RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Chapter 7C: Graphing Techniques

SYLLABUS INCLUDES

- Use of a graphing calculator to graph a given function
- Important characteristics of graphs such as symmetry, intersections with the axes, turning points and asymptotes of the following:
  $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$
  $y=\frac{ax+b}{cx+d}$
  $y=\frac{ax^2+bx+c}{dx+e}$
- Determining the equations of asymptotes, axes of symmetry, and restrictions on the possible values of x and/or y
- Simple parametric equations and their graphs

CONTENT

1 **Characteristics of graphs**
   1.1 Intersections with the axes
   1.2 Axes of symmetry
      1.2.1 Symmetry about the x-axis or y-axis
      1.2.2 Symmetry about other lines
   1.3 Restrictions on the possible values of x and/or y
   1.4 Asymptotes
   1.5 Shape of graphs

2 **Rational functions**
   2.1 Graphs of functions of the form $f(x)=\frac{ax+b}{cx+d}$, where $c \neq 0$, and the numerator and denominator have no common factor if $a \neq 0$
   2.2 Graphs of functions of the form $f(x)=\frac{ax^2+bx+c}{dx+e}$, where $a \neq 0$, $d \neq 0$ and the numerator and the denominator have no common factors

3 Conic Sections

3.1 Circle

3.2 Ellipse

3.3 Hyperbola

4 Parametric equations and their graphs

Appendix - Parabola

1 CHARACTERISTICS OF GRAPHS

In sketching graphs, certain features of a curve can be observed, or easily calculated, from the equation of the curve. Some of the important features that we can consider are intersections with the axes, axes of symmetry, asymptotes, restrictions on the possible values of x and/or y turning points etc.

1.1 Intersections with the axes

It is usually easy to determine whether the curve $y = f(x)$ meets the x-axis and/or y-axis. The x intercepts are found by setting $y = 0$ in the equation of the curve and solving for x. Similarly, the y intercepts are found by setting $x = 0$ in the equation of the curve and solving for y.

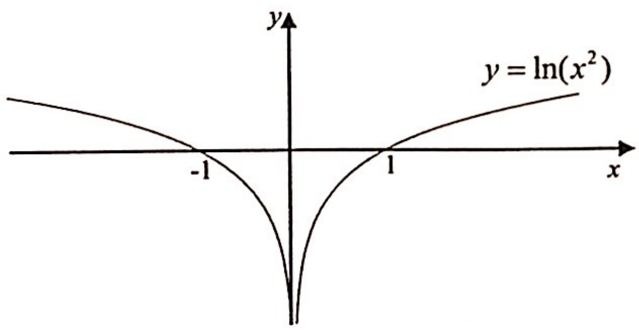
For example, the curve with equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ clearly intersects the x-axis at $(2,0)$ and $(-2,0)$ but does not cross the y-axis.

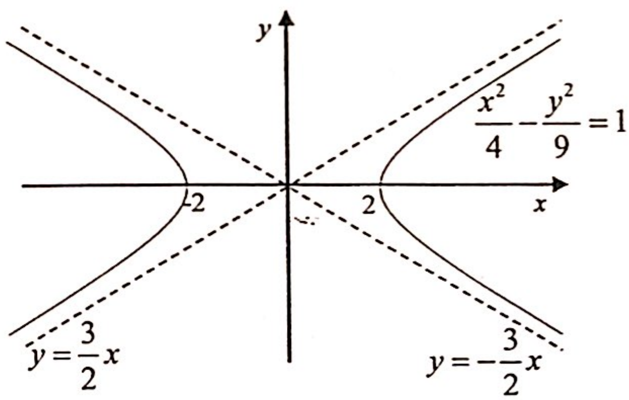
1.2 Axes of symmetry

1.2.1 Symmetry about the x-axis or y-axis

If the equation of the curve is unchanged when x is replaced by (-x), the graph is symmetrical about the y-axis. Similarly, if the equation of the curve is unchanged when y is replaced by (-y), the graph is symmetrical about the x-axis.

For example, the graph of the curve with equation $y = \ln(x^2)$ is symmetrical about the y-axis while the graph of the curve with equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is symmetrical both about the x-axis and y-axis.





Note:

The dotted lines $y=\frac{3}{2}x$ and $y=-\frac{3}{2}x$ are asymptotes to the curve $\frac{x^2}{4}-\frac{y^2}{9}=1$. In the case of the curve $y=\ln(x^2)$, the line $x=0$ is an asymptote to the curve. We will learn more about asymptotes in section **1.4.**

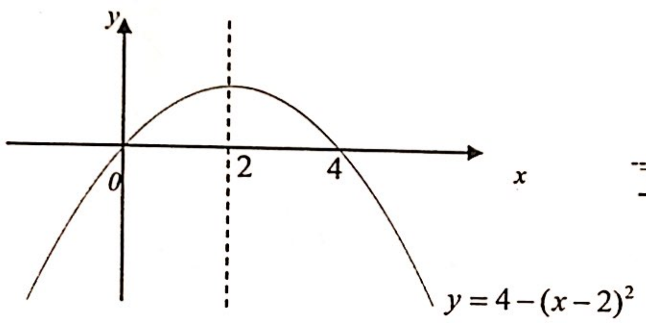
1.2.2 Symmetry about other lines

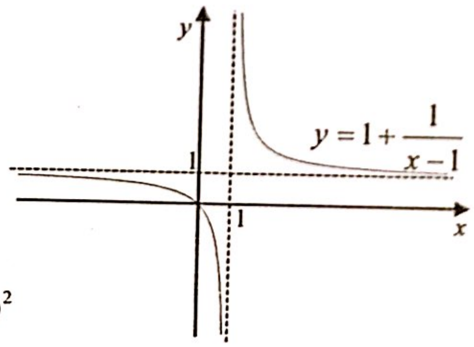
Besides symmetry about the axes, the graph of the curve could possibly have other axes of symmetry.

For the curve with equation $y=4-(x-2)^2$, the graph is symmetrical about the line $x=2$.

In the case of the curve with equation $y=1+\frac{1}{x-1}$, a line of symmetry is $y=x$.

also max point





$(2-2)^2$ 4 3 1 0

Recall: relationship between graphs of $y=f(x)$ and $y=f^{-1}(x)$

Note:
To determine if a curve is symmetrical in the line $y=x$, we can easily do so by replacing both x with y and y with x. If the equation of the curve is unchanged, then the line $y=x$ is a line of symmetry.

Exercise:
State a line of symmetry of the following curves whose equations are

(i) $y=\frac{8}{x^2-x+1}$, $(x-\frac{1}{2})^2+\frac{3}{4}$ complete the square (ii) $y=e^{-x^2}$.

Answer: (i) $x=\frac{1}{2}$, (ii) $x=0$

1.3 Restrictions on the possible values of x and/or y

It is sometimes useful to consider whether there are any restrictions on the possible values of x and/or y when we graph the curve $y = f(x)$.

For instance, the curve with equation $y = \ln(2-x)$ is only well-defined when $x < 2$. Thus the graph of $y = \ln(2-x)$ lies only in the region where $x < 2$.

In the case of the curve with equation $y^2 = x - 1$, we observe that since $y^2 \geq 0$ for all $y \in \mathbb{R}$, then $x - 1 \geq 0 \Rightarrow x \geq 1$. Thus the graph lies only in the region where $x \geq 1$.

Example 1

It is given that that $y = \dfrac{x^2+3}{x+1}$, $x \in \mathbb{R}, x \neq -1$. Without using a calculator, find the set of values that y can take.

Solution

$y = \dfrac{x^2+3}{x+1}$, $x \in \mathbb{R}$, $x \neq -1$

$yx + y = x^2 + 3$

$x^2 - yx + (3-y) = 0 \Rightarrow ax^2 + bx + c = 0$

$b^2 - 4ac \geq 0$

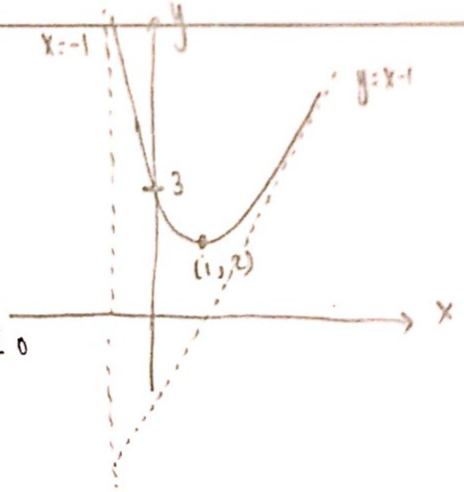
Since x is real, discriminant $(-y)^2 - 4(1)(3-y) \geq 0$

$y^2 + 4y - 12 \geq 0$

$(y+6)(y-2) \geq 0$

$y \leq -6$ or $y \geq 2$

$\therefore$ set of values of y is $(-\infty, 6] \cup [2, \infty)$



Note: conics

Graphs of circles, ellipses and hyperbolas (which will be dealt with in Section 3) have certain restrictions on values of x and y as well.

