RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2018 Year 5

Chapter 7B: Complex Numbers II – Complex Numbers in Polar Form

SYLLABUS INCLUDES

H2 Mathematics:

- representation of complex numbers in the Argand diagram
- magnitude
- calculation of modulus (r) and argument (θ) of a complex number
- complex numbers expressed in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ or $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.   + −   ÷ ×
- calculation of modulus (r) and argument (θ) of a complex number
- multiplication and division of two complex numbers expressed in polar form

cartesian

PRE-REQUISITES

- Trigonometry,
- Coordinate Geometry,
- Vectors,
- Indices and algebraic manipulation

CONTENT

1 Geometrical Representation of a Complex Number

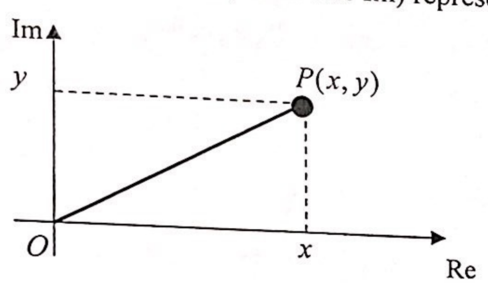
Recall that we represent real numbers on a number line.
For complex numbers, since there are imaginary numbers, it makes sense for us to have a imaginary number line as well.

1.1 Argand Diagram

A complex number $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$ can be represented by the point $P(x, y)$. This idea was formally introduced by the French mathematician Jean-Robert Argand, and hence the diagram which represents a complex number in this way was named after him.

An **Argand diagram** is similar to the Cartesian plane with the

- Horizontal axis (labeled as Re) representing the real part of z, and
- Vertical axis (labeled as Im) representing the imaginary part of z.



Remarks:

Points on the horizontal axis represent real numbers and points on the vertical axis represent purely imaginary numbers.

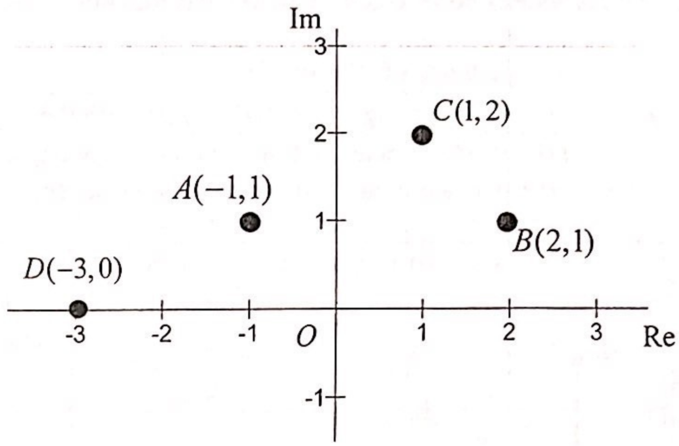
On an Argand diagram, complex numbers behave (in terms of additions and subtractions) like 2-D vectors, but they are NOT vectors. In particular, we can divide complex numbers, but we cannot do the same to vectors.

1.2 Geometrical Representation of Addition and Subtraction of Complex Numbers

Consider the complex numbers
$z_1 = -1+\mathrm{i}$ and $z_2 = 2+\mathrm{i}$.

Then $z_1 + z_2 = 1+2\mathrm{i}$
and $z_1 - z_2 = -3$.

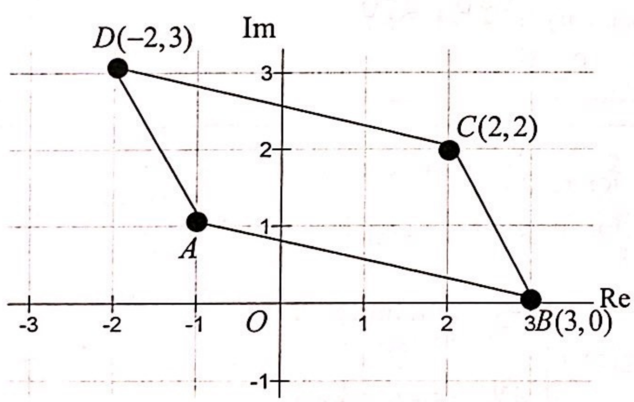
Let the points A, B, C and D represent z_1, z_2, z_1+z_2 and z_1-z_2 respectively on an Argand diagram.



Note that $OACB$ is a parallelogram.
As mentioned, the complex numbers behave like vectors on an Argand diagram.
(Recall how we proved $OACB$ is a parallelogram using vectors.)

