RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Chapter 4A: Complex Numbers I

SYLLABUS INCLUDES

- extension of the number system from real numbers to complex numbers
- complex roots of quadratic equations
- four operations of complex numbers expressed in the form $(x + iy)$
- equating real parts and imaginary parts
- conjugate roots of a polynomial equation with real coefficients
- representation of complex numbers in the Argand diagram
- calculation of modulus (r) and argument (θ) of a complex number

PRE-REQUISITES

- Trigonometry
- Coordinate Geometry
- Vectors

CONTENT

Introduction to the Imaginary Number i

We know that the solution to the equation $x^2+1=0$ cannot be a real number, as the square of a real number cannot be negative. We say $x^2=-1$ has no real roots.

In order to solve the above equation, we need to find a "number" whose square is -1.

Let's suppose such a "number" exists.

Since we imagined it, let's call this number the **imaginary number** i.

We define i as

$$\boxed{\mathrm{i}=\sqrt{-1}}$$

Hence the solutions to $x^2+1=0$ are $x=\pm\sqrt{-1}=\pm\mathrm{i}$.

Example 1

(a) If $\mathrm{i}=\sqrt{-1}$, simplify i^2, i^3, i^4, i^{2009}, i^{2010}, i^{2011}, i^{2012}.

Solution

$i = i$
$i^2 = -1$
$i^3 = (i)(i^2) = -i$
$i^4 = (i^2)(i^2) = 1$

there's a pattern of 4

$i^{2009} = i \Rightarrow [i^4]^{502}(i)$
$i^{2010} = -1$
$i^{2011} = -i$
$i^{2012} = 1$

Let's generalize: If k is a positive integer, then

$i^{4k}=1$, $i^{4k+1}=i$, $i^{4k+2}=-1$, $i^{4k+3}=-i$

(b) Perform the four basic operations on i:

(i) $\mathrm{i}+\mathrm{i}=$ $2i$

(ii) $5\mathrm{i}-\mathrm{i}=$ $4i$

(iii) $5\mathrm{i}\times 3\mathrm{i}=$ $15(i)^2$
$= -15$

(iv) $6\mathrm{i}\div 3\mathrm{i}=$ 2

(c) Solve for x if $x^2-2x+2=0$.

factorise, complete the square, quadratic formula

Solution

$x^2-2x+2=0$ complete the square

$x^2-2x+1=-1 \Leftrightarrow (x-1)^2=-1$

$x-1 = \cancel{\pm\sqrt{-1}} = \pm i$

$x = \cancel{1\pm\sqrt{-1}}, 1\pm i$

$\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{2\pm\sqrt{4-8}}{2}$

$= \dfrac{2\pm 2i}{2}$

$= 1\pm i$

2 COMPLEX NUMBERS

Notice that the solution to Example 1(c) is a combination of real numbers and imaginary numbers. Such numbers are called complex numbers.

2.1 Definition of a Complex Number

A **complex number** is of the form $x+\mathrm{i}y$ where x and y are real numbers and $\mathrm{i}=\sqrt{-1}$.

The set of complex numbers is denoted by $\mathbb{C}=\{z: z=x+\mathrm{i}y,\ x,\ y\in\mathbb{R}\}$.

x is known as the real part of z, denoted by $\mathrm{Re}(z)$, and

y is known as the imaginary part of z, denoted by $\mathrm{Im}(z)$.

coefficient!

Note that $\mathrm{Im}(z)$ **does not include** i.

$x+\mathrm{i}y$ is known as the **cartesian form** of the complex number z.

Example 2

Write down the real and imaginary parts of the following complex numbers:

z	Re(z)	Im(z)
$-2+3\mathrm{i}$	-2	3
$1-\mathrm{i}$	1	-1
-4	-4	0
$5\mathrm{i}$	0	5

Remarks:

- $\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}\subseteq\mathbb{C}$
- If $y=0$, then $z=x$ is a real number.
- If $x=0$, then $z=\mathrm{i}y$ is a purely imaginary number.
- Complex numbers cannot be ordered, i.e. given any two complex numbers z_1 and z_2, we cannot compare whether $z_1<z_2$ unless they are real numbers.

2.2 Operations on Complex Numbers

In this section, let $a,\ b,\ c,\ d \in \mathbb{R}$ and $z_1,\ z_2,\ z_3 \in \mathbb{C}$.