Consider the graph of $x^2 + y^2 = 9$, which is a circle with centre at the origin and radius 3 units. We note that $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

For the graph of hyperbola $x^2 - 2y^2 = 1$, we observe that as $x^2 = 1 + 2y^2$, we have $x^2 \geq 1$, since $y^2 \geq 0$ for all $y \in \mathbb{R}$. Thus $x \leq -1$ or $x \geq 1$.

1.4 Asymptotes

An asymptote of the curve with equation $y = \mathrm{f}(x)$ is a line or a curve that the graph of $y = \mathrm{f}(x)$ approaches when x or y is large. It gives an indication of the behaviour of the curve when $x \to \infty$ or $-\infty$ and/or when $y \to \infty$ or $-\infty$. In our syllabus, we will only concern ourselves with linear asymptotes.

Example 2

Find the equations of the asymptotes of the curve with equation $y = 2 - \frac{1}{x}$.

Solution

y can't divide by 0

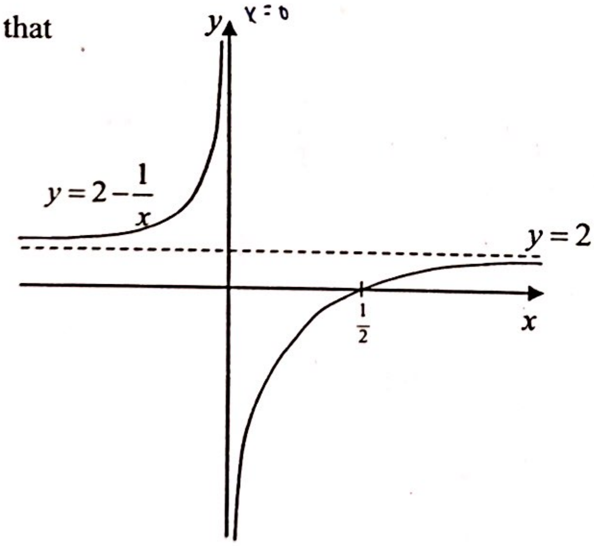
As $x \to \infty$, $\frac{1}{x} \to 0$, $y \to 2$ and hence $y = 2$ is a horizontal asymptote.

Note that, in fact, we also have as $x \to -\infty$, $y \to 2$.

As $x \to 0^-$ (i.e. approaching $x = 0$ from the left), $y \to \infty$. Thus $x = 0$ is a vertical asymptote.

Note also that as $x \to 0^+$ (i.e. approaching $x = 0$ from the right), $y \to -\infty$.

Graphically, we see that



Example 3

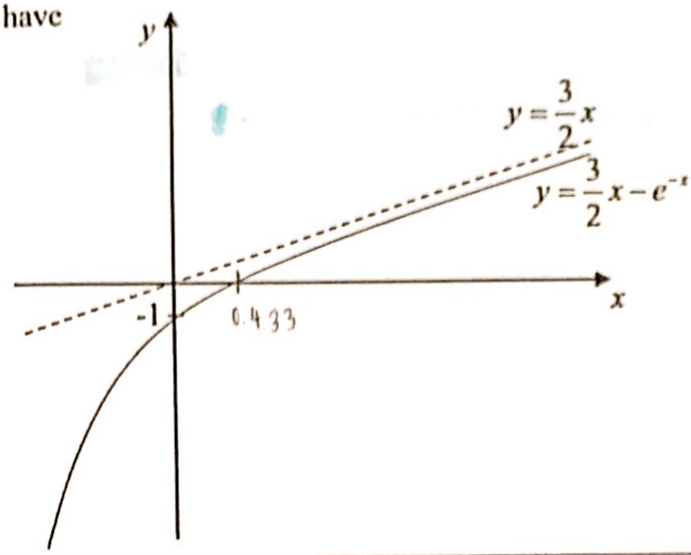
Find the equation(s) of the asymptote(s) of the curve with equation $y = \frac{3}{2}x - e^{-x}$.

Solution

As $x \to \infty$, $e^{-x} \to 0$ and hence $y \to \frac{3}{2}x$.

Thus $y = \frac{3}{2}x$ is an oblique asymptote.

Graphically, we have



Sketch asymptote before graph (need approaching trend)

Exercise:

State the equations of the asymptotes of the curve with equation $y=x-\frac{1}{x^2}$.

Answer: $y=x$, $x=0$

$x\to\infty$, $y\to x$ $y=x$

$x\to 0$, $y\to -\infty$ $x=0$

1.5 Shape of graphs

Having discussed intercepts and asymptotes of graphs, it remains for us to study the shape of the graph. Recall in Chapter 7B we have discussed the concept of strictly increasing (decreasing), concavity as well as stationary points.

These form the foundation to determine the shape of any given graph. Hence given a curve with equation $y=\mathrm{f}(x)$, it is important for us to look at $\mathrm{f}'(x)$ to determine the shape of the graph.

SUMMARY FOR CURVE SKETCHING

A simple way to start any curve sketching question is to remember the following mnemonic: **S** (Shape/Stationary Points) **I** (Intercepts) **A** (Asymptotes). In practice, we usually start from determining the asymptotes (if any), then the intercepts, followed by the shape, which includes the stationary points.

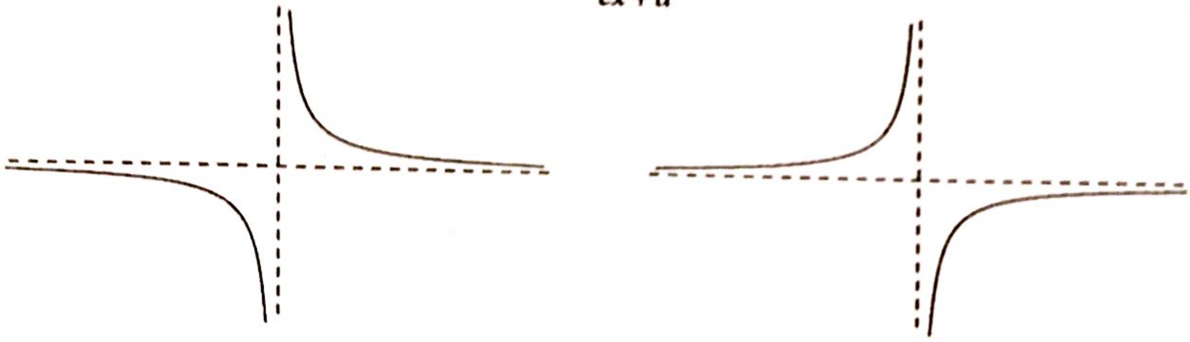
S	I	A
(Shape/Stationary Points)	(Intercepts)	(Asymptotes)

In the next section, we will learn how to sketch some standard curves such as graphs of rational functions and conic sections.

2 RATIONAL FUNCTIONS

2.1 Graphs of functions of the form $f(x)=\dfrac{ax+b}{cx+d}$, where $c \neq 0$, and the numerator and denominator have no common factor if $a \neq 0$

In general, the graph of $y=f(x)$ where $f(x)=\dfrac{ax+b}{cx+d}$ where $c \neq 0$ takes one of the following



The dotted lines are known as **asymptotes**. They are lines which the curve $y=f(x)$ approaches when x or y is large.

Note:

The graph of $y=\dfrac{ax+b}{cx+d}$ has no turning points. To understand this, we consider $\dfrac{dy}{dx}$ which can be easily shown to be given by $\dfrac{dy}{dx}=\dfrac{ad-bc}{(cx+d)^2}$.

When $ad-bc \neq 0$, $\dfrac{dy}{dx} \neq 0$. Hence the graph of $y=f(x)$ has no turning points.

When $ad-bc=0$, $\dfrac{dy}{dx}=0$ for all real values of x (except for $x=-\dfrac{d}{c}$). Hence $y=f(x)$ will be a horizontal line (with the point when $x=-\dfrac{d}{c}$ removed).

↳ no turning point either

Example 4

Sketch the graph of $y=\frac{x-1}{x+2}$, $x\neq -2$, labelling clearly any intersections with the axes, turning points and asymptotes.

Shape/Stationary
Intersect
Asymptote

Solution

To find axial intercepts: When $x=0$, $y=-\frac{1}{2}$ and when $y=0$, $x=1$.

To find the vertical asymptote:

We observe that as $x\to -2^-$, $y\to +\infty$. Also, we have as $x\to -2^+$, $y\to -\infty$.

