EJC

Q1

For $kx^2+(k-1)x+2k+1$ always negative,

Discriminant < 0 and $k<0$.

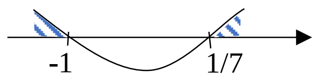
$b^2-4ac<0$

$(k-1)^2-4k(2k+1)<0$

$k^2-2k+1-8k^2-4k<0$

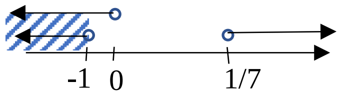
$-7k^2-6k+1<0$

$(7k-1)(k+1)>0$



$k<-1$ or $k>\dfrac{1}{7}$ and $k<0$

Combining solutions,



$k<-1$

Set of values of k is $\{k\in\mathbb{R}:k<-1\}$

Q2

(i)

Let the mass (in kg) of arabica, robusta and liberica coffee beans brought in by Starluck Coffee on International Coffee Day be x, y and z respectively.

$x+y+z=550$ ....(1)

$4x+1.5y+3.5z=1685$ ....(2)

$\frac{4}{5}x(4.00)+\frac{4}{9}y(1.50)+\left(\frac{4}{5}x-70\right)(3.50)=1315$

$3.2x+\frac{2}{3}y+2.8x-245=1315$

$6x+\frac{2}{3}y=1560$ ....(3)

Using the GC,

$x=240, y=180, z=130$

(ii)

Profit $=1315-240(3.70)-180(1.10)-130(3.20)=-187$

Starluck Coffee made a loss of \$187 on International Coffee Day

Q3
(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^{\frac{2}{3}}-\frac{2}{x}\right)^5$ $=5\left(3x^{\frac{2}{3}}-\frac{2}{x}\right)^4\left(2x^{-\frac{1}{3}}+\frac{2}{x^2}\right)$
(b) $\int_0^1 \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x$ $=\int_0^1 (3-2x)^{-\frac{1}{2}}\,\mathrm{d}x$ $=\left[\frac{(3-2x)^{\frac{1}{2}}}{(-2)(0.5)}\right]_0^1$ $=-\left[(3-2)^{\frac{1}{2}}-(3-0)^{\frac{1}{2}}\right]$ $=-[1-\sqrt{3}]$ $=\sqrt{3}-1$

Q4	
(a)	$2-px=0$ $x=\frac{2}{p}$
(b)	At A, y=0 $0=3-\ln(2-px)$ $\ln(2-px)=3$ $2-px=\mathrm{e}^3$ $x=\frac{2-\mathrm{e}^3}{p}$ x-coordinate of A is $\frac{2-\mathrm{e}^3}{p}$.
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{-p}{2-px}=\frac{p}{2-px}$ At point A,

$$\frac{dy}{dx} = \frac{p}{2 - p\left(\frac{2-e^3}{p}\right)}$$

$$= \frac{p}{2-2+e^3}$$

$$= \frac{p}{e^3}$$

Equation of tangent at A:

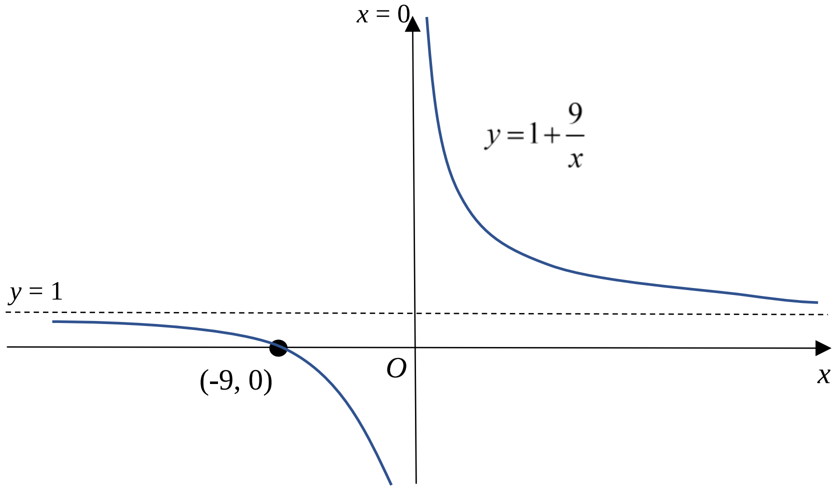
$$y - 0 = \frac{p}{e^3}\left(x - \frac{2-e^3}{p}\right)$$