Example 1

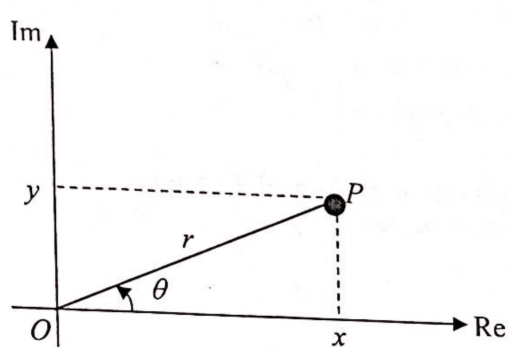
The points A, B, C and D represent four complex numbers z_1, $z_2=3$, $z_3=2+2\mathrm{i}$ and $z_4=-2+3\mathrm{i}$ respectively. Given that $ABCD$ forms a parallelogram, find z_1 by calculation (i.e. not by drawing).

Solution	
	Recall that if this is a vector question:

1.3 Modulus and Argument of a Complex Number

Let the point P represent the complex number $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$ on an Argand diagram. We are interested in 2 geometrical measurements:

- $r =$ distance of P from the origin O
- $\theta =$ directed angle that $\overrightarrow{OP}$ makes with the positive real axis
  ($\theta > 0$ when measured in an anti–clockwise sense;
  $\theta < 0$ when measured in a clockwise sense)



r is known as the **modulus** of z and is denoted by $|z|$, where

$$|z| = r = \sqrt{x^2 + y^2}\ .$$

Clearly $r \geq 0$.

θ is known as the (principal) **argument** of z and is denoted by $\arg(z)$, where

$$-\pi < \arg(z) \leq \pi\ .$$

$\arg(z)$ should be given in radians.

Questions

1. What is $\arg(z)$ if z is real? 0 , 180
2. What is $\arg(z)$ if z is purely imaginary? 90 , 270
3. What is $|z|$ and $\arg(z)$ if $z = 0$? 0 0

Answers

1. If z is real, $\arg(z) = \begin{cases} 0, & \text{for } \mathrm{Re}\, z > 0, \\ \pi, & \text{for } \mathrm{Re}\, z < 0. \end{cases}$

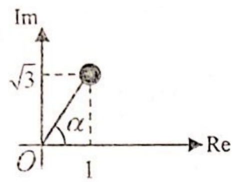
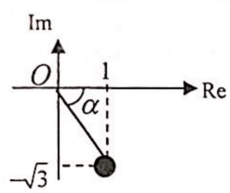
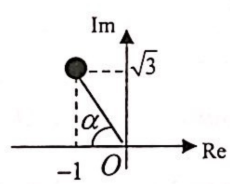
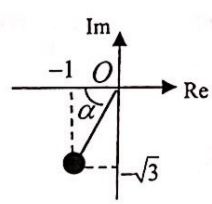
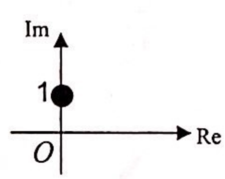
2. If z is purely imaginary, $\arg(z) = \begin{cases} \frac{\pi}{2}, & \text{for } \mathrm{Im}(z) > 0, \\ \frac{3\pi}{2}, & \text{for } \mathrm{Im}(z) < 0. \end{cases}$

3. If $z = 0$, $|z| = 0$ but $\arg(z)$ any value?

Example 2

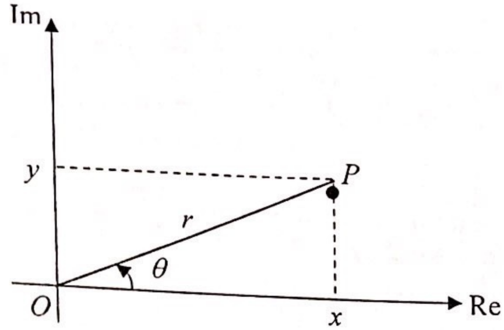
For each of the following complex number, find the exact value of the modulus and argument:

(i) $1+i\sqrt{3}$ **(ii)** $1-i\sqrt{3}$ **(iii)** $-1+i\sqrt{3}$ **(iv)** $-1-i\sqrt{3}$ **(v)** i

Solution (i) 	$\left\|1+i\sqrt{3}\right\| = \sqrt{\quad + \quad} =$ $\tan\alpha = \dfrac{\quad}{\quad} \Rightarrow \alpha = \qquad \Rightarrow \arg(1+i\sqrt{3}) = \alpha =$
(ii) 	$\left\|1-i\sqrt{3}\right\| = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$ $\arg(1-i\sqrt{3}) =$
(iii) 	$\left\|-1+i\sqrt{3}\right\| = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$ $\arg(-1+i\sqrt{3}) =$
(iv) 	$\left\|-1-i\sqrt{3}\right\| = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$ $\arg(-1-i\sqrt{3}) =$
(v) 	$\left\|i\right\| =$ $\arg(i) =$

2 Complex Numbers in Polar Form

Consider a point P on a Argand diagram representing the complex number, $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$ with $|z| = r$ and $\arg(z) = \theta,\ -\pi < \theta \le \pi$.



