HWA CHONG INSTITUTION
2021 JC2 Preliminary Examination

MATHEMATICS
Higher 3

9820/01

Paper 1

Monday **20 September 2021** **3 hours**

Additional materials: 12-page Answer Booklet
List of Formula (MF26)
4-page Additional Answer Booklet (upon request)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the 12-page Answer Booklet and any other additional 4-page Answer Booklets you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF26).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, slot any additional 4-page Answer Booklets used in your 12-page Answer Booklet and indicate on the 12-page Answer Booklet the number of additional 4-page Answer Booklets used (if any).

1. Let N be an integer greater than 1.

(i) Show that for any positive integer k, $\left(N+\sqrt{N^2-1}\right)^k+\dfrac{1}{\left(N+\sqrt{N^2-1}\right)^k}$ is an integer. [4]

(ii) Hence or otherwise, show that for any positive integer k, $\left(N+\sqrt{N^2-1}\right)^k$ differs from the integer closest to it by less than $\left(2N-\dfrac{1}{2}\right)^{-k}$. [4]

2. Let a, b and c be real numbers for all real x,

$$(1+ax)(1+bx)(1+cx)=1+qx^2+rx^3.$$

(i) Express q and r in terms of a, b and c, and show that $a+b+c=0$.

[3]

(ii) Find, in terms of q and r, the first four nonzero terms in the series expansion of $y=\ln\left(1+qx^2+rx^3\right)$. [2]

(iii) Find the coefficient of x^n, $n\geq 1$, in the series expansion of $y=\ln\left(1+qx^2+rx^3\right)$, leaving your answer in the form $(-1)^{n+1}T_n$, where T_n is in terms of a,b,c and n.

[2]

(iv) Using the results in parts **(ii)** and **(iii)**, show that

$$\frac{\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right)}{6}=\frac{a^5+b^5+c^5}{5}$$ [2]

3. Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^x$ both satisfy the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0. \qquad (1)$$ [3]

The general solution of (1) is given by $u = Ax + B\mathrm{e}^x$, where A and B are arbitrary constants.

(i) Show that the substitution $y = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$ transforms the differential equation

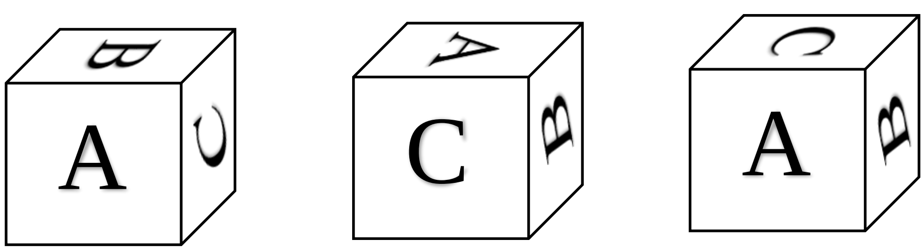
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x} \qquad (2)$$

into (1). [3]

(ii) Find the particular solution to (2) that satisfies $y = 2$ at $x = 0$. [3]

4. A factory has 7 colours of paint (Red, Orange, Yellow, Green, Blue, Indigo and Violet). The factory wants to produce different cubes such that all sides are painted and no 2 adjacent sides have the same colour. Cubes are considered different if and only if they cannot be rotated to form the same colour arrangement. For example in figure 1 below, the 2 cubes on the left are considered to be the same, while the cube on the right is different.

Figure 1:



Same cubes

Different cube

(i) Let the least number of different colours required to paint such a cube be n. State the value of n and find the number of different n-coloured cubes. [3]

(ii) Find the number of different cubes where all sides have different colours. [3]

(iii) Find the total number of different cubes. [4]

5. It is given that real numbers a and b, such that $0 \le a < b$, satisfy

$$\int_a^b x^2 \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x\right)^2.$$

(i) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$. [3]

It is given that $a = 1$.

(ii) Show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0.$$ [2]

(iii) By expressing p^2 in terms of q and considering $p^2 - q^2$, deduce that

$2 < b \le \dfrac{7}{3}$. [3]

6. For their school orientation, XYZJC ordered a large number of cushions with one of the letters $\{X, Y, Z, J, C\}$ printed on them.