$$y = \frac{p}{e^3}x - \frac{2-e^3}{e^3}$$

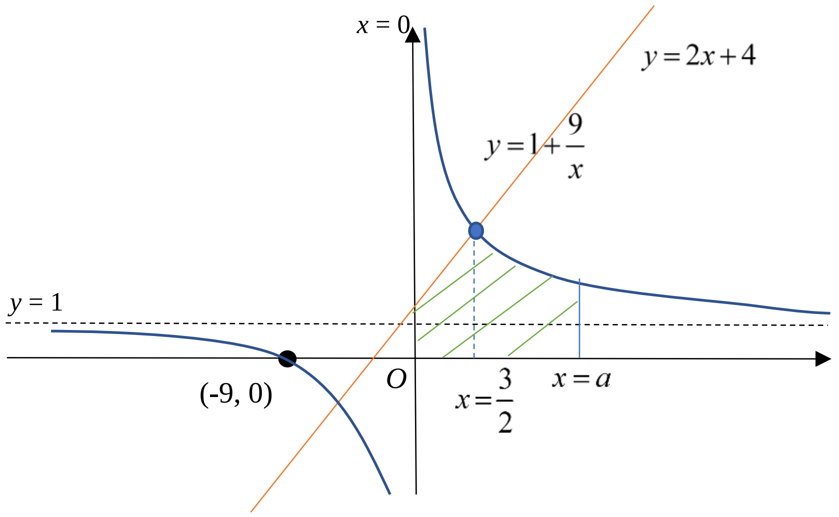
$$y = \frac{p}{e^3}x + 1 - \frac{2}{e^3}$$

Q5

(i)



(ii)



From the GC, the point of intersection is $(1.5,7)$

Area

$$=\int_0^{\frac{3}{2}} 2x+4 \text{ d}x+\int_{\frac{3}{2}}^{a} 1+\frac{9}{x} \text{ d}x$$

$$=8.25+[x+9\ln x]_{\frac{3}{2}}^{a}$$

$$=8.25+a+9\ln a-\frac{3}{2}-9\ln\frac{3}{2}$$

$$=6.75+a+9\ln a-9\ln\frac{3}{2}$$

$$=6.75+a+9\ln\frac{2a}{3}$$

Alternative (using area of trapezium)

$$=\frac{1}{2}\left(\frac{3}{2}\right)(4+7)+\int_{\frac{3}{2}}^{a} 1+\frac{9}{x} \text{ d}x$$

$$=8.25+[x+9\ln x]_{\frac{3}{2}}^{a}$$

$$=8.25+a+9\ln a-\frac{3}{2}-9\ln\frac{3}{2}$$

$$=6.75+a+9\ln a-9\ln\frac{3}{2}$$

$$=6.75+a+9\ln\frac{2a}{3}$$

Q6

(i)

When $t=0$,

$$C=2\mathrm{e}^{0.5(0)-1}-0.4(0)-\frac{1}{\mathrm{e}}=\frac{1}{\mathrm{e}}$$

$\frac{1}{\mathrm{e}}$ million dollars

(ii)

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 2\mathrm{e}^{0.5t-1}(0.5) - 0.4 = 0$$

$$\mathrm{e}^{0.5t-1} = 0.4$$

$$0.5t - 1 = \ln 0.4$$

$$t = 0.16742$$

$$t \approx 0.167$$

$$C = 0.36515 \approx 0.365$$

Method 1: Using 1st derivative test

t	0.167^-	0.167	0.167^+
$\frac{\mathrm{d}C}{\mathrm{d}t}$	E.g. use $t = 0.166$ $\frac{\mathrm{d}C}{\mathrm{d}t} = 2\mathrm{e}^{0.5(0.166)-1}(0.5) - 0.4$ $= -2.84 \times 10^{-4} < 0$	0	E.g. use $t = 0.168$ $\frac{\mathrm{d}C}{\mathrm{d}t} = 2\mathrm{e}^{0.5(0.168)-1}(0.5) - 0.4$ $= 1.16 \times 10^{-4} > 0$
Slope of tangent	\	-	/

$\therefore$ Value is a minimum

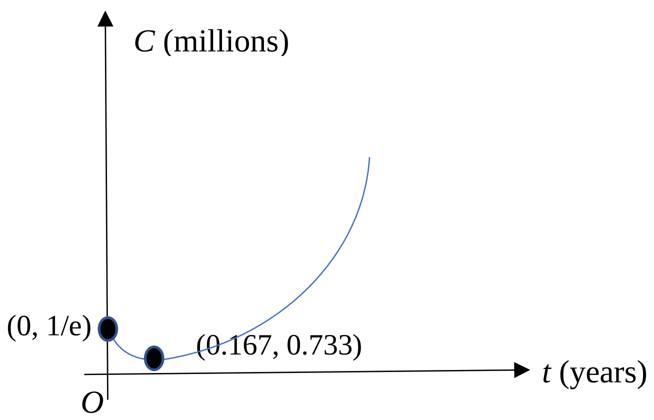
<u>Method 1: Using 2nd derivative test</u>

$$\frac{\mathrm{d}^2C}{\mathrm{d}t^2} = \mathrm{e}^{0.5t-1}(0.5)$$

when $t = 0.16742, \frac{\mathrm{d}^2C}{\mathrm{d}t^2} = 0.20000 > 0$

$\therefore$ Value is a minimum

(iii)



