National Junior College
2016 – 2017 H2 Further Mathematics
Topic: Further Special Discrete Probability Distributions **Tutorial Solutions**

1 **(i)** The mean number of emergency admissions to the hospital per minute is a constant.

Emergency admissions occur independently of one another throughout the entire day.

(ii) Let X denote the number of emergency admissions to the hospitals on a particular day. Then $X \sim \text{Po}(1.8)$.

Therefore, $\text{P}(X = 0) = 0.165$ (to 3 s.f.).

(iii) $\text{P}(X > 6) = 1 - \text{P}(X \le 6) = 0.00257$

(iv) Let Y be the number of emergency admissions in 2 consecutive days. Then $Y \sim \text{Po}(3.6)$.

Therefore, $\text{P}(Y \ge 3) = 1 - \text{P}(Y \le 2) = 0.697$ (to 3 s.f.).

2 **(i)** The mean number of typographical errors is constant for every page of the book.

Typographical errors occur independently of one another throughout every page of the book.

Let $\text{E}(X) = \lambda$. Then,

$$3\text{P}(X = 2) = 16\text{P}(X = 4) \Rightarrow 3\left(\frac{\cancel{e^{-\lambda}}\lambda^2}{2!}\right) = 16\left(\frac{\cancel{e^{-\lambda}}\lambda^4}{4!}\right)$$

$$\Rightarrow \frac{3\lambda^2}{2} = \frac{2\lambda^4}{3}$$

$$\Rightarrow \frac{9}{4} = \frac{\lambda^4}{\lambda^2} = \lambda^2$$

$$\Rightarrow \lambda = \frac{3}{2} = 1.5$$

(ii) $\text{P}(X = 0) = 0.223$ (to 3 s.f.).

(iii) Let Y be the number of errors in a section of the book. Then $Y \sim \text{Po}(6)$.

$$\begin{aligned} \text{P}(Y < 4) &= \text{P}(Y \le 3) \\ &= 0.15120 \\ &= 0.151 \text{ (to 3 s.f.).} \end{aligned}$$

(iv) Let W be the number of sections in the book with at least 4 errors each. Then $W \sim \text{B}(6, 1 - 0.15120) \Rightarrow W \sim \text{B}(6, 0.8488)$

$$\begin{aligned} \Rightarrow \text{P}(W \ge 4) &= 1 - \text{P}(W \le 3) \\ &= 0.952 \text{ (to 3 s.f.).} \end{aligned}$$

3 **(a)** **(i)** Let X denote the random variable for the number of demands per hour for a court in this sports hall on a weekend. Then $X \sim \text{Po}(7.2)$.

P(courts are fully booked on a particular time slot on a Saturday)

$= \text{P}(X \geq 6)$

$= 1 - \text{P}(X \leq 5)$

$= 0.7241025$

$= 0.724$ (to 3 s.f.)

(ii) Let Y denote the random variable for the number of hours on an entire weekend for which the courts are fully booked. Then $Y \sim \text{B}(30,\ 0.7241025)$.

P(the courts are fully booked on the entire weekend)

$= \text{P}(Y \geq 20)$

$= 1 - \text{P}(Y \leq 19)$

$= 0.81965$

$= 0.820$ (to 3 s.f.)

(b) **(i)** The likelihood that any one of the courts is booked is raised when another court has already been booked, since there are less courts now to fulfil any outstanding demands. Therefore, the event that each court is booked does not occur independently of one another, so a binomial model would probably not be valid.

[Note that any explanation that supports a claim that each court is not equally likely to be booked will not be accepted, as it is already given in the question that members have no preference between any one of the 6 courts, hence it is implicitly implied that the probability that each of the six courts is booked must be a constant.]

(ii) As people have to work during weekdays, the average number of demands on the weekdays will be fewer than that on the weekends. Hence the mean number of demands for each day is not likely to stay constant across an entire week, so a Poisson model would probably not be valid.