It can be easily seen from the diagram that $x = r\cos\theta$ and $y = r\sin\theta$.

So
$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r\cos\theta + \mathrm{i}(r\sin\theta) \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

Furthermore, it can be proven that

$$\boxed{\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta} \quad \text{(Euler's Formula)}}$$

(Refer to Appendix for further details.)

So any complex number can be written as

$$\begin{aligned} z &= x + \mathrm{i}y && \text{(cartesian form)} \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \\ &= r\,\mathrm{e}^{\mathrm{i}\theta} && \text{(polar form)} \end{aligned}$$
where $|z| = r$ and $\arg(z) = \theta,\ -\pi < \theta \le \pi$.

Recall that $\cos(\theta \pm 2k\pi) = \cos(\theta)$ and $\sin(\theta \pm 2k\pi) = \sin(\theta), k \in \mathbb{Z}^+$.
Thus, there is a need to have $-\pi < \arg(z) \le \pi$ for A level syllabus.

Example 3

(a) Express $2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ in the form $a+\mathrm{i}b$, $a,\ b\in\mathbb{R}$.

(b) If $|z|=4$ and $\arg(z)=\frac{2\pi}{3}$, express z in cartesian form.

(c) If $z_1=2+2\mathrm{i}$ and $z_2=\sqrt{3}-\mathrm{i}$, express z_1 and z_2 in polar form.

Solution

(a) $2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 2\left(\cos\left(\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$

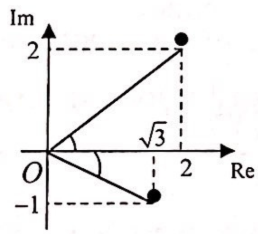
(b) $z=(4)\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$

$= 4\left(\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$

(c) $|z_1|=\sqrt{4+4}=\sqrt{8}$, $\arg(z_1)=\tan^{-1}\left(\frac{2}{2}\right)$

$|z_2|=$, $\arg(z_2)=$

$z_1=$ and $z_2=$



2.1 Multiplication and Division of Complex Numbers in Polar Form

It is straight forward to add or subtract complex numbers in cartesian form.
However, when it comes to multiplication and division, it is much easier to use the polar form.

Let $z_1=r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2=r_2\mathrm{e}^{\mathrm{i}\theta_2}$, where $r_1,\ r_2>0$ and $-\pi<\theta_1,\ \theta_2\le\pi$. For real numbers x and y we know that $\mathrm{e}^x\mathrm{e}^y=\mathrm{e}^{x+y}$, but this property also holds for complex numbers, so that we have

$$z_1z_2=r_1\mathrm{e}^{\mathrm{i}\theta_1}\times r_2\mathrm{e}^{\mathrm{i}\theta_2}=r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}\quad\text{(see Appendix for the proof)}$$

$$\frac{z_1}{z_2}=\frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}}=\frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$$

$$z_1^{\,n}=\underbrace{z_1\times z_1\times\ldots\times z_1}_{n\text{ times}}=\underbrace{r_1\mathrm{e}^{i\theta_1}\times r_1\mathrm{e}^{i\theta_1}\times\ldots\times r_1\mathrm{e}^{i\theta_1}}_{n\text{ times}}=\underbrace{r_1\times\ldots\times r_1}_{n\text{ times}}\times\underbrace{\mathrm{e}^{i\theta_1}\times\ldots\times\mathrm{e}^{i\theta_1}}_{n\text{ times}}=r_1^{\,n}\mathrm{e}^{in\theta_1}$$

Hence

(1) $|z_1 z_2| = |z_1||z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$, provided $-\pi < \theta_1 + \theta_2 \le \pi$

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$, provided $-\pi < \theta_1 - \theta_2 \le \pi$

(3) For $n \in \mathbb{Z}^+$, $|z^n| = |z|^n$

$\arg(z_1{}^n) = n\arg(z_1) = n\theta_1$, provided $-\pi < n\theta_1 \le \pi$

Note that it is tedious to derive the above results if we use cartesian form of the complex numbers.

Example 4

(a) If $z_1 = 2 + 2\mathrm{i}$ and $z_2 = \sqrt{3} - \mathrm{i}$, express $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in polar form.

(b) If $z = -\sqrt{3} + \mathrm{i}$, find the modulus and argument of z^2.