(a) Equality of Two Complex Numbers

2 complex numbers are equal if and only if their corresponding real and imaginary parts are equal, i.e. $a + \mathrm{i}b = c + \mathrm{i}d \Leftrightarrow a = c$ and $b = d$.

For example, if $x + \mathrm{i}y = 5 - 3\mathrm{i}$, we have $x = 5$ and $y = -3$

(b) Addition of Complex Numbers

$$(a+\mathrm{i}b)+(c+\mathrm{i}d)=(a+c)+\mathrm{i}(b+d)$$

Addition of complex numbers is commutative: $z_1 + z_2 = z_2 + z_1$

Addition of complex numbers is associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

For example, $(3+4\mathrm{i})+(1-\mathrm{i}) =$ $4+3\mathrm{i}$

(c) Subtraction of Complex Numbers

$$(a+\mathrm{i}b)-(c+\mathrm{i}d)=(a-c)+\mathrm{i}(b-d)$$

For example, $(3+4\mathrm{i})-(1-\mathrm{i}) =$ $2+5\mathrm{i}$

(d) Multiplication of Complex Numbers

$$(a+\mathrm{i}b)(c+\mathrm{i}d) = ac+\mathrm{i}ad+\mathrm{i}bc+\mathrm{i}^2bd = ac+\mathrm{i}ad+\mathrm{i}bc+(-1)bd$$
$$=(ac-bd)+\mathrm{i}(ad+bc)$$

Multiplication of complex numbers is commutative: $z_1 z_2 = z_2 z_1$

Multiplication of complex numbers is associative: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Multiplication of complex numbers is distributive over addition: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

For example, $(3+4\mathrm{i})(1-\mathrm{i}) =$ $3-4\mathrm{i}^2+\mathrm{i} = 7+\mathrm{i}$ (note that $\mathrm{i}^2 = -1$)

$(3+4\mathrm{i})^2 = (3+4\mathrm{i})(3+4\mathrm{i}) =$ $9+16\mathrm{i}^2+24\mathrm{i} = -7+24\mathrm{i}$

Remarks: The above manipulations of complex numbers are the same as algebraic manipulations of expressions such as $(a+b)(c+d), (a+b)\pm(c+d)$ and so on. The only additional consideration is $\mathrm{i}^2 = -1$.

We can use the GC to perform operations on complex numbers

Operation	Example	GC Screen
Multiplication of complex numbers (Press [2nd][.] to get i)	$(3+4\mathrm{i})(1-\mathrm{i})$	(3+4i)(1-i) 7+i
Square of a complex number	$(3+4\mathrm{i})^2$	(3+4i)² -7+24i
Division of complex numbers (Only works in Classic, not in MathPrint)	$\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$	(3+4i)/(1-i) -.5+3.5i $\frac{3+4i}{1-i}$ Error

How did the GC obtain $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}=-0.5+3.5\mathrm{i}$?

(e) Division of Complex Numbers

Recall when we tried to simplify $\dfrac{2-\sqrt{3}}{1+\sqrt{2}}$, we multiply it by $\dfrac{1-\sqrt{2}}{1-\sqrt{2}}$ so that we could **rationalize the denominator.**

For $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$, we will multiply it by $\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$, so that

$$\frac{3+4\mathrm{i}}{1-\mathrm{i}} = \frac{(1+i)(3+4i)}{1-i^2} = \frac{3+4i^2+7i}{2} = \frac{7i-1}{2}$$

In general, $\dfrac{a+\mathrm{i}b}{c+\mathrm{i}d}=\dfrac{a+\mathrm{i}b}{c+\mathrm{i}d}\times\dfrac{c-\mathrm{i}d}{c-\mathrm{i}d}=\dfrac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2-\mathrm{i}^2d^2}=\dfrac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2+d^2}$

Remark: Note that $c-\mathrm{i}d$ is chosen based on the denominator of $\dfrac{a+\mathrm{i}b}{c+\mathrm{i}d}$.

We call $c-\mathrm{i}d$, the conjugate of $c+\mathrm{i}d$.

Example 3

Find the square roots of 24 – 10i.
By completing the square, or otherwise, solve $z^2 - 6z = 15 - 10i$.

