac{-1}{\sin\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ $= \frac{1}{9}\left(\frac{-1}{\sin\frac{\pi}{4}} + \frac{1}{\sin\frac{\pi}{6}}\right)$ $= \frac{2-\sqrt{2}}{9}$
(b)	$\int \ln\left(x^2+4\right)\mathrm{d}x$ $= \int 1\cdot\ln\left(x^2+4\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int x\left(\frac{2x}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int\left(\frac{2\left(x^2+4\right)-8}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int\left(2 - \frac{8}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - 2x + 4\tan^{-1}\left(\frac{x}{2}\right) + c$ Let $u = \ln(x^2+4) \quad v = 1$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x}{x^2+4} \qquad \int v\,\mathrm{d}x = x$

Qn8	**2015/PJC Prelim/1/9**
(a)(i)	Consider $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right)=2x\mathrm{e}^{x^2}$ Therefore, $\int x\mathrm{e}^{x^2}\,\mathrm{d}x=\frac{1}{2}\mathrm{e}^{x^2}+C$
(a)(ii)	$\int x^3\mathrm{e}^{x^2}\,\mathrm{d}x$ $=\int x^2\left(x\mathrm{e}^{x^2}\right)\mathrm{d}x$ $=\frac{1}{2}x^2\mathrm{e}^{x^2}-\int xe^{x^2}\,\mathrm{d}x$ $=\frac{1}{2}x^2\mathrm{e}^{x^2}-\frac{1}{2}\mathrm{e}^{x^2}+C$ $=\frac{1}{2}x^2\mathrm{e}^{x^2}-\frac{1}{2}\mathrm{e}^{x^2}+C$ Let $u=x^2$ $\quad v=x\mathrm{e}^{x^2}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=2x$ $\quad \int v\,\mathrm{d}x=\frac{1}{2}\mathrm{e}^{x^2}$
(b)	$\int\sqrt{\frac{1-u}{u}}\,\mathrm{d}u$ $=\int\sqrt{\frac{1-\sin^2 x}{\sin^2 x}}\left(2\sin x\cos x\right)\,\mathrm{d}x$ $=\int\frac{\cos x}{\sin x}\left(2\sin x\cos x\right)\,\mathrm{d}x$ $=\int 2\cos^2 x\,\mathrm{d}x$ $=\int \cos 2x+1\,\mathrm{d}x$ $=\frac{\sin 2x}{2}+x+C$ $=\sin x\cos x+x+C$ $=\sqrt{u}\sqrt{1-u}+\sin^{-1}\sqrt{u}+C$ $=\sqrt{u-u^2}+\sin^{-1}\sqrt{u}+C$ $u=\sin^2 x\Rightarrow \sin x=\sqrt{u}$ $\Rightarrow x=\sin^{-1}\sqrt{u}$ $\sin x=\frac{\sqrt{u}}{1}$ $\cos x=\frac{\sqrt{1-u}}{1}$ 

Qn 9	**2017/JJC Prelim/1/2**
(a)	$\int \sin\left(3\theta\right)\cos\left(3\theta\right)\mathrm{d}\theta$ $=\frac{1}{2}\int 2\sin 3\theta\cos 3\theta\ \mathrm{d}\theta$ $=\frac{1}{2}\int \sin 6\theta\ \mathrm{d}\theta$ $=-\frac{1}{12}\cos 6\theta+C$
(b)	$\theta=\sqrt{\pi}\Rightarrow\sqrt{x}=\sqrt{\pi}\Rightarrow x=\pi$ $\theta=\sqrt{\frac{\pi}{2}}\Rightarrow\sqrt{x}=\sqrt{\frac{\pi}{2}}\Rightarrow x=\frac{\pi}{2}$ $\theta=\sqrt{x}\ \Rightarrow\ \frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$. $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}}\ \theta^3\cos\left(\theta^2\right)\mathrm{d}\theta$ $=\int_{\frac{\pi}{2}}^{\pi}\ x\sqrt{x}\left(\cos x\right)\left(\frac{1}{2\sqrt{x}}\right)\mathrm{d}x$ $=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\ x\cos x\ \mathrm{d}x$ $=\frac{1}{2}\left[\left[x\sin x\right]_{\frac{\pi}{2}}^{\pi}-\int_{\frac{\pi}{2}}^{\pi}1\left(\sin x\right)\mathrm{d}x\right]$ Let $u=x$ $\quad v=\sin x$ $\frac{\mathrm{d}u}{\mathrm{d}x}=1$ $\quad \int v\ \mathrm{d}x=-\cos x$ $=\frac{1}{2}\left(0-\frac{\pi}{2}+\left[\cos x\right]_{\frac{\pi}{2}}^{\pi}\right)$ $=\frac{1}{2}\left[-\frac{\pi}{2}+\left(-1-0\right)\right]$ $=-\frac{1}{2}-\frac{\pi}{4}$

Qn 10	**2017/NYJC Prelim/1/4**
(i)	$x-1=3\tan\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\sec^2\theta$

Qn 10	**2017/NYJC Prelim/1/4**
	$\displaystyle\int \frac{1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\left(x-1\right)^2+3^2}}\,\mathrm{d}x$ $\displaystyle= \int \frac{1}{\sqrt{\left(3\tan\theta\right)^2+3^2}} \cdot 3\sec^2\theta\ \mathrm{d}\theta$ $\displaystyle= \int \frac{1}{3\sec\theta} \cdot 3\sec^2\theta\ \mathrm{d}\theta$ $\displaystyle= \int \sec\theta\ \mathrm{d}\theta$ $\displaystyle= \ln\left\|\sec\theta+\tan\theta\right\|+C$ $\displaystyle= \ln\left\|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\|+C$
(ii)	$\displaystyle x+3=\frac{1}{2}\left(2x-2\right)+4$ $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ $\displaystyle= \int \frac{\frac{1}{2}\left(2x-2\right)+4}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ $\displaystyle= \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,\mathrm{d}x+\int \frac{4}{\sqrt{\left(x-1\right)^2+3^2}}\,\mathrm{d}x$ $\displaystyle= \frac{1}{2}\frac{\sqrt{x^2-2x+10}}{\frac{1}{2}}+4\int \frac{1}{\sqrt{\left(x-1\right)^2+3^2}}\,\mathrm{d}x$ $\displaystyle= \sqrt{x^2-2x+10}+4\ln\left\|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\|+C$