(iv)

$$\int_1^3 2e^{0.5t-1} - 0.4t - \frac{1}{e}\,dt = 1.8330 \approx 1.83$$

The total cost incurred between 1st Jan 2023 to 1st Jan 2025 is 1.83 million dollars.

(v)

Using GC, when $t = \frac{5}{12}$,

$$\frac{dP}{dt} = 2.53453 \approx 2.53$$

Rate of increase of total profit is 2.53 million dollars per year.

Q7

(i)

Number of ways = ${}^{4}C_2 \times {}^{6}C_3 \times {}^{8}C_5 = 6720$

(ii)

Number of ways to arrange 17 units $(4+5+7+1 \text{ pair of sisters}) = 17!$

Number of ways to arrange within the group of sisters $= 2!$

Probability $= \frac{17! \times 2!}{18!} = \frac{1}{9}$

(iii)

$$\mathrm{P}(\text{4 students from School of Engineering are all separated} \mid \text{2 sisters are next to each other})$$

$$= \frac{\mathrm{P}(\text{4 students from School of Engineering are all separated} \cap \text{2 sisters are next to each other})}{\mathrm{P}(\text{2 sisters are next to each other})}$$

Number of ways to arrange all students except those from the School of Engineering $(5+7+1 \text{ pair of sisters}) = 13!$

Number of ways to arrange within the group of sisters $= 2!$

Number of ways to slot the students from the School of Engineering $= {}^{14}C_4 \times 4!$

$$\text{Probability} = \frac{\frac{13! \times 2! \times {}^{14}C_4 \times 4!}{18!}}{\frac{1}{9}} = \frac{143}{340}$$

Alternatively,

$$\text{Probability} = \frac{\frac{13! \times 2! \times {}^{14}P_4}{18!}}{\frac{1}{9}} = \frac{143}{340}$$

Q8

(a) $\mathrm{P}(A'|B)$ represents the conditional probability that event A do not occur given that event B has already occurred.

(b) $\mathrm{P}(A'|B)=0.8$

$$\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(B)}=0.8$$

$$\mathrm{P}(A'\cap B)=1.6p$$

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(A'\cap B)$$

$$0.728=p+1.6p$$

$$p=0.28 \quad \text{(Shown)}$$

(c) $\mathrm{P}(A\cap B)=\mathrm{P}(B)-\mathrm{P}(A'\cap B)$

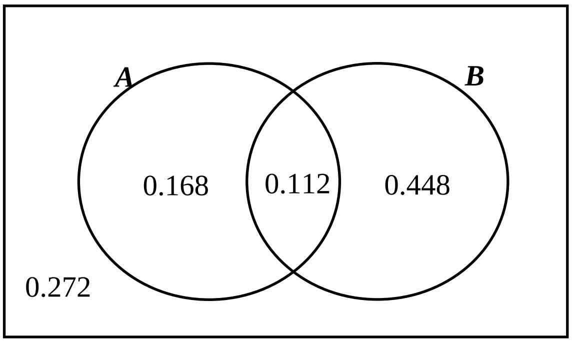
$$=2p-1.6p$$

$$=0.4p$$

$$=0.112$$

Since $\mathrm{P}(A\cap B)=0.112\neq 0$, events A and B are not mutually exclusive.

(d)



Q9

(a) P(student is in Year 1 and takes up sports) $=\dfrac{320}{1100}=\dfrac{16}{55}$

(b) P(student is in Year 2 $\cup$ takes up sports) $=\dfrac{285+154+61+320}{1100}=\dfrac{41}{55}$

Alternatively,

P(student is in Year 2 $\cup$ takes up sports)

=P(student in Year 2) + P(takes up sports) – P(in Year 2 and takes up sports)

$$= \frac{500}{1100} + \frac{605}{1100} - \frac{285}{1100}$$
$$= \frac{41}{55}$$

(c) P(student in Year 2 | does not take up performing arts) $= \frac{346}{800} = \frac{173}{400}$ (=0.4325)

(d) Required Probability $= {}^{3}C_{2} \left(\frac{195}{1100}\right)\left(\frac{194}{1099}\right)\left(\frac{905}{1098}\right)$

= 0.0774

Q10

(i)
Let X represent the number of rotten avocados in a box of 16 avocados.