Solution:

let the square roots of 24 – 10i be $x + yi$, $x, y \in \mathbb{R}$

square both sides $24 - 10i = (x+yi)^2$

$= x^2 + y^2 i^2 + 2xyi$

$= x^2 - y^2 + 2xyi$

comparing real and imaginary parts,

$24 = x^2 - y^2$ — (1)

$-10 = 2xy$ — (2)

from (2):

$y = \frac{-5}{x}$ — (3)

sub (3) in (1): $24 = x^2 - \left(\frac{-5}{x}\right)^2$

$= x^2 - \frac{25}{x^2}$

$x^4 - 24x^2 - 25 = 0$

$(x^2 - 25)(x^2 + 1) = 0$

$x = 5, -5, i$ or $-i$
(NA) (NA)
since x is real

when $x = 5$, when $x = -5$,
$y = -1$ $y = 1$

∴ the square roots of $24 - 10i$ are $5 - i$ or $i - 5$ //

GC only gives one root, other root is the negation of it

$z^2 - 6z = 15 - 10i$

$\Rightarrow (z-3)^2 - 3^2 = 15 - 10i$

$\Rightarrow (z-3)^2 = 24 - 10i$

$\Rightarrow z - 3 = \sqrt{24 - 10i}$

$= \pm(5 - i)$ (from above result)

$z = 3 + 5 - i$ or $z = 3 - 5 + i$
$= 8 - i$ $= i - 2$ //

use gc to check answers: sub $z = 8 - i$ and $z = -2 + i$ into $z^2 - 6z$, get $15 - 10i$

2.3 Complex Conjugates

The **complex conjugate** of $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, is denoted by z^* and defined as

$$z^* = x - \mathrm{i}y$$

Observe that $\mathrm{Re}(z^*) = x = \mathrm{Re}(z)$ and $\mathrm{Im}(z^*) = -y = -\mathrm{Im}(z)$.
Note that $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$ are conjugates of each other, and we call them a conjugate pair.

2.4 Some Properties of Complex Conjugates

The following properties can be easily proven by letting $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ where $x, y, u, v \in \mathbb{R}$.

Properties	Proofs	Example
(a) $(z^*)^* = z$	$[(x+\mathrm{i}y)^*]^* = (x-\mathrm{i}y)^* = x+\mathrm{i}y$	$[(1-3i)^*]^*$ $= (1+3i)^* = 1-3i$
(b) $z + z^* = 2\,\mathrm{Re}(z)$	$(x+\mathrm{i}y)+(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $=2x$ $= 2\mathrm{Re}(z)$	$(1-3i)+(1-3i)^*$ $=(1-3i)+(1+3i)$ $= 2$ $= 2\mathrm{Re}(z)$
(c) $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	$(x+\mathrm{i}y)-(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $=2\mathrm{i}y$	$(1-3i)-(1-3i)^*$ $=(1-3i)-(1+3i)$ $= -6i$
(d) $zz^* = x^2 + y^2$	$(x+\mathrm{i}y)(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)(x-\mathrm{i}y)$ $=x^2-(\mathrm{i}y)^2$ (difference of 2 squares) $=x^2-\mathrm{i}^2y^2$ $(\mathrm{i}^2=-1)$ $=x^2+y^2$	$(1-3i)(1-3i)^*$ $=(1-3i)(1+3i)$ $= 1-9i^2$ $= 10$ $=$
(e) $z = z^* \Leftrightarrow z \in \mathbb{R}$	$z = z^*$ $\Leftrightarrow x+\mathrm{i}y = x-\mathrm{i}y$ $\Leftrightarrow 2\mathrm{i}y = 0$ $\Leftrightarrow \mathrm{Im}(z) = 0$ $\Leftrightarrow z$ is real	$(1-3i) = (1-3i)^*$ $(1+3i)$ $0 = 6i$ $\mathrm{Im}(z) = 0$ $\therefore z$ is real
(f) $(z+w)^* = z^* + w^*$	$(x+\mathrm{i}y+u+\mathrm{i}v)^* = (x+u+\mathrm{i}(y+v))^*$ $= x+u-\mathrm{i}(y+v)$ $=(x-\mathrm{i}y)+(u-\mathrm{i}v)$ $=z^*+w^*$	