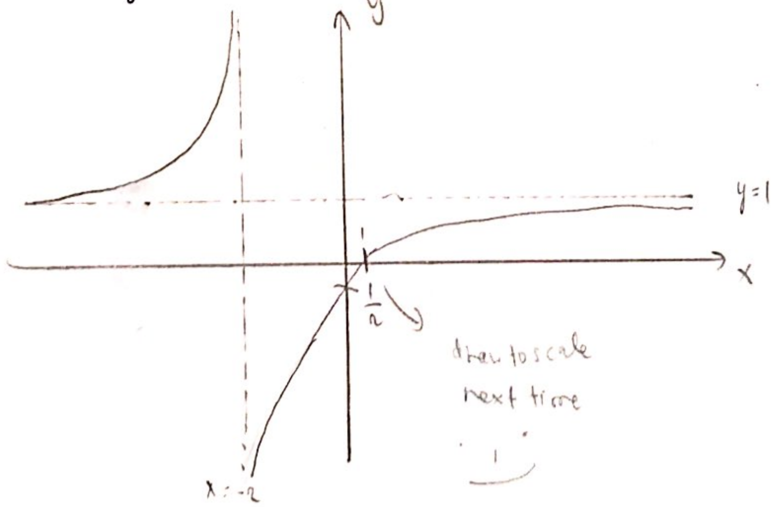
$\therefore$ The line $x=-2$ is a vertical asymptote for the graph of $y=\frac{x-1}{x+2}$.

Simply put, vertical asymptotes for rational functions occur at points where the denominator is zero, provided the numerator and denominator have no common factors.

To find the horizontal asymptote:
We need to consider what happens to y as $x\to\pm\infty$.
Express $y=\frac{x-1}{x+2}$ in the form $y=1-\frac{3}{x+2}$.

As $x\to\pm\infty$, $\frac{3}{x+2}\to 0$, $y\to 1$, i.e. the graph "approaches" the line $y=1$ as $x\to\pm\infty$.

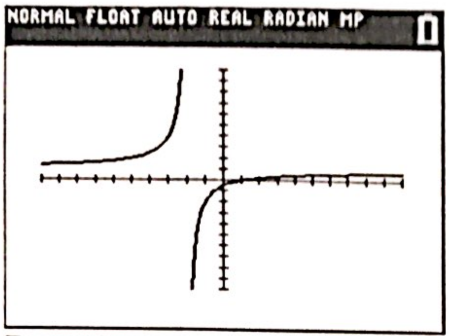
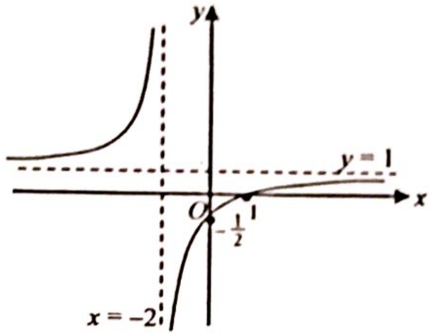
$\therefore$ The line $y=1$ is a horizontal asymptote for the graph of $y=\frac{x-1}{x+2}$.

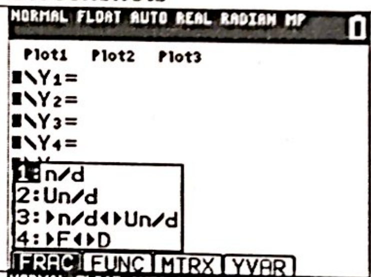
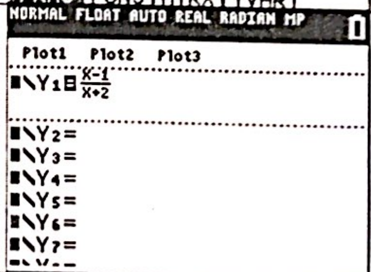
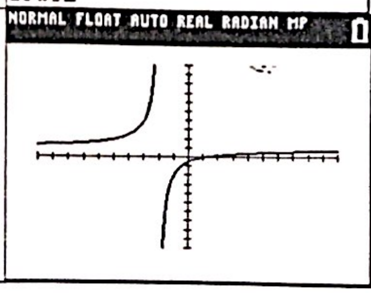


Note :
The GC gives the shape of the graph. In many cases, you may need to resize the window to capture the entire graph, or to change window settings to capture critical features of the graph.

We can use it to find intersections with axes and turning points, but **it does not draw asymptotes!** To determine if a graph has any asymptotes, you need to examine its equation.

To illustrate this point, we shall investigate a case by sketching the graph of $y = \dfrac{x-1}{x+2}$.

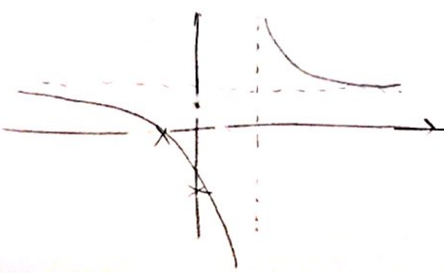
What you see	What it should be
$y = \dfrac{x-1}{x+2}$  **Remarks** : There are 2 asymptotes, but both are not shown.	$y = \dfrac{x-1}{x+2}$ 

GC Keystrokes	***Screenshots***
1. Press $\boxed{\text{Y=}}$ followed by $\boxed{\text{ALPHA}}\boxed{\text{Y=}}$ 2. Select **1: n / d**	
3. Key in the numerator and denominator	
4. Press $\boxed{\text{ZOOM}}$ 5. Select **6: ZStandard** (standard window of $-10 \le x \le 10,\ -10 \le y \le 10$)	

$y = \dfrac{2x+1}{x-3}$ $\dfrac{2x-6+7}{x-3}$

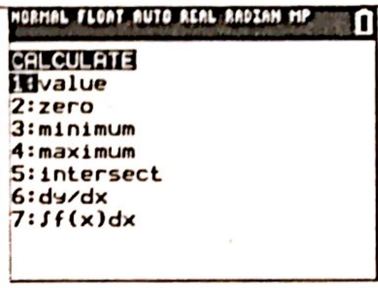
$2 + \dfrac{7}{x-3}$

asymptotes $x = 3$
$y = 2$



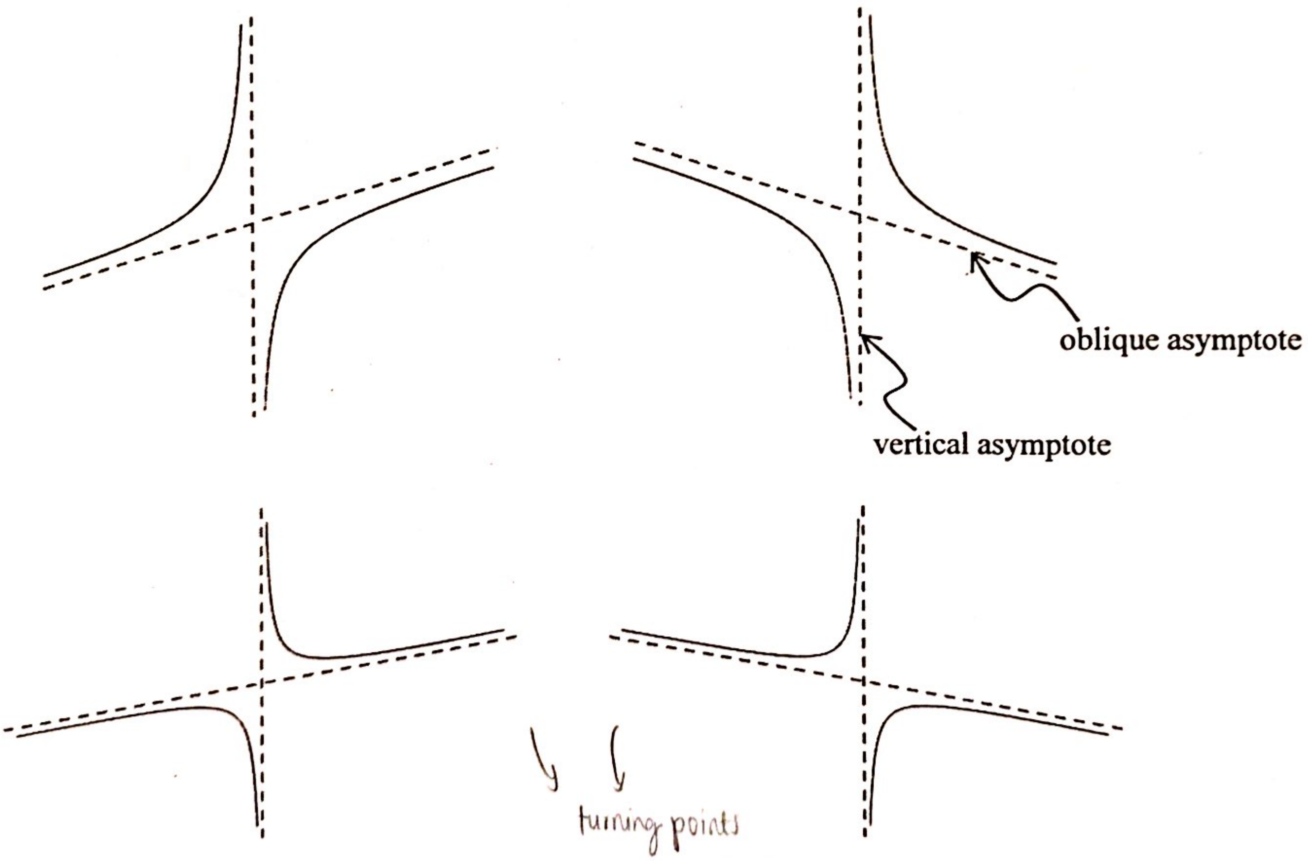
6. Press $\boxed{\text{2nd}}\boxed{\text{TRACE}}$

7. Select **1: value** (to calculate y-intercept)
2: zero (to calculate x-intercept)
3: minimum (to calculate min. turning pt.)
4: maximum (to calculate max. turning pt.)