$X \sim B\left(16, \frac{p}{100}\right)$

Using mean $= np$

$16\left(\frac{p}{100}\right) = 3.52$

$p = 22$ (Shown)

(ii)

$X \sim B(16, 0.22)$

$P(X > 3) = 1 - P(X \leq 3)$
$= 0.48143$
$= 0.481$

(iii)
Let Y represent the number of boxes with more than 3 rotten avocados, out of 15.

$Y \sim B(15, 0.48143)$

$P(Y < 5) = P(Y \leq 4)$
$= 0.078234$
$= 0.0782$

(iv)
Probability
$= (1 - 0.48143)^6$
$= 0.019446$
$= 0.0194$

Q11

(i)
Let X denote the mass of one oatmeal cookie, $X \sim N(40, 1.2^2)$ in grams.
Let Y denote the mass of one white chip cookie, $Y \sim N(45, 0.4^2)$ in grams.

$P\left(X > 1.02(40)\right) = P\left(X > 40.8\right)$ $= 0.25249$ $= 0.252$
(ii) $P(Y < 44.9) \times P(Y < 44.9)$ $= 0.40129^2$ $= 0.16103$ $= 0.161$
(iii) Let $A = 0.04\left(X_1 + X_2 + X_3 + ... + X_6\right)$ $\mathrm{E}(A) = 0.04\left(40 + 40 + 40 + ... + 40\right) = 9.6$ $\mathrm{Var}\left(A\right) = 0.04^2\left(1.2^2 + 1.2^2 + 1.2^2 + ... + 1.2^2\right) = 0.013824$ $A \sim \mathrm{N}\left(9.6,\ 0.013824\right)$ Let $B = 0.06\left(Y_1 + Y_2 + Y_3 + ... + Y_{10}\right)$ $\mathrm{E}(B) = 0.06\left(45 + 45 + 45 + ... + 45\right) = 27$ $\mathrm{Var}\left(A\right) = 0.06^2\left(0.4^2 + 0.4^2 + 0.4^2 + ... + 0.4^2\right) = 0.00576$ $B \sim \mathrm{N}\left(27,\ 0.00576\right)$ $A + B \sim \mathrm{N}\left(36.6,\ 0.019584\right)$ $P\left(A + B < 36.5\right) = 0.23743$ $= 0.237$
(iv) $\mathrm{E}(T) = 10$ $\mathrm{Var}\left(T\right) = 2.5^2$ $\overline{T} = \frac{1}{80}\left(T_1 + T_2 + ... + T_{80}\right)$ Since $n = 80$ is large, by Central Limit Theorem, $\overline{T} \sim \mathrm{N}\left(10,\ \frac{2.5^2}{80}\right)$ approximately. $\mathrm{P}\left(0 \le \overline{T} - 10 \le 0.5\right) = \mathrm{P}\left(10 \le \overline{T} \le 10.5\right)$ $= 0.46318$ $= 0.463$

Q12
(i) Using GC, unbiased estimate of the population mean $\overline{x} = 37.15$ unbiased estimate of the population standard deviation $s = 4.0734$ unbiased estimate of the population variance $s^2 = 4.0734^2 = 16.593 \approx 16.6$
(ii)

The probability of any student being selected for the sample is the same and the selection of any student is independent of the selection of other students.

(iii)

Let X denote the studying time, in hours, of one randomly chosen student

Let μ be the population's mean studying time.

Test $\mathrm{H}_0 : \mu = 38$
against $\mathrm{H}_1 : \mu < 38$

at 10% Level of significance

Under H_0, since n = 40 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(38, \frac{4.0734^2}{40}\right)$ approximately.

Using a one-tailed test, $\overline{x} = 37.15$ gives $p - \text{value} = 0.093459 \approx 0.0935$

Since $p - \text{value} = 0.0935 < 0.1$, we reject H_0. Hence, there is sufficient evidence at 10% level of significance to conclude that the mean studying times of the students are less than 38 hours/ management has overstated the mean studying times of the students. The teacher's suspicion is valid.