(a) Adam wanted to arrange the cushions in order so that only an odd number of consecutive same letters are allowed. For example when $n = 4$, *XYYY* is a valid arrangement of four cushions while *XXYY* is an invalid arrangement. Let $A(n)$ be the number of ways to form such an n-lettered word.
Find the recurrence relation between $A(n)$, $A(n+1)$ and $A(n+2)$, justifying your answer. Hence find $A(5)$. [4]

(b) Becky took a dozen cushions. How many ways can she select 12 cushions such that all letters are selected and no letter is chosen more than 4 times? [3]

(c) Chandra takes 7 cushions randomly. Find the probability that at least 1 of each letter is chosen. [3]

(d) There are 10 misprinted identical cushions with no letter printed on them. How many ways can Dion pack them to return to the manufacturer if she can use up to 3 identical boxes, assuming each box has a capacity of 10 cushions? [2]

7. A function $\mathrm{f}(x)$ is said to be concave downwards on an interval (a,b) if, for any x in the interval (a,b), $\mathrm{f}''(x) < 0$.

(i) Show algebraically that the functions

$$\mathrm{g}(x) = \sin x, \qquad 0 < x < \pi,$$

$$\mathrm{h}(x) = \ln x, \qquad x > 0$$

are concave downwards on their domains. [2]

Jensen's inequality states that, given a function f that is concave downwards on an interval (a,b), for any $x_1, x_2, x_3, \ldots x_n$ in the interval (a,b),

$$\frac{1}{n}\sum_{k=1}^{n} \mathrm{f}(x_k) \le \mathrm{f}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right),$$

and the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Using Jensen's inequality,

(ii) show that for the three inner angles α, β and γ of any triangle,

$$\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}; \qquad [2]$$

(iii) with the use of a suitable function, show that for any positive numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \ldots x_n}{n} \ . \qquad [3]$$

Using the result in part **(iii)**,

(iv) find the minimum value of $x^5 + y^5 + z^5 - 20xyz$ for any positive integer x, y and z. State the values of x, y and z that this minimum value is obtained. [3]

8. Let $\mathrm{f}:\mathbb{Q}\to\mathbb{Q}$ be a function satisfying the functional equation

$$\mathrm{f}(x+y)=\mathrm{f}(x)+\mathrm{f}(y), \qquad \forall x,y\in\mathbb{Q}$$

(i) Verify that $\mathrm{f}(0)=0$. [1]

A function g is known as an odd function if it satisfies $\mathrm{g}(-x)=-\mathrm{g}(x)$.

(ii) Show that f is an odd function. [2]

(iii) Using Mathematical Induction, show that $\mathrm{f}(nx)=n\mathrm{f}(x)$ for all $n\in\mathbb{Z}^+$. [3]

(iv) Using the result in part **(iii)** or otherwise, show that $\mathrm{f}\left(\frac{x}{m}\right)=\frac{1}{m}\mathrm{f}(x)$ for all $m\in\mathbb{Z}^+$. [2]

(v) Given that the graph $y=\mathrm{f}(x)$ passes through $(1,3)$, show that $\mathrm{f}(x)=3x,\ x\in\mathbb{Q}$ is the solution to the functional equation. [2]

9. Let $I_n=\int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t$, for $n\in\mathbb{Z}^+$.

(i) Show that $I_{n+1}\le\frac{1}{2}I_n$. [3]

(ii) Show also that $I_{n+1}=-\frac{1}{n2^n}+I_n$, and deduce that $I_n\le\frac{1}{n2^{n-1}}$. [3]

(iii) Using part **(ii)**, prove that $\ln 2=\sum_{r=1}^{n}\frac{1}{r2^r}+I_{n+1}$. [3]

(iv) Using parts **(ii)** and **(iii)**, show that $\ln 2\approx 0.69$. [3]

10. For any positive integer *N* whose unique prime factorisation is given by $N = p_1^{k_1} p_2^{k_2} \dots p_l^{k_l}$, with p_i being the prime factors, the function $\mathrm{f}(N)$ is given by

$$\mathrm{f}(N) = N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_l}\right), \text{ with } \mathrm{f}(1)=1.$$

For example, with $60 = 2^2 \times 3 \times 5$, we have $\mathrm{f}(60) = 60\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 16$.

(i) Evaluate $\mathrm{f}(15)$ and $\mathrm{f}(180)$. [2]

(ii) Show that $\mathrm{f}(N)$ is an integer for any positive integer *N*. [2]

(iii) Determine whether the following statements are true, justifying your answers.

(a) For any positive integers *a* and *b*,

$$\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b).$$ [2]

(b) For any positive integers *a* and *b*,

$$\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b)$$

if and only if *a* and *b* are coprime to each other. [4]

(iv) Show that if *p* is a prime number and *k* is a positive integer, then $\mathrm{f}\left(p^k\right) = n$, where *n* is the number of positive integers that are less than or equal to p^k and are coprime to p^k.

[2]

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