2.2 Graphs of functions of the form $f(x)=\dfrac{ax^2+bx+c}{dx+e}$, where $a\neq 0$, $d\neq 0$ and the numerator and the denominator have no common factors

In general, the graph of $y=f(x)$ where $f(x)=\dfrac{ax^2+bx+c}{dx+e}$ where $c\neq 0$ takes one of the following forms.



The dotted lines are the **asymptotes**. There is a vertical asymptote and an oblique asymptote. The graph of $y=f(x)$ may or may not have turning points depending on the values of a, b, c, d and e. We will now look at an example to illustrate how to sketch graphs of this form.

Example 5

Sketch the graph of $y=\dfrac{x^2-3x-4}{x-1}$, $x\neq 1$, labelling clearly any intersections with the axes, turning points and asymptotes.

Solution

To find axial intercepts: When $x=0$, $y=4$ and when $y=0$, $x=-1,\ 4$.

To find the vertical asymptote: As $x\to 1^-$, $y\to +\infty$ and as $x\to 1^+$, $y\to -\infty$

Hence the graph has a vertical asymptote $x=1$

As $x\to\infty$, y does not approach a constant, so there is no horizontal asymptote. However, we have an oblique asymptote.

To find the oblique asymptote :

Consider what happens to y as $x\to\pm\infty$.

We first express $y=\dfrac{x^2-3x-4}{x-1}$ in the form $y=x-2-\dfrac{6}{x-1}$.

As $x\to\pm\infty$, $y\to x-2$ (since $\dfrac{6}{x-1}\to 0$), i.e. the graph "approaches" the line $y=x-2$ as $x\to\pm\infty$.

$\therefore$ The line x-2 is an oblique asymptote for the graph of $y=\dfrac{x^2-3x-4}{x-1}$.

To find stationary points:

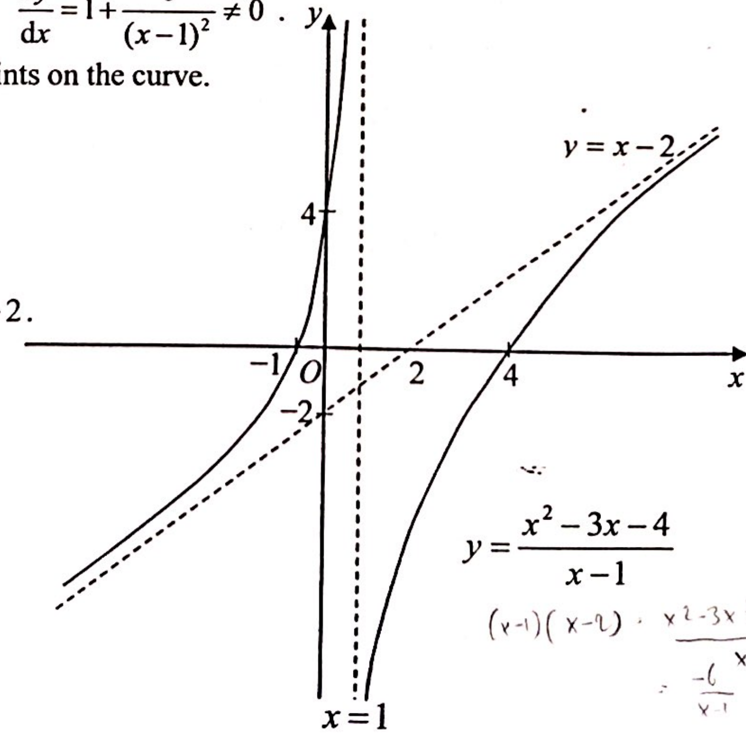
optional?

$y=\dfrac{x^2-3x-4}{x-1}=x-2-\dfrac{6}{x-1} \Rightarrow \dfrac{dy}{dx}=1+\dfrac{6}{(x-1)^2}\neq 0$.

Hence, there are no stationary points on the curve.

$y=\dfrac{x^2-3x-4}{x-1}=x-2-\dfrac{6}{x-1}$

Asymptotes are $x=1$ and $y=x-2$.



$(x-1)(x-2) = \dfrac{x^2-3x+2-6}{x-1}$

$= \dfrac{-6}{x-1}$

Example 6

Sketch the graph of $y=\dfrac{2x^2+5x+10}{x+2}$, $x\neq -2$, labelling clearly any intersections with the axes, turning points and asymptotes.

Solution

$y=\dfrac{2x^2+5x+10}{x+2}=\dfrac{(x+2)(2x+a)+b}{x+2}$

$\dfrac{2x^2+ax+4x+2a+b}{x+2}$

$\Rightarrow a=1$

$=(2x+1)+\dfrac{8}{x+2}$

as $x\to -2$, $y\to\infty$

The line $x=-2$ is a vertical asymptote.

as $x\to\pm\infty$, $y\to 2x+1$,

The line $y=2x+1$ is an oblique asymptote.

intercepts: when $y=0$, $b^2-4ac<0$

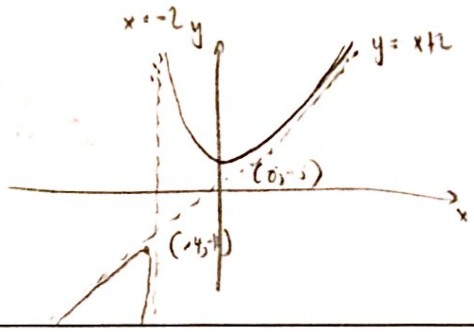
there is not x-intercept → must have turning point

when $x=0$, $y=5$

y-intercept $=5$

$\dfrac{dy}{dx}=\dfrac{(x+2)(4x+5)-(2x^2+5x+10)}{(x+2)^2}$ use GC to find turning point

$=4x^2+$



Note:
In general, if the equation of the curve is known, we can use the GC to find the shape, intersections with axes and turning points of the graph. However, we need to examine the equation to determine the asymptotes.

Example 7

$x=k$, $y=x+k$

The curve C has equation $y=x+k+\dfrac{k+2}{x-k}$, where k is a constant. Find the range of values of k for which the curve C cuts the x-axis at two distinct points.

Given that $k=2$,

(i) Sketch the curve C, stating its asymptotes, stationary points and axial intercepts if any.

(ii) The two asymptotes of C intersect at point P. Show that P lies on the line $y=mx+(4-2m)$ for all real values of m.

Hence, state the range of values of m for which the line $y=mx+(4-2m)$ does not cut the curve C.

Solution

$0=\dfrac{x^2-k^2+k+2}{x-k}$

$b^2-4ac>0$

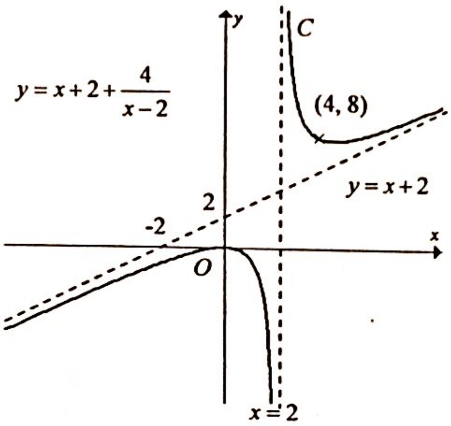
$x^2=k^2-k-2$

$k^2-k-2>0$

$(k+1)(k-2)>0$

$k<-1$ or $k>2$

(i)



given $k = 2$,
$y = x + 2 + \frac{4}{x-2}$
oblique asymptote : $x+2$
vertical asymptote : $x = 2$ (since denominator $\neq 0$, $x \neq 2$)
axes - intercepts: when $x = 0$, $y = 0$
when $y = 0$, $\frac{4}{x-2} = -x-2$
$4 = (-x-2)(x-2)$
$= -x^2 + 4$
(no solution)

ii) $y = x+2$
$x = 2$
$y = 4$
$\therefore$ coordinates of $P = (2,4)$
sub $x = 2, y = 4$ into $y = mx + (4-2m)$
$4 = 2x + 4 - 2m$
$4 = 4$ //(shown)
$\therefore m \in (-\infty, \infty)$

← check answer not legit

by observation,
from the graph, if line does not intersect C, $m \leq 1$,
↓
same gradient or gentler gradient than
follow the equation
$y = x+2$
$y = mx + 4 - 2m$
$\therefore y = mx + (4-2m)$ passes through P for all real values of M.