(c) Given that $|\mathrm{i}z| = 3$ and $\arg(\mathrm{i}z) = \dfrac{\pi}{4}$, express z in cartesian form.

$\arg(iz) = \arg(i) + \arg(z) = \frac{\pi}{4}$

$|iz| = |i||z| = 3$

$|z| = 3$, $\arg(z) = -\frac{\pi}{4}$

$z = 3e^{-\frac{\pi}{4}i}$

$~~= 3(\cos -\frac{\pi}{4} + i\sin\frac{\pi}{4})~~$

$= 3\left(\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right)$

Solution

(a) From Example 3(c), $z_1 = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ and $z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$

$z_1 z_2 = \left(2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\right) = 4\sqrt{2}\, e^{i\frac{\pi}{12}}$

$\dfrac{z_1}{z_2} = \dfrac{2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}} = \sqrt{2}\, e^{i\frac{5\pi}{12}}$

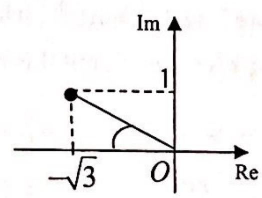
(b) $|z| = \sqrt{3+1} = 2$, $\arg(z) = \dfrac{5\pi}{6}$

$|z^2| = |z|^2 = 4$, $\arg(z^2) = 2\arg(z)$ -2π

$= 2\left(\frac{5\pi}{6}\right)$ -2π

not in range $-\pi < \theta \le \pi$ → $= \frac{5\pi}{3}$ -2π

$= -\frac{\pi}{3}$



OR

$z^2 = \left(2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}\right)^2 = 4e^{i\frac{5\pi}{3}} = 4e^{i\left(\frac{5\pi}{3} - 2\pi\right)} = 4e^{-i\frac{\pi}{3}}$

$|z^2| = 4$ and $\arg(z^2) = -\dfrac{\pi}{3}$

$\arg(z) = \frac{\pi}{6}$ $z = re^{i\theta}$, $z^2 = r^2 e^{i2\theta}$

$\arg(z^2) = \frac{\pi}{3}$ ✗ $-\frac{\pi}{3}$

$|z| = 2$, $|z^2| = 4$ ✓

(c) $|iz| = |i||z| = 1\times|z| = 3 \;\Rightarrow\; |z| = 3$

$\arg(iz) = \frac{\pi}{4} \;\Rightarrow \arg(i) + \arg(z) = \frac{\pi}{4}$

$\Rightarrow \frac{\pi}{2} + \arg(z) = \frac{\pi}{4}$

$\Rightarrow \arg(z) = -\frac{\pi}{4}$

$\therefore z = 3e^{-i\frac{\pi}{4}} = 3\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

$= 3\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

$= \frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}$

OR $|iz| = 3,\ \arg(iz) = \frac{\pi}{4}$

$\Rightarrow iz = 3e^{i\frac{\pi}{4}}$

$\Rightarrow z = \frac{3e^{i\frac{\pi}{4}}}{i} = \frac{3e^{i\frac{\pi}{4}}}{e^{i\frac{\pi}{2}}} = 3e^{i\left(\frac{\pi}{4}-\frac{\pi}{2}\right)} = 3e^{-i\frac{\pi}{4}}$

$\therefore z = 3e^{-i\frac{\pi}{4}}$

$= 3\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

$\therefore z = \quad e^{i(\;\;)} = \quad\left[\cos\left(\;\;\right) + i\sin\left(\;\;\right)\right]$

Example 5 [HCJC 9233 Prelim/2003/02/Q4] [Self-Read]

The complex number z is given by $z = \frac{(1+i)^3}{\sqrt{2}(a+i)^2}$, where $a > 0$. ~~$|z| = \frac{\sqrt{2}^3 e^{i\frac{3\pi}{4}}}{\sqrt{2}}$~~ $\frac{\sqrt{2}^3}{\sqrt{2}(\sqrt{a^2+1})^2} = \frac{2}{a^2+1}$

Given that $|z| = \frac{1}{2}$, find the value of a and show that $\arg(z) = \frac{5\pi}{12}$.