Properties	Proofs	Example
(g) $(zw)^* = z^* w^*$ $(z^2)^* = z^* z^* = (z^*)^2$ $(z^n)^* = (z^*)^n$ $n \in \mathbb{Z}$	$((x+iy)(u+iv))^* = (xu - yv + i(xv+yu))^*$ $= xu - yv - i(xv + yu)$ $(x+iy)^*(u+iv)^* = (x-iy)(u-iv)$ $= xu - yv - i(xv + yu)$	$((1+3i)(1-2i))^*$ $= (1+i-6i^2)^*$ $= (7+i)^* = 7-i$ $(1+3i)^*(1-2i)^*$ $= (1-3i)(1+2i)$ $= 1-i-6i^2$ $= 7-i$

Example 4 [RJC Prelim 9233/2005/01/Q1(i)]

The complex numbers z and w are such that $z = -1 + 2i$ and $w = 1 + bi$, where $b \in \mathbb{R}$. Given that the imaginary part of $\frac{w}{z}$ is $-\frac{3}{5}$, find the value of b.

Solution

$\frac{w}{z} = \frac{1+bi}{-1+2i} \times \frac{-1-2i}{-1-2i}$

$= \frac{-1-2i-bi-2bi^2}{1-4i^2}$

$= \frac{-1-(2+b)i+2b}{5}$

$= \frac{2b-1}{5} - \frac{(2+b)i}{5}$

$\frac{-(2+b)}{5} = \frac{3}{5}$

$b = 1$

Given $\text{Im}\left(\frac{w}{z}\right) = -\frac{3}{5} \Rightarrow \frac{-2-b}{5} = -\frac{3}{5} \Rightarrow b = 1$

Example 5

Solve the simultaneous equations $z + w = -1$, $2z - iw = -1$.

Solution

$z + w = -1$ ----- (1) $\quad 2z - iw = -1$ ----- (2)

① × i + ② gives $(i+2)z = -i-1 \Rightarrow z = \frac{-i-1}{i+2} \times \frac{i-2}{i-2}$

$= \frac{-i^2+2i-i+2}{i^2-4}$

$= \frac{-i^2+i+3}{-5}$ $\quad z = -\frac{3}{5} - \frac{1}{5}i$

$= \frac{4+i}{-5}$

$w = -1-z$

$= \frac{5}{-5} - \frac{4+i}{-5} = \frac{1+i}{-5}$

$w = \frac{2}{5} + \frac{1}{5}i$

Example 6

(a) Let $z = 1 + ia$ and $w = 1 + ib$, where $a, b \in \mathbb{R}$ and $a > 0$. If $zw^* = 3 - 4i$, find the exact values of a and b.

(b) Without the use of GC, find $z = c + id$, where $c, d \in \mathbb{R}$ such that $z^2 = -8 - 6i$.

Solution

(a) $zw^* = (1+ia)(1-ib)$

$= 1 - i^2ab + ia - ib$

$= 1+ab + (a-b)i = 3-4i$

$\therefore ab+1 = 3$

$ab = 2$

$a-b = -4$

$b = a+4$

$\therefore (a)(a+4) = 2$

$a^2+4a-2=0$

$(a+2)^2-6=0$

$(a+2)^2 = 6$

$a = -2+\sqrt{6}$ or $-2-\sqrt{6}$ (NA, $a>0$)

$b = 4-2+\sqrt{6}$

$= 2+\sqrt{6}$ //

(b) $z^2 = (c+id)^2 = (c^2-d^2)+2cdi = -8-6i$

Equating real and imaginary parts, $c^2 - d^2 = -8$ ----- (1)

$2cd = -6 \Rightarrow d = -\dfrac{3}{c}$ ----- (2)

Subst (2) into (1), $c^2 - \left(-\dfrac{3}{c}\right)^2 = -8$

$c^4 + 8c^2 - 9 = 0$

$(c^2-1)(c^2+9) = 0$

$c^2 = 1$ or $c^2 = -9$ (NA since $c \in \mathbb{R}$ hence $c^2 \ge 0$)

$c = \pm 1$

same as example 3

When $c = 1$, $d = -3$; When $c = -1$, $d = 3$

Hence $z = 1 - 3i$ or $z = -1 + 3i$

3 Complex Roots of Polynomial Equations

With complex numbers, we have the following theorem.

Fundamental Theorem of Algebra:

A polynomial equation of degree n has n roots (real or non-real).

Thus, taking non-real roots into account, a quadratic (degree 2) equation always has 2 roots, a cubic (degree 3) equation always has 3 roots, and so on.

Furthermore, if the coefficients of the polynomial equation are real, we have the following result:

Non-real roots of a polynomial equation with <u>real coefficients</u> occur in conjugate pairs.