In summary,

$y = \dfrac{ax+b}{cx+d}$, where $c \neq 0$, numerator and denominator have no common factor	Express $y = \dfrac{ax+b}{cx+d}$ in the form $y = p + \dfrac{q}{cx+d}$, where p and q are constants. • Vertical asymptote : $x = -\dfrac{d}{c}$ • Horizontal asymptote : $y = p = \dfrac{a}{c}$ If $a = 0$, the horizontal asymptote is $y = 0$, i.e. x-axis
$y = \dfrac{ax^2+bx+c}{dx+e}$ where $a \neq 0$, $d \neq 0$, numerator and denominator have no common factor	Express $y = \dfrac{ax^2+bx+c}{dx+e}$ in the form $y = (px+q) + \dfrac{r}{dx+e}$, where p, q and r are constants, $p \neq 0$. • Vertical asymptote : $x = -\dfrac{e}{d}$ • Oblique asymptote : $y = px + q$

Note:

In general when we deal with graphs of rational functions, we can apply similar strategies discussed earlier to determine the equations of the asymptotes. That is,

- vertical asymptotes occur at values of x where the denominator is zero, provided the numerator and denominator have no common factors.
- horizontal or oblique asymptotes can be determined by writing the rational expression in a form of a polynomial plus a proper fraction.

Example 8

Determine the equations of any asymptotes of the following curves with equations

(a) $y=\dfrac{1}{x^2-1}$, **(b)** $y=\dfrac{x(2x+3)}{x^2-1}$, **(c)** $y=\dfrac{x^3-8}{x^2-2}$

and sketch the curves on separate diagrams.

Solution

(a) $y=\dfrac{1}{x^2-1}$

$\dfrac{dy}{dx} = \dfrac{-(2x)}{(x^2-1)}$

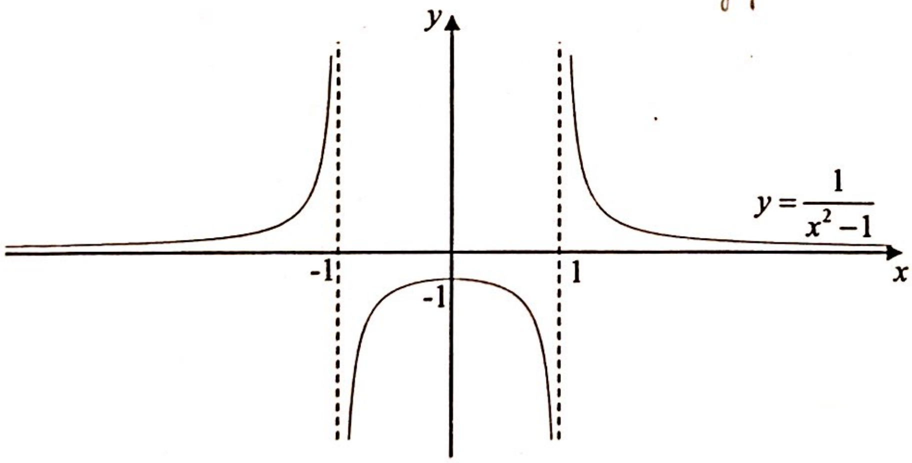
$= \dfrac{-2x}{x^2-1}$

$y=0$

Asymptotes are $x=-1$ and $x=1$

turning point at $x=0$

* use GC to find graphs

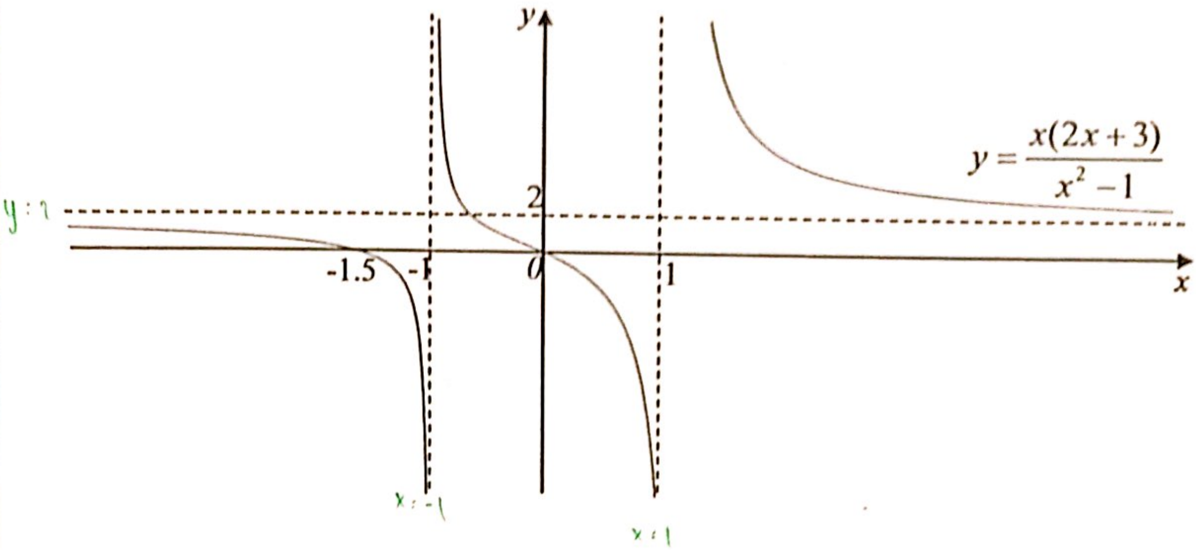


(b) $y=\dfrac{x(2x+3)}{x^2-1}=\dfrac{2(x^2-1)+3x+2}{x^2-1} = 2+\dfrac{3x+2}{x^2-1}$

when $x \to \infty$, $y \to 2$

axes-intercepts:
let $y=0$
$x(2x+3)=0$
$x=0$ or $x=-\frac{3}{2}$

Asymptotes are $x=1$, $x=-1$, $y=2$

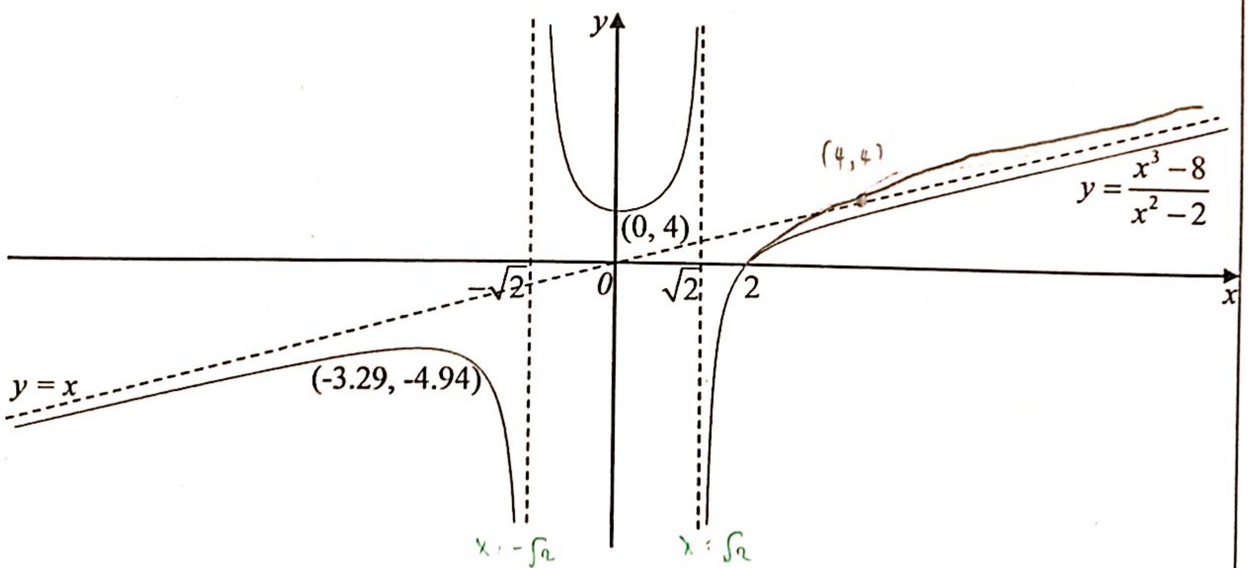


(c) $y = \dfrac{x^3-8}{x^2-2} = \dfrac{(x^2-2)(x) - 8 + 2x}{x^2-2} = x + \dfrac{2x-8}{x^2-2}$

when $x \to \infty$, $y \to x$

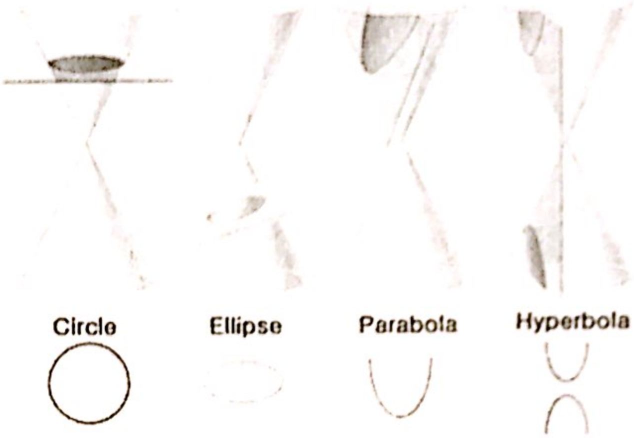
Asymptotes are $x = \sqrt{2}$, $x = -\sqrt{2}$, $y = x$

when $x=4$, $y=4$
the curve cuts the asymptote at
$y=x$ when $x>4$
we have $(x-4)>0$ and $x^2-2>0$
$\therefore y>x$ for $x>4$



3 CONIC SECTIONS

Conics are curves that can be obtained from the intersections of a cone and a plane. In this section, we will look at the equations of circle, ellipse, and hyperbola and some of their characteristics.