Solution

$$|z| = \left|\frac{(1+i)^3}{\sqrt{2}(a+i)^2}\right| = \frac{|(1+i)^3|}{|\sqrt{2}(a+i)^2|} = \frac{|1+i|^3}{|\sqrt{2}||a+i|^2} = \frac{(\sqrt{2})^3}{\sqrt{2}(\sqrt{a^2+1^2})^2} = \frac{2}{a^2+1}$$

$$\therefore \frac{2}{a^2+1} = \frac{1}{2} \;\Rightarrow a^2 = 3 \;\Rightarrow a = \sqrt{3} \text{ (since } a > 0)$$

$(1+i) = \sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$ and $(\sqrt{3}+i) = 2e^{i\left(\frac{\pi}{6}\right)}$

$(1+i)^3 = 2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ and $(\sqrt{3}+i)^2 = 4e^{i\left(\frac{\pi}{3}\right)}$

$$z = \frac{(1+i)^3}{\sqrt{2}(\sqrt{3}+i)^2} = \frac{2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}}{4\sqrt{2}e^{i\left(\frac{\pi}{3}\right)}}$$

$$= \frac{1}{2}e^{i\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)} = \frac{1}{2}e^{i\left(\frac{5\pi}{12}\right)}$$

$\arg(z) = \frac{5\pi}{12}$ (shown)

OR

$\arg(1+i) = \frac{\pi}{4}$

$\arg(\sqrt{3}+i) = \frac{\pi}{6}$

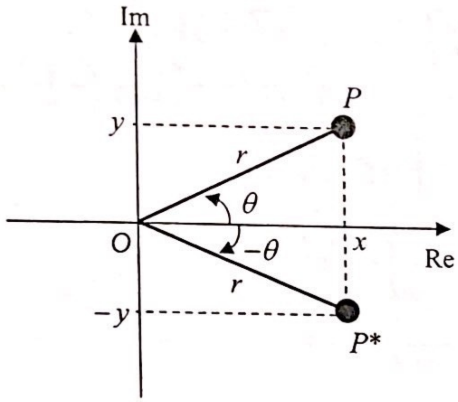
$$\arg(z) = \arg\left(\frac{(1+i)^3}{\sqrt{2}(\sqrt{3}+i)^2}\right)$$

$$= 3\arg(1+i) - \left[\arg(\sqrt{2}) + 2\arg(\sqrt{3}+i)\right]$$

$$= 3\left(\frac{\pi}{4}\right) - \left[0 + 2\left(\frac{\pi}{6}\right)\right] = \frac{5\pi}{12} \text{ (shown)}$$

2.2 Relationships between $\arg(z)$ and $\arg(z^*)$, $|z|$ and $|z^*|$

Let the points P and P^* represent the complex numbers $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, $x, y \in \mathbb{R}$ respectively on an Argand diagram.



Observe that P and P^* are reflections of each other about the real axis.

From the diagram, it is clear that

$$|z^*| = |z| \text{ and } \arg(z^*) = -\arg(z).$$

Also,

$$zz^* = x^2 + y^2 = |z|^2.$$

In polar form,

if $z = re^{\mathrm{i}\theta}$, then $z^* = re^{-\mathrm{i}\theta}$.

Example 6

A complex number z is such that $|z^*| = \sqrt{2}$ and $\arg(z^*) = \frac{3\pi}{4}$. $\arg(z) = -\frac{3\pi}{4}$

Find the modulus and argument of the complex number z^2. $z = \sqrt{2}e^{-i\frac{3\pi}{4}}$

$z^2 = 2e^{-i\frac{3\pi}{2}}$

Solution

$|z| = |z^*| = \sqrt{2}$, $\arg(z) = -\arg(z^*) = -\frac{3\pi}{4} \Rightarrow z = \sqrt{2}e$

$z^2 = \left(\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^2 = 2e^{-i\frac{3\pi}{2}} = 2e^{i(-\frac{3\pi}{2}+2\pi)} = 2e^{i\frac{\pi}{2}}$

$\Rightarrow |z^2| = 2$ and $\arg(z^2) = \frac{\pi}{2}$

Alternatively, $|z^2| = |z|^2 = (\sqrt{2})^2 = 2$, and $\leftarrow |z| = |z^*| = \sqrt{2}$

$\arg(z^2) = 2\arg(z) + 2\pi = 2(-\frac{3\pi}{4}) + 2\pi$

$= -\frac{3\pi}{2} + 2\pi$

$= \frac{\pi}{2}$

$-\pi < \arg(z) < \pi$

2.3 Complex Numbers in Cartesian Form vs Polar Form

In section 5.1, it seems that multiplication and division of complex using polar form is less tedious. However, when we perform additions and subtractions, it is more direct to use Cartesian form. In essence, we have to be flexible when dealing with problems involving complex numbers.