(iv)

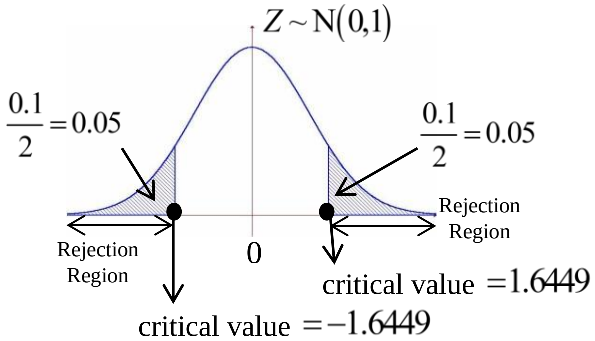
Sample mean, $\overline{x} = 38.4$

Test $\mathrm{H}_0 : \mu = 38$
against $\mathrm{H}_1 : \mu \neq 38$

at 10% Level of significance

Under H_0, since n = 90 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(38, \frac{\sigma^2}{90}\right)$ approximately.

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathrm{N}(0,1)$$



Using a two-tailed test, reject H_0 if $z_{cal} \le -1.6449$ or $z_{cal} \ge 1.6449$

$$z_{cal} = \frac{38.4-38}{\sqrt{\sigma^2/90}} = \frac{0.4}{\sigma/\sqrt{90}} = \frac{0.4\sqrt{90}}{\sigma}$$

Since we do not reject H_0,

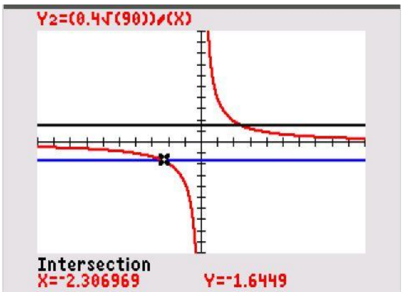
$-1.6449 < z_{cal} < 1.6449$

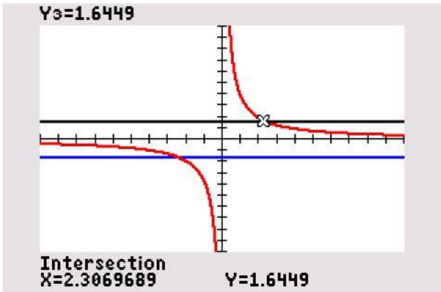
$-1.6449 < \dfrac{0.4\sqrt{90}}{\sigma} < 1.6449$

Using graph,

$\sigma < -2.3070$ (rejected $\because \sigma > 0$) or $\sigma > 2.3070$

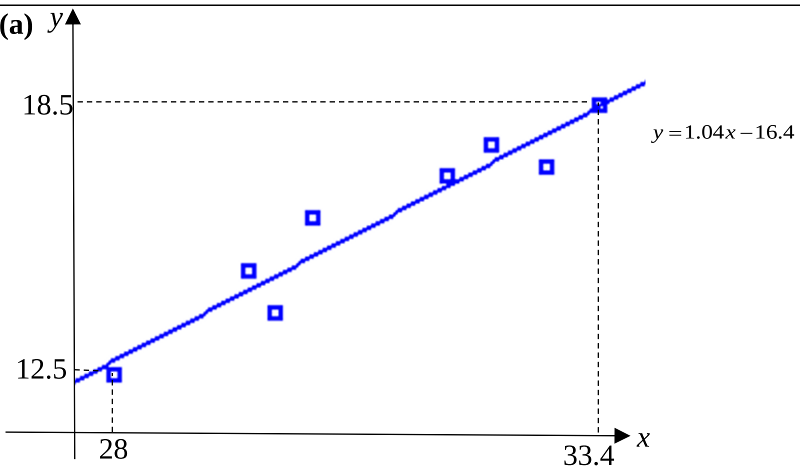
$\therefore \sigma > 2.31$





Q13

(a)



(b) $r = 0.957$ (3 s.f)

Since the value of r is positive and close to 1, there is a strong positive linear correlation between the average temperature and the sales from ice-cream. Furthermore, the scatter diagram reveals a strong positive linear relationship.

(c) $y = 1.044348185x - 16.41007632$

$y = 1.04x - 16.4$ (3 s.f)

(d) $y = 1.044348185x - 16.41007632$

$$\begin{aligned} y &= 1.044348185(31) - 16.41007632 \\ &= 15.964717 \\ &= 16.0 \quad \text{(3 s.f)} \end{aligned}$$

An estimate of sales from ice cream on a day with average temperature of $31°C$ is $1600.

The estimate is reliable as there is a strong linear correlation (|r|=0.957 is close to 1) and the data points lies close to regression line. $x = 31$ lies within the data range $28 \le x \le 33.4$, the estimate is obtained by interpolation.

(e) There will be no change in the value of m as the sales of ice-cream falls by the same amount, p on all days. m which is the rate of increase of sales of ice-cream with average temperature remains the same.

c will decrease by the same fixed amount, p as it is the sales from ice-cream when average temperature is $0°C$.