In other words, if β is a non-real root of a polynomial equation with <u>real coefficients</u>, then β^* is also a non-real root of the equation.

Note that real coefficients include the constant term as well.

[Refer to Appendix for the proof of this result]

From **Example 1(c)** we obtained $x = 1 \pm \mathrm{i}$ as conjugate pair solutions to $x^2 - 2x + 2 = 0$.

Example 7 (Quadratic)
Find the roots of the equation $z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$.

Solution

$z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$

$z = \dfrac{-(-1+4i) \pm \sqrt{(-1+4i)^2 - 4(-5+i)}}{2}$

$= \dfrac{(1-4i) \pm \sqrt{5+12i}}{2}$

$= \dfrac{(1-4i) \pm (3-2i)}{2}$

$= \dfrac{4-6i}{2}$ or $-1-i$ //

$= 2-3i$ //

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

(evaluate $(-1+4i)^2$ using GC

$\sqrt{5-12i}$ evaluate using GC

$3-2i$ or $-3+2i$

Question: Why are the roots not in conjugate pairs?
Answer: because not all the coefficients of the quadratic equation are real

Example 8 (Cubic)

Find the exact roots of the equation $z^3 - 2z^2 + 2z - 1 = 0$.

Solution

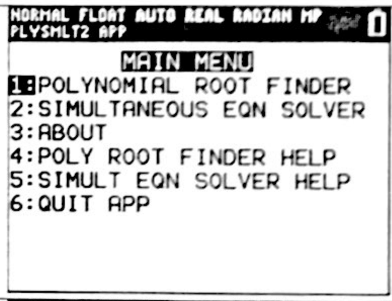
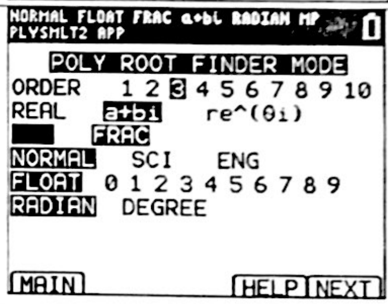
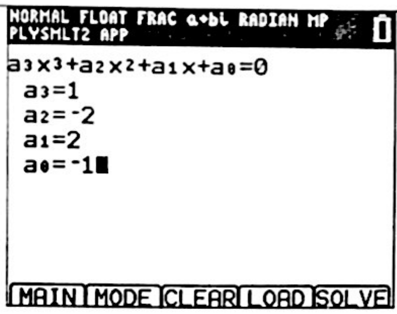
Since the cubic equation has real coefficients, it has either 3 real roots or 1 real root and 1 conjugate pair of non-real roots.

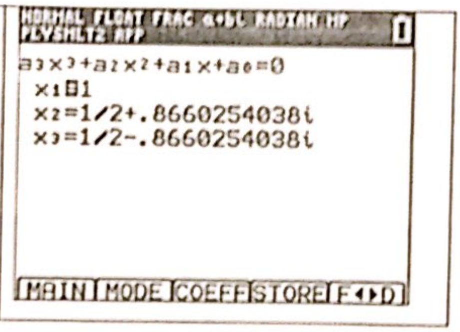
$z = 1$ is clearly a solution of $z^3 - 2z^2 + 2z - 1 = 0$.

Observe that $z^3 - 2z^2 + 2z - 1 = (z-1)(z^2 - z + 1)$.

Hence
$$\begin{aligned} z^3 - 2z^2 + 2z - 1 = 0 &\Rightarrow (z-1)(z^2 - z + 1) = 0 \\ &\Rightarrow z - 1 = 0 \text{ or } z^2 - z + 1 = 0 \\ &\Rightarrow z = 1 \text{ or } z = \frac{1 \pm \sqrt{1-4}}{2} \\ \text{i.e. } z = 1 \text{ or } z = \frac{1 + \mathrm{i}\sqrt{3}}{2} &\text{ or } z = \frac{1 - \mathrm{i}\sqrt{3}}{2} \end{aligned}$$

Note that the **Polynomial Root Finder** of the **PlySmlt2** app in the GC can be used to solve equations with real coefficients.

1. Press **[APPS]** and select **PlySmlt2**. 2. Select **1: Polynomial Root Finder**.	
3. Adjust the settings as depicted on the screen to solve a cubic equation. Remember to select "a+bi" or "re^(θi)" for the GC to display all roots, not just the real roots. 4. Press **[GRAPH]** which is the button below **NEXT**.	
5. Key in the values of the coefficients a_3, a_2, a_1 and a_0. . 6. Press **[GRAPH]** to solve the system of equations.	