		$z = x + \mathrm{i}y$	$z = r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$
(a)	$\|z\|$	$\sqrt{x^2+y^2}$	r
(b)	z^*	$x - \mathrm{i}y$	$r\mathrm{e}^{-\mathrm{i}\theta}$
(c)	$(z^*)^* = z$	$[(x+\mathrm{i}y)^*]^*$ $= (x-\mathrm{i}y)^*$ $= x+\mathrm{i}y$	$[(r\mathrm{e}^{\mathrm{i}\theta})^*]^*$ $= (r\mathrm{e}^{-\mathrm{i}\theta})^*$ $= r\mathrm{e}^{\mathrm{i}\theta}$
(d)	zz^*	x^2+y^2	r^2
(e)	$z + z^* = 2\,\mathrm{Re}(z)$	$(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $= 2x$	$r(\cos\theta+\mathrm{i}\sin\theta)+r(\cos(-\theta)+\mathrm{i}\sin(-\theta))$ $= 2r\cos\theta$
(f)	$z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	$(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $= 2\mathrm{i}y$	$r(\cos\theta+\mathrm{i}\sin\theta)-r(\cos(-\theta)+\mathrm{i}\sin(-\theta))$ $= \mathrm{i}(2r\sin\theta)$

Q: Given that $z = \dfrac{1}{1+\mathrm{e}^{\mathrm{i}\theta}}$, how do we express z in cartesian form?

$\arg(z) = \arg(1) - \arg(1+e^{i\theta})$

Method 1: multiply by conjugate

$$\frac{1}{1+e^{i\theta}} = \frac{1}{1+e^{i\theta}} \times \frac{1+e^{-i\theta}}{1+e^{-i\theta}}$$

$$= \frac{1+e^{-i\theta}}{1+e^{i\theta}+e^{-i\theta}+1}$$

$$= \frac{1+\cos\theta - i\sin\theta}{2+2\cos\theta}$$

$$= \frac{1+\cos\theta}{2(1+\cos\theta)} - i\frac{\sin\theta}{2(1+\cos\theta)}$$

$=$

Method 2: split into conjugate pair

$$\frac{1}{1+e^{i\theta}} = \frac{1}{e^{i0}+e^{i\theta}}$$

$$= \frac{1}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}\right)}$$

$=$

Example 7

If n is a positive integer, prove that $(-1+\mathrm{i}\sqrt{3})^n+(-1-\mathrm{i}\sqrt{3})^n=2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$.

Solution

$\left|-1+\mathrm{i}\sqrt{3}\right|=2$, $\arg(-1+\mathrm{i}\sqrt{3})=\frac{2\pi}{3}$ $\Rightarrow$ $-1+\mathrm{i}\sqrt{3}=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$

Since $-1-\mathrm{i}\sqrt{3}$ and $-1+\mathrm{i}\sqrt{3}$ are conjugates of each other, $-1-\mathrm{i}\sqrt{3}=2\mathrm{e}^{-\mathrm{i}\left(\frac{2\pi}{3}\right)}$

Now $(-1+\mathrm{i}\sqrt{3})^n+(-1-\mathrm{i}\sqrt{3})^n=\left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^n+\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\right)^n$

$=2^n\left(e^{i\frac{2n\pi}{3}}\right)+2^\pi\left(e^{-i\frac{2\pi n}{3}}\right)$

$|z|=\sqrt{4}=2$

$\arg(z)=\tan^{-1}(-\sqrt{3})=\frac{2\pi}{3}$

$(-1+i\sqrt{3})^n+(-1-i\sqrt{3})^n$

$=2^n e^{i\frac{2n\pi}{3}}+2^n e^{-\frac{i2n\pi}{3}}$

$=2^n\left(e^{i\frac{2\pi n}{3}}+e^{-\frac{i2n\pi}{3}}\right)$

$=2^n\left(\cos\frac{2\pi n}{3}+i\sin\frac{2n\pi}{3}+\cos\frac{2n\pi}{3}+i\sin\left(-\frac{2n\pi}{3}\right)\right)$

$=2^n\left(2\cos\frac{2n\pi}{3}\right)$

$=2^{n+1}\left(\cos\frac{2n\pi}{3}\right)$

3 Geometrical Effect of Multiplying Two Complex Numbers

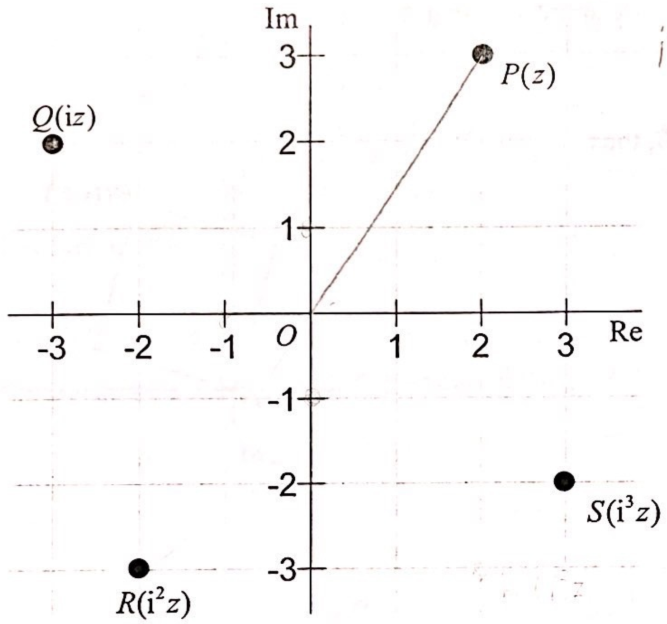
3.1 Geometrical Effect of Multiplying a Complex Number by i