4 **(i)** $$\frac{P(X=x+1)}{P(X=x)}=\frac{\left[\frac{\cancel{e^{-\lambda}}\lambda^{x+1}}{(x+1)!}\right]}{\left[\frac{\cancel{e^{-\lambda}}\lambda^{x}}{x!}\right]}=\frac{\lambda^{x+1}}{x!(x+1)}\times\frac{x!}{\lambda^x}=\frac{\lambda}{x+1}$$

(ii) Since m is a mode of X, $P(X=m)\geq P(X=m-1)$ and $P(X=m)\geq P(X=m+1)$. Hence

$$P(X=m)\geq P(X=m-1)\Rightarrow\frac{P(X=m)}{P(X=m-1)}\geq 1$$
$$\Rightarrow\frac{\lambda}{(m-1)+1}\geq 1 \text{ (from the result in part (i))}$$
$$\Rightarrow\frac{\lambda}{m}\geq 1 \text{ (from the result in part (i))}$$
$$\Rightarrow m\leq\lambda$$

and

$$P(X=m)\geq P(X=m+1)\Rightarrow\frac{P(X=m+1)}{P(X=m)}\leq 1$$
$$\Rightarrow\frac{\lambda}{m+1}\leq 1 \text{ (from the result in part (i))}$$
$$\Rightarrow m+1\geq\lambda$$
$$\Rightarrow m\geq\lambda-1$$

Therefore, $\lambda-1\leq m\leq\lambda.$

However, since λ is not an integer, it is not a possible value for the random variable X (which is integer-valued). Hence $m\neq\lambda$ and $m\neq\lambda-1,$ so $\lambda-1<m<\lambda$ (shown).

(iii) Suppose λ is an integer and there is only one mode. Then,

$$P(X=m)>P(X=m-1) \text{ and } P(X=m)>P(X=m+1).$$

This would imply that $\lambda-1<m<\lambda.$ However, as λ is an integer, there is no integer that lies between $\lambda-1$ and λ (exclusive), which contradicts the assumption that there is only one mode. Therefore there must be more than one mode.

Following from an argument similar to that in part **(ii)**, if λ is an integer, then $\lambda-1\leq m\leq\lambda.$ Hence, both $\lambda-1$ and λ are modes of X in this case.

5 **(i)** Let X denote the number of black dots that appear in region R. Then $X \sim \text{Po}\left(\frac{r\lambda}{s}\right)$, since the number of black dots appear at a constant average rate throughout the square frame.

Therefore, $\text{P}(X = x) = \frac{\text{e}^{-\frac{r\lambda}{s}}}{x!}\left(\frac{r\lambda}{s}\right)^x$.

(ii) Let Y and W denote the number of black dots that appear inside the square frame and the number of black dots that appear inside the square frame but outside region R respectively. Then

$$Y \sim \text{Po}(\lambda) \text{ and } W \sim \text{Po}\left(\frac{(s-r)\lambda}{s}\right).$$

$$\begin{aligned}
\text{P}(X = x \mid Y = n) &= \frac{\text{P}(X = x \cap Y = n)}{\text{P}(Y = n)} = \frac{\text{P}(X = x \cap W = n-x)}{\text{P}(Y = n)} = \frac{\text{P}(X = x)\times\text{P}(W = n-x)}{\text{P}(Y = n)} \\
&= \frac{\dfrac{\text{e}^{-\frac{r\lambda}{s}}}{x!}\left(\dfrac{r\lambda}{s}\right)^x \times \dfrac{\text{e}^{-\frac{(s-r)\lambda}{s}}}{(n-x)!}\left(\dfrac{(s-r)\lambda}{s}\right)^{n-x}}{\dfrac{\text{e}^{-\lambda}\lambda^n}{n!}} \\
&= \frac{\text{e}^{-\frac{r\lambda}{s}}\times\text{e}^{-\frac{(s-r)\lambda}{s}}}{\text{e}^{-\lambda}} \times \frac{n!}{x!(n-x)!} \times \frac{\lambda^s \times \lambda^{n-s}}{\lambda^n} \times \left(\frac{r}{s}\right)^x \times \left(1-\frac{r}{s}\right)^{n-x} \\
&= 1\cdot\binom{n}{x}\cdot 1\cdot\left(\frac{r}{s}\right)^x\left(1-\frac{r}{s}\right)^{n-x} = \binom{n}{x}\left(\frac{r}{s}\right)^x\left(1-\frac{r}{s}\right)^{n-x},
\end{aligned}$$

for $x = 0, 1, 2, \ldots, n$ (since $0 \le x \le n$).

Therefore the conditional distribution of X given $Y = n$ is $\text{B}\left(n, \frac{r}{s}\right)$ (shown).