7. Read off the answers. x_1, x_2 and x_3 give the 3 roots of z. Note : The GC cannot display the roots in exact form.	

Example 9 (Cubic)

Given that $1-\mathrm{i}$ is a root of the equation $z^3-5z^2+8z+p=0$, where $p\in\mathbb{R}$, find the other 2 roots and the value of p.

$(x-A)(x-B) = x^2-(A+B)x+AB$

$x^3+Mx^2+Nx+D=0$

- sum of roots! - product of roots

the IETA D

Solution

Method 1:

Since the cubic equation has real coefficients, 1+i is also a root.

The third root must be a real number k.

$z^3-5z^2+8z+p = [(z-(1+i))][z-(1-i)](z-k)$

$= [(z-1)+i][(z-1)-i](z-k)$

$= [(z-1)^2-i^2](z-k)$

$= (z^2-2z+2)(z-k)$

$= z^3-(k+2)z^2+(2k+2)z-2k$

Comparing coefficient of z, $\cancel{k=3}$ $2k+2=8 \Rightarrow k=3$

Comparing the constant, $p = -2k$

$= -6$

The other 2 roots are $1+i$ and 3, and the value of p is -6.

Method 2:

Alternatively, since $1-\mathrm{i}$ is a root of the given equation, an alternative way of finding p is by substituting the root into the equation.

$(1-i)^3-5(1-i)^2+8(1-i)+p=0$

$(-2-2i)-5(-2i)+8(1-i)+p=0$

$p=-6$

Since the cubic equation has real coefficients, $1+\mathrm{i}$ is also a root.
The third root must be a real number k.

$$z^3-5z^2+8z-6=[z-(1+\mathrm{i})]\,[z-(1-\mathrm{i})](z-k)$$
$$=(z^2-2z+2)(z-k)$$
$$=z^3-(k+2)z^2+(2k+2)z-2k$$
Comparing the constant, $-6=-2k \Rightarrow k=3$
The other 2 roots are $1+\mathrm{i}$ and 3.

4 Geometrical Representation of a Complex Number

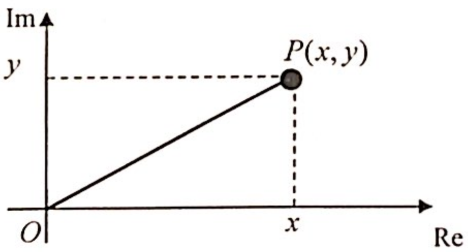
Recall that we represent real number on a number line.
For complex numbers, since there are imaginary numbers, it makes sense for us to have a imaginary number line as well.

4.1 Argand Diagram

A complex number $z=x+\mathrm{i}y$, x, $y\in\mathbb{R}$ can be represented by the point $P(x,y)$. This idea was formally introduced by the French mathematician Jean-Robert Argand, and hence the diagram which represents a complex number in this way was named after him.

An **Argand diagram** is similar to the Cartesian plane with the

- Horizontal axis (labeled as Re) representing the real part of z, and
- Vertical axis (labeled as Im) representing the imaginary part of z.



Remarks:

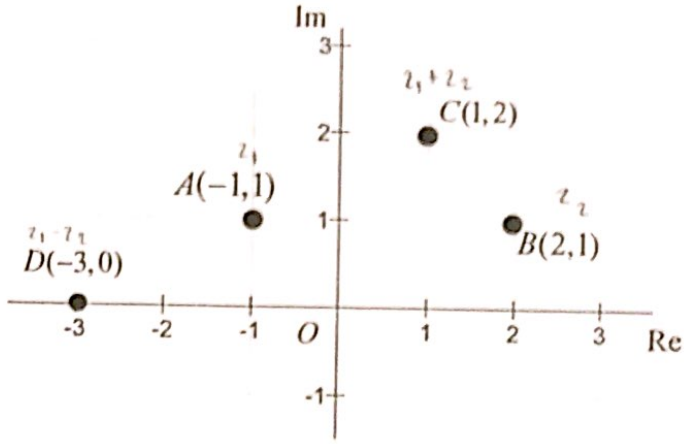
Points on the horizontal axis represent real numbers and points on the vertical axis represent purely imaginary numbers.