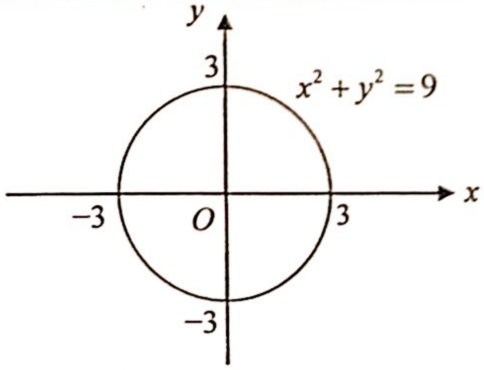
3.1 Circle

The equation of a circle with centre (0,0) and radius r units is given by $$x^2+y^2=r^2, \quad \text{where } r>0.$$

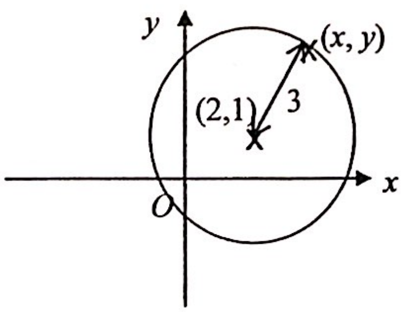
Consider the circle with equation $x^2+y^2=9$.

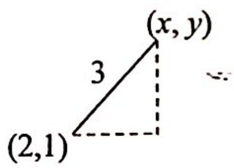
This is a circle with centre $(0,0)$ and radius 3 units.

$y = \sqrt{1-x^2}$



Now, let us find out the equation of a circle with centre (2,1) and radius 3 units.





Using Pythagoras' Theorem, we know the equation of a circle with centre (2,1) and radius 3 units is $(x-2)^2+(y-1)^2=3^2=9$.

> The standard form of the equation of a circle with centre (h,k) and radius r units is given by
> $$(x-h)^2+(y-k)^2=r^2, \quad \text{where } r>0.$$

Note:

1. The coefficients of x^2 and y^2 are equal.
2. When sketching a circle,
   - the scale for both axes should be the same, and ***use a compass***
   - label the centre and radius clearly
   - check if the origin is in the circle, on the circle, or outside the circle. (How?)

Example 9

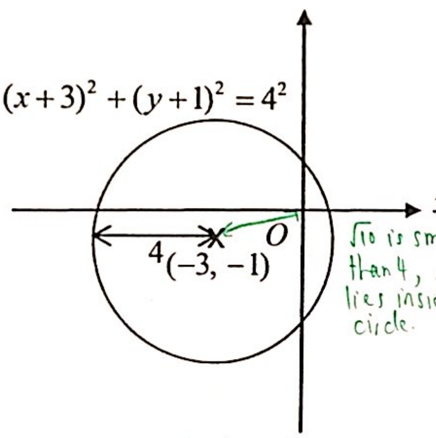
Sketch, on separate diagrams, the circles given by the equations

(i) $(x+3)^2+(y+1)^2=16$, (ii) $x^2+y^2+2x-4y=0$.

$x^2+2xy+y^2$

Solution

(i)



$\sqrt{10}$ is smaller than 4, $\therefore$ origin lies inside the circle.

Circle with centre $(-3,-1)$ and radius 4 units.

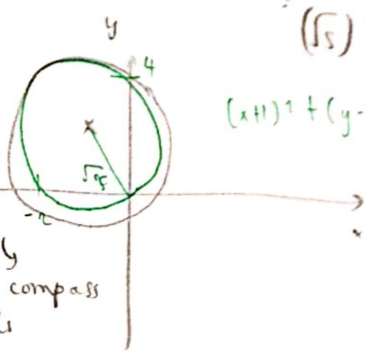
(ii)

$(x^2+2x+1)+(y^2-4y+4)-5=0$

$(x+1)^2+(y-2)^2=5$

$(\sqrt{5})^2$

$(x+1)^2+(y-2)^2=(\sqrt{5})^2$



↳ use compass pls

circle with centre $(-1,2)$ and radius $\sqrt{5}$ units ✓

* Try using the GC to sketch the circle to check your answers.

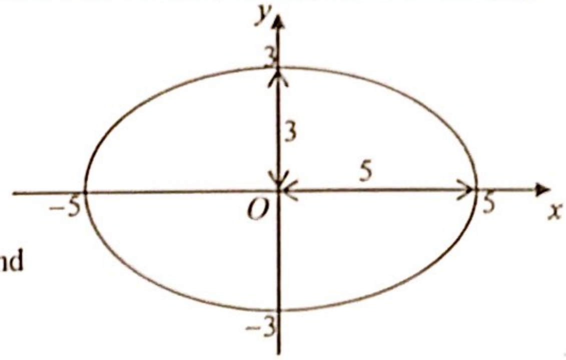
↳ $\boxed{\text{apps}}$

3.2 Ellipse

The equation of an ellipse centred at (0, 0) is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0, b>0$.

Consider the ellipse with equation $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$.

This is an ellipse with centre (0,0).
The graph is symmetrical about the x-axis and y-axis.
The length of semi-major axis (horizontal) is 5 units and the length of semi-minor axis (vertical) is 3 units.



Note:

1. When $a=b$, the equation becomes the equation of a circle of radius a and centre $(0, 0)$. Hence, a circle is a special ellipse.
2. When $a \neq b$, the x^2 and y^2 terms have the same sign but different coefficients.
3. When sketching an ellipse,
   - the scale for both axes should be the same
   - note symmetry about $x=0$ and $y=0$
   - label the centre and lengths of semi-major and semi-minor axes clearly
   - check if the origin is in the ellipse, on the ellipse, or outside the ellipse.

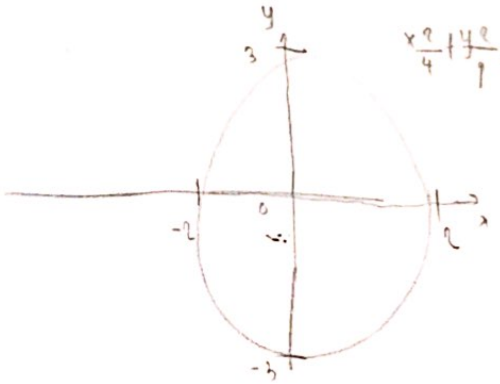
Example 10
Sketch the ellipse with equation $9x^2+4y^2=36$.

$\frac{y^2}{9} = 1-\frac{x^2}{4}$
$\frac{y}{3} = \sqrt{1-\frac{x^2}{4}}$
$y = 3\sqrt{1-\frac{x^2}{4}}$

Solution

$9x^2+4y^2=36 \rightarrow$ rearranging to standard form, we get
$\frac{x^2}{4}+\frac{y^2}{9}=1$

$\frac{x^2}{4}+\frac{y^2}{9}=1$



3.3 Hyperbola

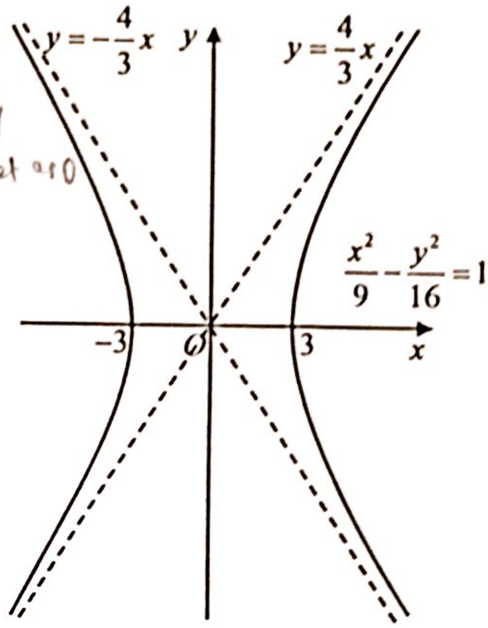
The equation of a hyperbola with centre (0,0) is given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Consider the hyperbola with equation $\frac{x^2}{9}-\frac{y^2}{16}=1$.