Consider the complex number $z = 2 + 3\mathrm{i}$.
Draw the points P, Q, R and S representing z, iz, $\mathrm{i}^2 z$ and $\mathrm{i}^3 z$ respectively on the same Argand diagram.

$-z$

$-2-3i \quad -i(2+3i)$

$i^3 z = -2i + 3$



$i(2+3i)$
$i(2i-3)$
$i(-3i-2)$
$3-2i$

Now $\mathrm{i}z = -3 + 2\mathrm{i}$, $\mathrm{i}^2 z = -2 - 3\mathrm{i}$, $\mathrm{i}^3 z = 3 - 2\mathrm{i}$

Notice that if P is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain Q.

If Q is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain R.

If R is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain S.

Why is this so? In general, if $z = r\mathrm{e}^{\mathrm{i}\theta}$ and we multiply it by i, we have

$$\mathrm{i}z = \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = r\mathrm{e}^{\mathrm{i}\left(\theta+\frac{\pi}{2}\right)}$$

Thus, the geometrical effect of multiplying a complex number z by i is an <u>anti-clockwise rotation</u> of P through an angle of $\frac{\pi}{2}$ radians about O. Note that the modulus is still the same.

Example 8

In an Argand diagram, the points A, B and C represent the complex numbers a, $6+8\mathrm{i}$ and c respectively. $OABC$ is a square described in an anti-clockwise sense, where O is the origin. Give a reason why $c = \mathrm{i}a$ and a reason why $a+c = 6+8\mathrm{i}$. Find a and c by calculation (i.e. not by geometry).

Solution

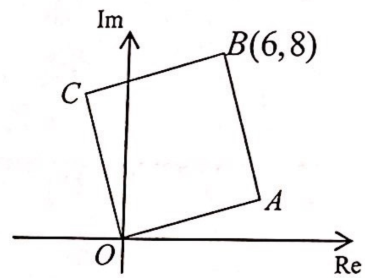
When A is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we get C.

Hence $c =$ ia ----- (1)

OC is parallel and equal in length to AB, thus

$c - 0 =$ $6+8i-a$

Thus, $a+c = 6+8i$ ----- (2)



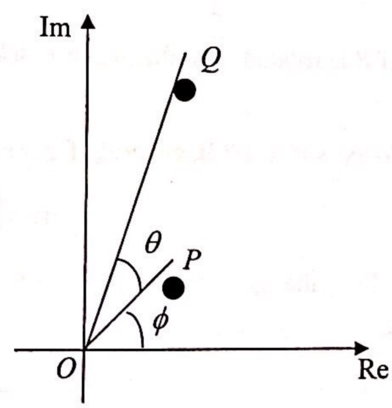
Subst (1) into (2),

Subst into (1), $c =$

3.2 Geometrical Effect of Multiplying a Complex Number by $re^{i\theta}$

Let P and Q represent the complex numbers z_1 and $z_1 z_2$ respectively, where $z_1 = 2\mathrm{e}^{\mathrm{i}\phi}$ and $z_2 = r\mathrm{e}^{\mathrm{i}\theta}$.

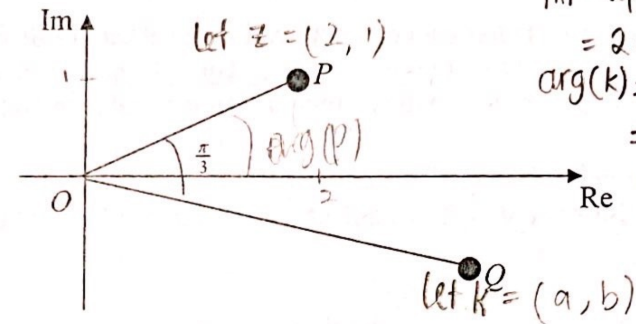
Now $z_1 z_2 = \left(2\mathrm{e}^{\mathrm{i}\phi}\right)\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = 2r\mathrm{e}^{\mathrm{i}(\theta+\phi)}$



Thus, Q is obtained by scaling a factor r of the length OP, followed by an anti-clockwise rotation through an angle of θ radians about O.