Let $p = \frac{r}{s}$. Then

$$\begin{aligned}
\text{P}(X = 2 \mid Y = 4) = 0.0486 &\Rightarrow \binom{4}{2}p^2(1-p)^2 = 0.0486 \\
&\Rightarrow p^2(1-p)^2 = \frac{0.0486}{6} = 0.0081 \\
&\Rightarrow p(1-p) = \sqrt{0.0081} = 0.09 \\
&\Rightarrow p^2 - p + 0.09 = 0 \\
&\Rightarrow p = \frac{1 \pm \sqrt{(-1)^2 - 4(0.09)}}{2} = \frac{1 \pm \sqrt{0.64}}{2} = \frac{1 \pm 0.8}{2} = 0.1 \text{ or } 0.9
\end{aligned}$$

Hence the possible proportions of the area of the square frame taken up by region R is either $\frac{1}{10}$ or $\frac{9}{10}$.

6 **(i)** Note that A's n^{th} shot is the $(2n-1)^{\text{th}}$ shot of the game.

$$\text{P(A destroys his phone on his } n^{\text{th}} \text{ shot)} = \left(\frac{5}{6}\right)^{2n-2}\left(\frac{1}{6}\right)$$
$$= \frac{6}{25}\left(\frac{25}{36}\right)^n$$

(ii) $$\text{P(A loses the game)} = \sum_{n=1}^{\infty} \frac{6}{25}\left(\frac{25}{36}\right)^n$$
$$= \frac{\frac{6}{25}\left(\frac{25}{36}\right)}{1-\frac{25}{36}} = \frac{6}{11}$$

(iii) $$\text{P(B loses the game)} = 1 - \text{P(A loses the game)}$$
$$= 1 - \frac{6}{11} = \frac{5}{11}$$

(iv) Note that this is equivalent to finding the expected number of shots taken if A plays the game on his own, i.e. A and B are the same person. The problem then becomes finding the expectation of a geometric random variable with probability of success $\frac{1}{6}$, which is 6.

7 Suppose X is a discrete random variable with the memoryless property, i.e.

$$\text{P}(X > m+n \mid X > n) = \text{P}(X > m).$$

Rewriting, we have $\dfrac{\text{P}\left(X > m+n \cap X > n\right)}{\text{P}\left(X > n\right)} = \text{P}(X > m)$

$$\frac{\text{P}\left(X > m+n\right)}{\text{P}\left(X > n\right)} = \text{P}(X > m)$$
$$\text{P}\left(X > m+n\right) = \text{P}(X > m)\text{P}\left(X > n\right)$$

In particular, we note that $\text{P}\left(X > n\right) = \text{P}(X > n-1)\cdot\text{P}\left(X > 1\right)$

$$= \text{P}(X > n-2)\cdot\text{P}\left(X > 1\right)\cdot\text{P}\left(X > 1\right)$$
$$= \cdots$$
$$= \text{P}(X > 1)^n$$

Therefore, $\text{P}\left(X = n\right) = \text{P}\left(X > n-1\right) - \text{P}\left(X > n\right)$

$$= \text{P}(X > 1)^{n-1} - \text{P}(X > 1)^n$$
$$= \text{P}(X > 1)^{n-1}\left[1 - \text{P}(X > 1)\right]$$
$$= (1-p)^{n-1}p, \text{ where } p = \text{P}(X = 1).$$

Hence, X is geometric. (shown)

8 **(a)** Let N denote the number of days taken for the light bulb to fail. Then $N \sim \text{Geo}\left(0.05\right)$.

$$\begin{aligned}\mathrm{P}(N=n \mid N>n-1) &= \frac{\mathrm{P}(N=n \cap N>n-1)}{\mathrm{P}(N>n-1)} \\ &= \frac{\mathrm{P}(N=n)}{\mathrm{P}(N>n-1)} \\ &= \frac{0.05(1-0.05)^{n-1}}{(1-0.05)^{n-1}} = 0.05\end{aligned}$$

Therefore, $\mathrm{P}(N=n \mid N>n-1)$ is independent of n.

(b) $(\Rightarrow)$ Suppose X is geometric, say with parameter p. Then its rate function is given by

$$\begin{aligned}\mathrm{r}(x) &= \frac{\mathrm{P}(X=x)}{\mathrm{P}(X>x-1)} \\ &= \frac{p(1-p)^{n-1}}{(1-p)^{n-1}} = p\end{aligned}$$

Therefore, its rate function is constant.