On an Argand diagram, complex numbers behave (in terms of additions and subtractions) like 2-D vectors, but they are NOT vectors. In particular, we can divide complex numbers, but we cannot do the same to vectors.

Consider the complex numbers
$z_1 = -1 + \mathrm{i}$ and $z_2 = 2 + \mathrm{i}$.

Then $z_1 + z_2 = 1 + 2\mathrm{i}$
and $z_1 - z_2 = -3$.

Draw the points A, B, C and D representing z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ respectively on an Argand diagram.



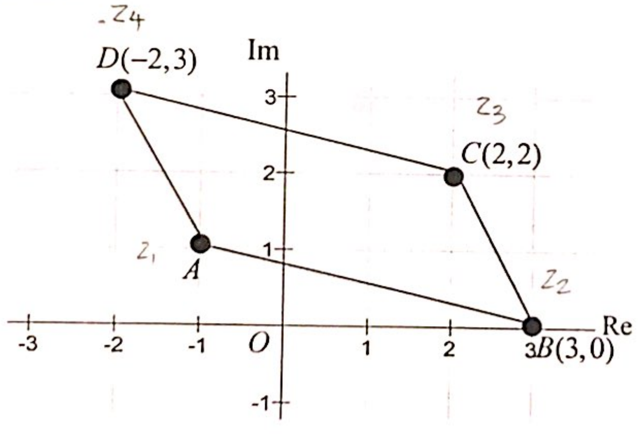
Note that $OACB$ is a parallelogram.
As mentioned, the complex numbers behave like vectors on an Argand diagram.
(Recall how we proved $OACB$ is a parallelogram using vectors.)

$OC = OA + OB$

Example 10

The points A, B, C and D represent four complex numbers z_1, $z_2 = 3$, $z_3 = 2 + 2\mathrm{i}$ and $z_4 = -2 + 3\mathrm{i}$ respectively. Given that $ABCD$ forms a parallelogram, find z_1 by calculation (i.e. not by drawing).

Solution



$z_1 - z_4 = z_2 - z_3$
$\Rightarrow z_1 = z_2 - z_3 + z_4$
$z_2 - z_3 = 3 - 2 - 2i$
$= 1 - 2i = z_1 - z_4$
$z_4 + 1 - 2i = -2 + 3i + 1 - 2i$
$= i - 1 = z_1$

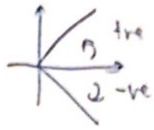
Recall that if this is a vector question:

$\vec{OA} = \vec{CB}$
$\Rightarrow \vec{OA} - \vec{OD} = \vec{OB} - \vec{OC}$
$\Rightarrow \vec{OA} = \vec{OB} - \vec{OC} + \vec{OD}$

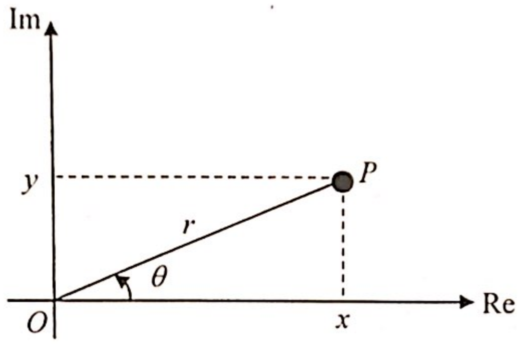
4.3 Modulus and Argument of a Complex Number

Let the point P represent the complex number $z = x + \mathrm{i}y$, x, $y \in \mathbb{R}$ on an Argand diagram. We are interested in 2 geometrical measurements:

- $r =$ distance of P from the origin O
- $\theta =$ directed angle that $\overrightarrow{OP}$ makes with the positive real axis
  ($\theta > 0$ when measured in an anti–clockwise sense;
  $\theta < 0$ when measured in a clockwise sense)



directed

	r is known as the **modulus** of z and is denoted by $\lvert z\rvert$, where $\lvert z\rvert = r = \sqrt{x^2+y^2}$. Clearly $r \geq 0$. θ is known as the (principal) **argument** of z and is denoted by $\arg(z)$, where $-\pi < \arg(z) \leq \pi$. $\arg(z)$ should be given in radians.

Questions

1. What is $\arg(z)$ if z is real?
2. What is $\arg(z)$ if z is purely imaginary?
3. What is $\lvert z\rvert$ and $\arg(z)$ if $z = 0$?