This graph is symmetrical about the x-axis and y- axis.

It has 2 oblique asymptotes, $y=-\frac{4}{3}x$ and $y=\frac{4}{3}x$. ✓ set as 0

When $y=0$, $\frac{x^2}{9}=1 \Rightarrow x=\pm 3$



To find the asymptotes of a hyperbola :

Consider $\frac{x^2}{h^2}-\frac{y^2}{k^2}=1$. Rewriting, we obtain $\frac{y^2}{k^2}=\frac{x^2}{h^2}-1$.
Observe that as $x \to \pm\infty$, 1 becomes insignificant.
Hence, $\frac{y^2}{k^2}\approx\frac{x^2}{h^2}$ for large values of x.

Consequently, we get $y=\pm\frac{k}{h}x$ as the oblique asymptotes.

Similarly for $\frac{y^2}{k^2}-\frac{x^2}{h^2}=1$.

<u>Not in syllabus: Formal proof of obtaining the asymptotes of a hyperbola</u>

Consider $\frac{x^2}{h^2}-\frac{y^2}{k^2}=1$.

Rewriting, we obtain $y=\pm\frac{k}{h}x\sqrt{1-\frac{h^2}{x^2}}$. As $x \to \pm\infty$, $\frac{h^2}{x^2}\to 0$. Hence, $y \to \pm\frac{k}{h}x$.

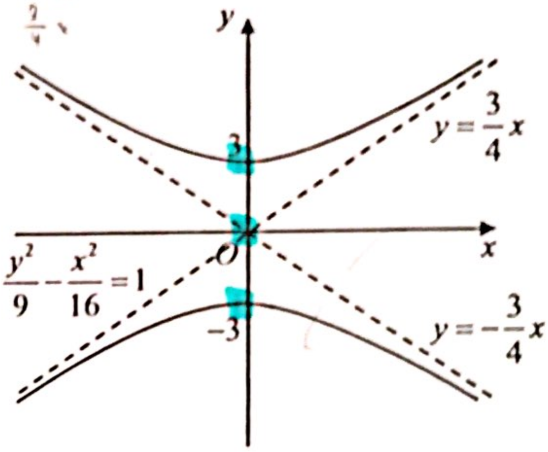
Consequently, we get $y=\pm\frac{k}{h}x$ as the oblique asymptotes. (Similarly for $\frac{y^2}{k^2}-\frac{x^2}{h^2}=1$.)

What about the graph of a hyperbola with equation $\frac{y^2}{9}-\frac{x^2}{16}=1$? Using the same approach as above, we observe that this graph is symmetrical about the x-axis and y- axis.

It also has 2 oblique asymptotes, $y=-\frac{4}{3}x$ and $y=\frac{4}{3}x$. However, in this case, the curve intersects the y-axis but not the x-axis. $-\frac{3}{4}x$ $\frac{3}{4}x$

Symmetry about x-axis and y-axis

$(x^2 = -x)$ $(y^2 = -y)$



Note :

1. The x^2 and y^2 terms have opposite signs.
2. When drawing a hyperbola,
   - the scale for both axes should be the same
   - draw curve approaching the oblique asymptotes
   - label the centre and vertices of the hyperbola
   - label the equations of the asymptotes.

Example 11

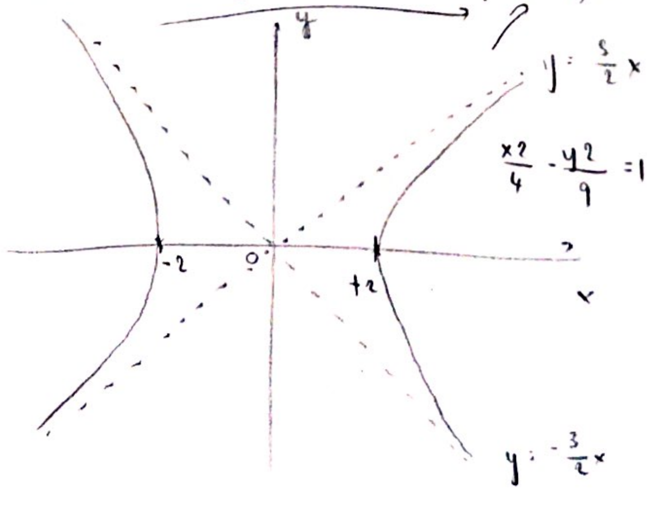
Sketch the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$, indicating clearly the equations of asymptotes.

Solution

asymptotes : $y = \pm \frac{3}{2}x$

pls be symm

when $y = 0$, $x^2 = 9$, $x = \pm 2$



$\frac{x^2}{4} - \frac{y^2}{9} = 1$ $\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$

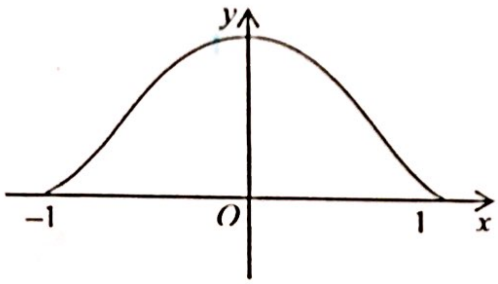
4 PARAMETRIC EQUATIONS AND THEIR GRAPHS

Recall that we have studied in Chapter 7A how to find the gradient of a curve defined parametrically at any given point. Here, we shall study how we can sketch such parametric curves and how to use the sketches to work out the gradient.

Example 12

Sketch the curve with parametric equations $x=\cos\theta, y=\sin^3\theta, 0\le\theta\le\pi$. Find the gradient at the point where $\theta=\dfrac{\pi}{3}$.

Solution

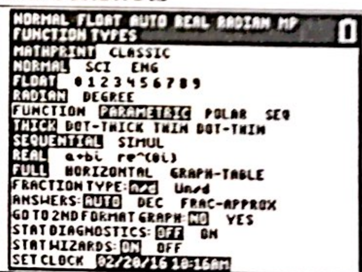
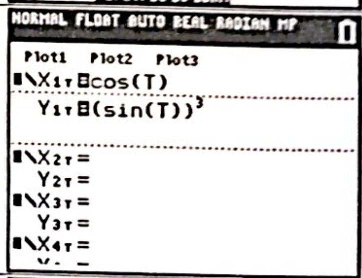


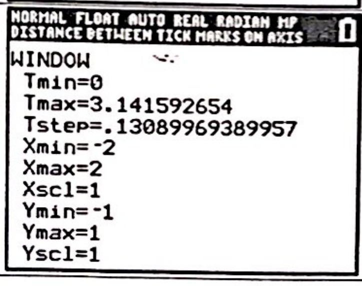
$\theta=0: x=1, y=0$

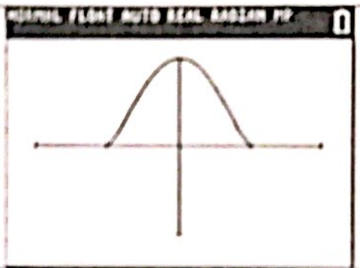
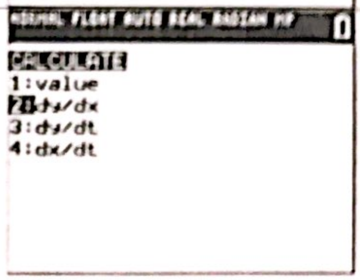
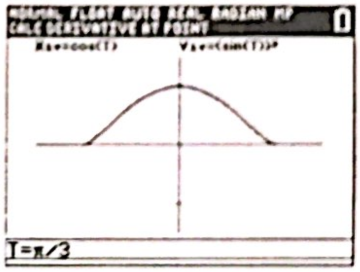
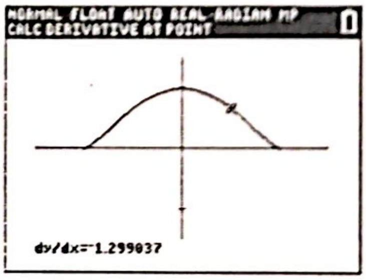
$\theta=\dfrac{\pi}{2}: x=0, y=1$

$\theta=\pi: x=-1, y=0$

From GC, gradient at the point where $\theta=\dfrac{\pi}{3}$ is -1.30 (3sf).

<table>
<tr><th>GC Keystrokes</th><th>Screenshots</th></tr>
<tr><td>1. Change the mode to PAR (parametric mode).</td><td>
</td></tr>
<tr><td>2. In the $\boxed{\text{Y=}}$ screen, enter the functions.</td><td>
</td></tr>
<tr><td>3. Need to adjust window settings.