Example 9

P and Q are 2 points on an Argand diagram such that $OQ = 2OP$ and $\angle QOP = \dfrac{\pi}{3}$ as shown in the diagram below. If P represents the complex number $2+\mathrm{i}$, find the complex number represented by Q in the form $a+\mathrm{i}b$, where the exact real values of a and b are to be found.



$|k| = 2|z|$
$= 2\sqrt{4+1} = 2\sqrt{5}$
$\arg(k) = \frac{\pi}{3} - \arg(z)$
$= -\left(\frac{\pi}{3} - \tan^{-1}\left(\frac{1}{2}\right)\right) = -0.5844$
$k = 2\sqrt{5}e^{-i0.584}$
$= 2\sqrt{5}(\cos(-0.584) + i\sin(-0.584))$
$= 3.734 - 2.47i$ ✓

Solution

If $OQ = 2OP$ and $\angle QOP = \dfrac{\pi}{3}$, we can obtain Q by a scaling of factor 2 of the length OP, followed by a clockwise rotation thru an angle of $\frac{\pi}{3}$ rad about O

So we have

$|P| = |2+i| = \sqrt{2^2+1} = \sqrt{5}$
$|Q| = |a+ib| = \sqrt{a^2+b^2}$
$\sqrt{a^2+b^2} = 2\sqrt{5}$
$a^2+b^2 = 20$ —— (1)
$\tan \arg(P) = \frac{1}{2}$

Hence

$\arg(P) = 0.46365$
$\arg(Q) = -\left(\frac{\pi}{3} - 0.46365\right) = -0.58354$
$\tan 0.58354 = \frac{-b}{a} = 0.52823$ —— (2)
Using GC, $b = -2.083$, $a = 3.957$
$\therefore Q = 3.96 - 2.08i$
$3.86 - 2.25i$

$q = p\left(2e^{-i\frac{\pi}{3}}\right)$

$q = (2+i)\left(2e^{-i\frac{\pi}{3}}\right)$
$= (2+i)(2)\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$
$= (2+i)(2)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$
$= (2+i)(1 - \sqrt{3}i)$
$= 2 - 2\sqrt{3}i + i - \sqrt{3}i^2$
$= 2 + (1-2\sqrt{3})i + \sqrt{3}$
$= 2+\sqrt{3} + (1-2\sqrt{3})i$ ✓

CONCLUSION

Basically, we've added another dimension to the real number system so that we can evaluate the square root of negative numbers. Complex numbers are represented geometrically on an Argand diagram. In the Argand diagram, complex numbers behave like vectors, but complex numbers are different from vectors, because vectors cannot be divided or raised to a power, but complex numbers can.

Complex numbers have essential concrete applications in a variety of sciences and related areas such as signal processing (creation of fractals), control theory, electromagnetism, quantum mechanics, cartography, vibration analysis, fluid dynamics, and many others.

If you are interested in the history of imaginary number i, you may want to refer to the book "An Imaginary Tale, the Story of $\sqrt{-1}$", by Paul J.Nahin, Princeton University Press.

APPENDIX

Euler's Formula

$$\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$$

Proof:

From the series expansions of e^{θ}, $\cos\theta$ and $\sin\theta$ (which has been covered in Chapter 6C and can be found in MF 26), we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta} &= 1+\mathrm{i}\theta+\frac{(\mathrm{i}\theta)^2}{2!}+\frac{(\mathrm{i}\theta)^3}{3!}+\frac{(\mathrm{i}\theta)^4}{4!}+\frac{(\mathrm{i}\theta)^5}{5!}+\ldots \\ &= 1+\mathrm{i}\theta-\frac{\theta^2}{2!}-\mathrm{i}\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\mathrm{i}\frac{\theta^5}{5!}+\ldots\end{aligned}$$

(using the result that if $k\in\mathbb{Z}^+$, $\mathrm{i}^{4k}=1$, $\mathrm{i}^{4k+1}=\mathrm{i}$, $\mathrm{i}^{4k+2}=-1$, $\mathrm{i}^{4k+3}=-\mathrm{i}$)

$$\begin{aligned}&= \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\mathrm{i}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right) \\ &= \cos\theta+\mathrm{i}\sin\theta\end{aligned}$$

Proof of result $r_1\mathrm{e}^{\mathrm{i}\theta_1}\times r_2\mathrm{e}^{\mathrm{i}\theta_2}=r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$.

Proof:

Let $z_1=r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2=r_2\mathrm{e}^{\mathrm{i}\theta_2}$, where $r_1,\ r_2>0$ and $-\pi<\theta_1,\ \theta_2\le\pi$.

Then

$$\begin{aligned} r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Therefore $z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

SUMMARY