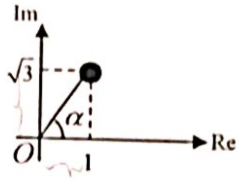
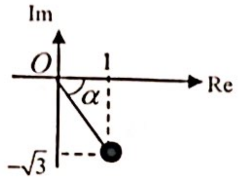
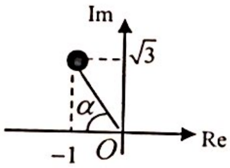
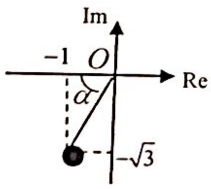
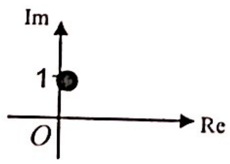
Answers

1. If z is real,
   $\arg(z) = \begin{cases} 0, & \text{for } z > 0 \\ \pi & \text{for } z < 0 \end{cases}$
2. If z is purely imaginary,
   $\arg(z) = \begin{cases} \frac{\pi}{2} & \text{for } \mathrm{Im}(z) > 0 \\ -\frac{\pi}{2} & \text{for } \mathrm{Im}(z) < 0 \end{cases}$
3. If $z = 0$, $\lvert z\rvert = 0$ but $\arg(z)$ is not defined

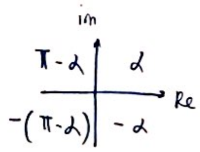
Example 11

For each of the following complex number, find the exact value of the modulus and argument:

(i) $1+i\sqrt{3}$ (ii) $1-i\sqrt{3}$ (iii) $-1+i\sqrt{3}$ (iv) $-1-i\sqrt{3}$ (v) i

Solution	
(i) 	$\lvert 1+i\sqrt{3}\rvert = \sqrt{1+3}$ $= 2$ $\tan\alpha = \sqrt{3}$ $\alpha = \frac{\pi}{3} \Rightarrow \arg(1+i\sqrt{3}) = \frac{\pi}{3}$
(ii) 	$\lvert 1-i\sqrt{3}\rvert = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \arg(1-i\sqrt{3}) = -\frac{\pi}{3}$
(iii) 	$\lvert -1+i\sqrt{3}\rvert = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \arg(-1+i\sqrt{3}) = \frac{2\pi}{3}$
(iv) 	$\lvert -1-i\sqrt{3}\rvert = \sqrt{1+3} = 2$ $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$ $\Rightarrow \arg(-1-i\sqrt{3}) = -\frac{2\pi}{3}$
(v) 	$\lvert i\rvert = \sqrt{1} = 1$ $\arg(i) = \frac{\pi}{2}$

to find arg(z)
① sketch z on an Argand diagram
② find basic angle α



APPENDIX
PROOF OF RESULT THAT NON-REAL ROOTS OF A POLYNOMIAL EQUATION WITH REAL COEFFICIENTS OCCUR IN CONJUGATE PAIRS

Consider the equation

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z + a_0 = 0,$$

where a_n, a_{n-1}, $a_{n-2}, \ldots$, a_1, $a_0 \in \mathbb{R}$, $a_n \neq 0$, $n \in \mathbb{Z}^+$.

Suppose β is a non-real root of the equation,

i.e. $$a_n \beta^n + a_{n-1} \beta^{n-1} + a_{n-2} \beta^{n-2} + \ldots + a_1 \beta + a_0 = 0.$$

Taking conjugates on both sides of the equation,

$$(a_n \beta^n + a_{n-1} \beta^{n-1} + a_{n-2} \beta^{n-2} + \ldots + a_1 \beta + a_0)^* = 0^*$$

$$(a_n \beta^n)^* + (a_{n-1} \beta^{n-1})^* + (a_{n-2} \beta^{n-2})^* + \ldots + (a_1 \beta)^* + a_0^* = 0^*$$

Now $(\beta^k)^* = (\beta^*)^k$ and $(a_k)^* = a_k$ since $a_k \in \mathbb{R}$.

Clearly $0^* = 0$ since $0 \in \mathbb{R}$.

Thus we have

$$a_n (\beta^*)^n + a_{n-1} (\beta^*)^{n-1} + a_{n-2} (\beta^*)^{n-2} + \ldots + a_1 (\beta^*) + a_0 = 0,$$

i.e. β^* is also a non-real root of the given equation.

SUMMARY