Limitation :
The default setting is $T_{\min}=0, T_{\max}=2\pi$.</td><td>
</td></tr>
</table>

4. Press [GRAPH] for a sketch of the curve.	
5. To find the gradient of the curve at the point where $\theta = \frac{\pi}{3}$, (a) press [2nd][TRACE] (b) Select **2: dy/dx** (c) Type $\frac{\pi}{3}$, and press enter.	  

Note :
The GC gives the shape of the graph. In many cases, you may need to resize the window to capture the entire graph, or to change window settings to capture critical features of the graph.

We can use it to find intersections with axes and turning points, but **it does not draw asymptotes!**

To determine if a graph has any asymptotes, you need to examine its equation. Sometimes, a vertical line may appear like a vertical asymptote, but actually it is only connecting one point off the top of the screen with the next point off the bottom. The GC does not know if there is any undefined value between the values it is graphing.

To illustrate this point, we shall use the following example.

Example 13

A curve C has parametric equations $x = t-2$ and $y = t-2+\frac{1}{t-4}$ for $t \neq 4$.

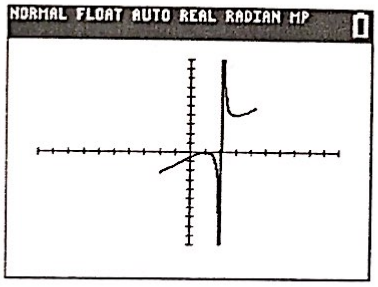
(i) Find a Cartesian equation of C.
(ii) Hence sketch the curve C.

Solution

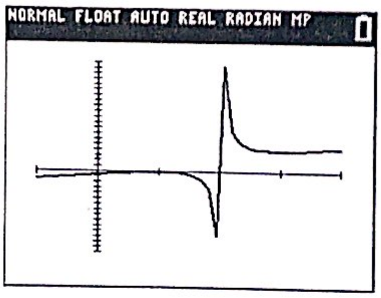
(i) $y = x + \frac{1}{x-2}$

<table>
<tr><th>What you see if you use Parametric mode to sketch the graph</th><th>What it should be</th></tr>
<tr><td>

$x = t-2$ and $y = t-2+\frac{1}{t-4}$



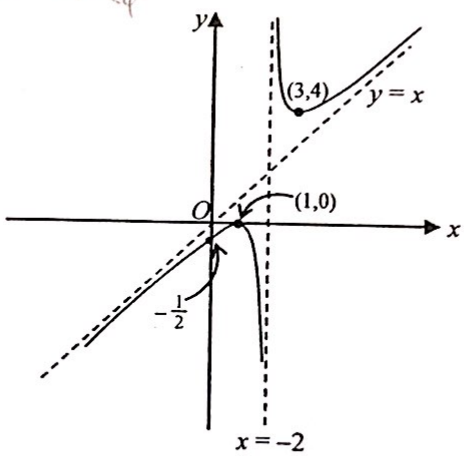
<u>Remarks</u> : There are actually 2 asymptotes ($y = x$ and $x = 2$). The vertical line to the right of y-axis appears like a vertical asymptote, but actually it is only connecting one point off the top of the screen with the next point off the bottom. The GC does not know if there is any undefined value between the values it is graphing. This is made more obvious in another window setting as shown below :



</td><td>

(ii) $y = x + \frac{1}{x-2}$

$y = -2 + \frac{1}{-4}$



↳ $x = 2$

</td></tr>
</table>

Example 14

A curve has parametric equations

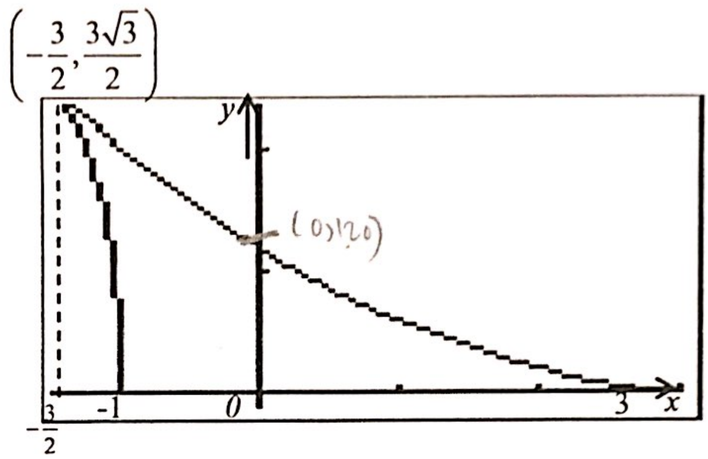
$$x = 2\cos\theta + \cos 2\theta, \quad y = 2\sin\theta - \sin 2\theta, \quad 0 \le \theta \le \pi.$$

(i) Sketch the curve.

(ii) Express x in the form $x = 2(\cos\theta + a)^2 + b$ where a and b are constants to be determined. Hence, find the least value of x and the value of θ when it occurs.

Solution

(i)



(ii)

$y = \frac{1}{2}$

$x = 2\cos\theta + 2\cos^2\theta - 1$

$= 2\left(\cos^2\theta + \cos\theta + \frac{1}{4}\right) - \frac{3}{2}$

$= 2\left(\cos\theta + \frac{1}{2}\right)^2 - \frac{3}{2}$

$\Downarrow$

$a = \frac{1}{2}, \ b = -\frac{3}{2}$

$x \geq -\frac{3}{2}$

$x = -\frac{3}{2}$

$\left(\cos\theta + \frac{1}{2}\right)^2 = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{3}$

when $\theta = 0$, $x = 3, y = 0$.

when $\theta = \pi$, $x = -1, y = 0$

let $x = 0$

$\Rightarrow 2\cos\theta + \cos 2\theta = 0$

$\Rightarrow 2\cos\theta + 2\cos^2\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{-2 \pm \sqrt{12}}{4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ (reject $\cos\theta = -\frac{1}{2} - \frac{\sqrt{3}}{2}$ (<-1))

$\theta = 1.196$

$x = 0, y = 1.20$

---

least value of x is -1.5 when $\theta = \cancel{\left(-\frac{\pi}{2}\right)}\ \frac{2\pi}{3}$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

to determine the coordinates of the sharp point at the 2nd quadrant.

when $\theta = \frac{2\pi}{3}$,

$y = 2\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{4\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{3\sqrt{3}}{2}$ //

CONCLUSION

Graphs have many practical uses in everyday life. For example, the exponential curve is useful in modelling the rate of growth of the population of a species, or the rate of decay of a radioactive substance. The applications of conics are related to their reflective properties and to planetary motion. For example, the reflectors on automobile headlights and dish antennas are parabolic, while the orbits of planets are elliptic.

It is useful to recognise the equations of basic functions and understand the properties of their graphs. When sketching graphs, one should take note of characteristics such as symmetry, intersections with the axes, turning points and asymptotes.

A graphing calculator is a useful tool in sketching graphs. However, it has certain limitations. For example, it cannot draw vertical asymptotes, and the graphs obtained are only an approximation. A GC also does not know what critical features of a graph to display – that is dependent on you.

SUMMARY

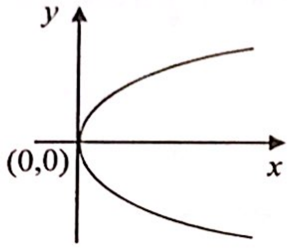
Appendix – Parabola

A parabola is a symmetrical open plane curve formed by the intersection of a cone with a plane parallel to its side. The path of a projectile under the influence of gravity follows a curve of this shape.

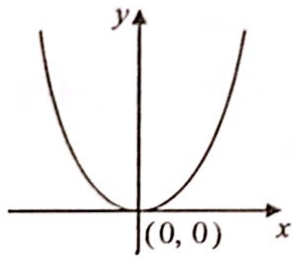
The equation of parabola with vertex (0,0) is given by

$$y^2 = cx \qquad \text{or} \qquad x^2 = cy.$$

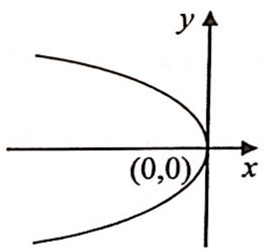
$c > 0$



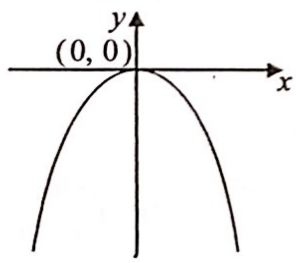
$c > 0$



$c < 0$



$c > 0$



This graph is symmetrical about x-axis.

This graph is symmetrical about y-axis.

The standard form of the equation of a parabola with vertex $(0,0)$ is given by $y^2 = cx$ or $x^2 = cy$ where $c, d \neq 0$.