$(\Leftarrow)$ Suppose X has a constant rate function, say $\mathrm{r}(x)=r$. Note that

$$\begin{aligned}\frac{\mathrm{P}(X=x)}{\mathrm{P}(X>x-1)} = r &\Rightarrow \mathrm{P}(X=x) = r\mathrm{P}(X>x-1) \\ &\Rightarrow \mathrm{P}(X>x-1) - \mathrm{P}(X>x) = r\mathrm{P}(X>x-1) \\ &\Rightarrow \mathrm{P}(X>x) = (1-r)\mathrm{P}(X>x-1) \\ &\qquad = (1-r)(1-r)\mathrm{P}(X>x-2) \\ &\qquad = \cdots \\ &\qquad = \underbrace{(1-r)(1-r)\cdots(1-r)}_{x-1 \text{ times}}\mathrm{P}(X>1) \\ &\qquad = (1-r)^{x-1}\left(1-\mathrm{P}(X=1)\right) \\ &\qquad = (1-r)^{x-1}\left(\frac{1-\mathrm{P}(X=1)}{\mathrm{P}(X>0)}\right) \text{ (since } \mathrm{P}(X>0)=1) \\ &\qquad = (1-r)^{x-1}(1-r) \\ &\qquad = (1-r)^{x}\end{aligned}$$

$$\begin{aligned}\text{Hence, } \mathrm{P}\left(X=x\right) &= \mathrm{P}(X>x-1) - \mathrm{P}(X>x) \\ &= \left(1-r\right)^{x-1} - \left(1-r\right)^{x} \\ &= \left(1-r\right)^{x-1}\left[1-\left(1-r\right)\right] \\ &= r\left(1-r\right)^{x-1}\end{aligned}$$

Therefore, X is geometric with parameter r (proven).

9 **(a)** If the r^{th} success occurs on the y^{th} trial, then out of the first $y-1$ trials, the event of success occurs in exactly $r-1$ of the trials. Therefore,

$$\mathrm{P}\left(Y=y\right)=\underbrace{\binom{y-1}{r-1}}_{\substack{\text{Select } r-1 \text{ trials} \\ \text{to be successful}}}\underbrace{p^{r-1}}_{r-1\text{ successes}}\underbrace{(1-p)^{(y-1)-(r-1)}}_{(y-1)-(r-1)\text{ failures}}\cdot\underbrace{p}_{\substack{\text{success on} \\ \text{the } y^{\text{th}} \text{ trial}}}$$

$$=\binom{y-1}{r-1}p^{r}(1-p)^{y-r},\ \text{for } y=r,r+1,r+2,\ldots$$

(b) Note that $Y=X_1+X_2+\cdots+X_r$, where $X_1,X_2,\ \ldots$ and X_r are independent geometric random variables with parameter p. Therefore,

$$\begin{aligned}\mathrm{E}(Y)&=\mathrm{E}\left(X_1+X_2+\cdots+X_r\right)\\&=\mathrm{E}\left(X_1\right)+\mathrm{E}\left(X_2\right)+\cdots+\mathrm{E}\left(X_r\right)\\&=\frac{r}{p}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(Y)&=\mathrm{Var}\left(X_1+X_2+\cdots+X_r\right)\\&=\mathrm{Var}\left(X_1\right)+\mathrm{Var}\left(X_2\right)+\cdots+\mathrm{Var}\left(X_r\right)\\&=\frac{r\left(1-p\right)}{p^2}\end{aligned}$$

Further Practice Questions

1 $Y \sim \text{Po}(\lambda)$

(i) $\text{P}(Y=0)=0.25$

$\Rightarrow \dfrac{\text{e}^{-\lambda}\lambda^0}{0!}=0.25$

$\Rightarrow \text{e}^{-\lambda}=0.25$

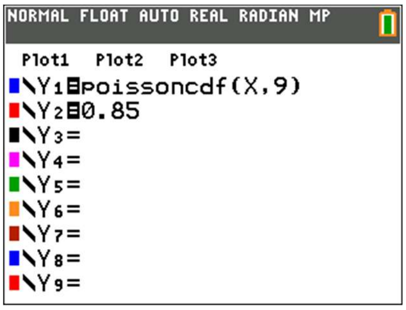
$\Rightarrow -\lambda=\ln 0.25$

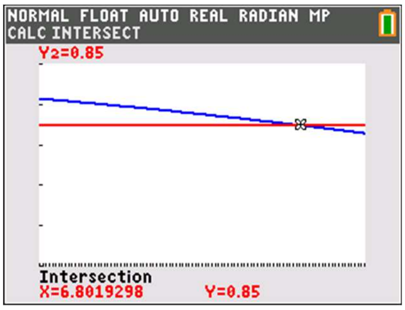
$\therefore \lambda=1.39$ (to 3 s.f.)

(ii) $\text{P}(Y<10)=0.850$

$\Rightarrow \text{P}(Y\le 9)=0.850$

Graph Y_1 and Y_2 and adjust the window setting or use Zoom box,





Solving for intersection of Y_1 and Y_2, $\lambda=6.80$ (to 3 s.f.)

2 Let X be the number of times a radio set breaks down in a year. Then $X \sim \text{Po}(1)$.

(a) $\text{P}(X>2)=1-\text{P}(X\le 2)=0.0803$ (to 3 s.f.)

(b) Let Y be the number of times a radio set breaks down in 3 years. Then $Y \sim \text{Po}(3)$.

$\text{P}(Y=0)=\text{e}^{-3}=0.0498$ (to 3 s.f.)

(c) Let R be the number of times a radio set breaks down in one week. Then $R \sim \text{Po}\left(\dfrac{1}{52}\right)$.

$\text{P}(R=2)=0.000181$ (to 3 s.f.)

3 Let X be the number of letters the man receives in the post per day. Then $X \sim \text{Po}(1.5)$.

(i) $\text{P}(X = 0) = \text{e}^{-1.5} = 0.223$ (to 3 s.f.)

(ii) $\text{P}(X > 3) = 1 - \text{P}(X \le 3) = 0.0656$ (to 3 s.f.)

Let Y be the number of letters the man receives in a period of 3 days. Then $Y \sim \text{Po}(4.5)$.
P(the man receives 3 letters on Sunday) = $\text{P}(Y = 3) = 0.169$ (to 3 s.f.).

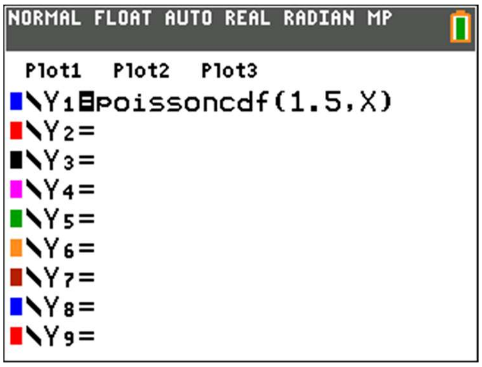
4 **(i)** Let X denote the number of demands for vans in a day. Then $X \sim \text{Po}(1.5)$.
$\text{P}(X \ge 2) = 1 - \text{P}(X \le 1) = 0.44217 = 0.442$ (to 3 s.f.)

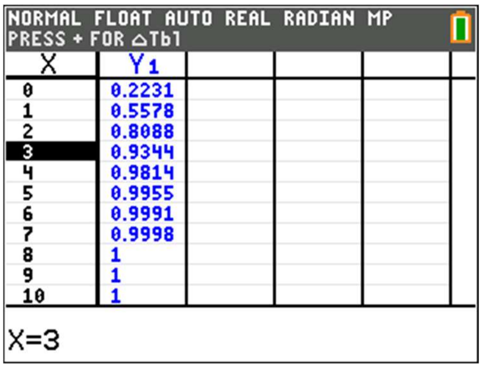
(ii) Let R denote the no. of days on which both vans are in use. Then $R \sim \text{B}(5, 0.44217)$.
$\text{P}(R \ge 2) = 1 - \text{P}(R \le 1) = 0.73191 = 0.732$ (to 3 s.f.)

(iii) $\text{P}(X > n) < 0.1$

$1 - \text{P}(X \le n) < 0.1$

$\text{P}(X \le n) > 0.9$





From the GC, $\text{P}(X \le 2) = 0.809 < 0.9$

$\text{P}(X \le 3) = 0.934 > 0.9$

Therefore least value of $n = 3$.

(iv) Let Y denote the number of demands for cars in a day. Then

$$Y \sim \text{Po}(2) \text{ and } X + Y \sim \text{Po}(3.5).$$

$\text{P}(\text{no demand for car} \mid X + Y = 6)$

$= \dfrac{\text{P}(X = 6 \text{ and } Y = 0)}{P(X + Y = 6)} = 0.00620$ (to 3 s.f.)

(v) $\text{P}(X \ge 1 \text{ and } Y \ge 1 \mid X + Y = 4)$

$= \dfrac{[\text{P}(X = 1 \text{ and } Y = 3] + [\text{P}(X = 2 \text{ and } Y = 2)] + [\text{P}(X = 3 \text{ and } Y = 1)]}{\text{P}(X + Y = 4)}$

$= 0.860$ (to 3 s.f.)