2013 H1 Mathematics 8864/01

Section A

1. Find the set of values of *k* for which the equation $x^2-\left(k-2\right)x+\left(2k+1\right)=0$ has no real roots. [4]

$$x^2+(k-2)x+(2k+1)=0$$
$$D<0$$
$$(k-2)^2-4(1)(2k+1)<0$$
$$k^2-4k+4-8k-4<0$$
$$k^2-12k<0$$
$$k(k-12)<0$$

$$\{k\in\mathbb{R}:0<k<12\}$$

2. (i) Differentiate $\ln\left(1+2x^2\right)$. [2]

(ii) Use a non-calculator method to find the exact value of $\int_{-1}^{0}\frac{1}{\left(1-3x\right)^4}\,\mathrm{d}x$. [4]

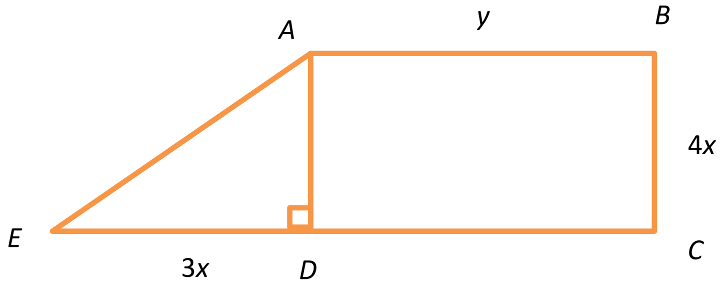
(i) $\frac{d}{dx}\ln\left(1+2x^2\right)$

$$=4x\times\frac{1}{1+2x^2}$$
$$=\frac{4x}{1+2x^2}$$

(ii)

$$\int_{-1}^{0}\frac{1}{\left(1-3x\right)^4}\,dx$$
$$=\int_{-1}^{0}\left(1-3x\right)^{-4}\,dx$$
$$=-\frac{1}{3}\int_{-1}^{0}(-3)\left(1-3x\right)^{-4}\,dx$$
$$=-\frac{1}{3}\left[\frac{\left(1-3x\right)^{-3}}{-3}\right]_{-1}^{0}$$
$$=\frac{1}{9}\left[1-\frac{1}{4^3}\right]$$
$$=\frac{7}{64}\qquad(0.109375)$$

3.



A piece of card has the shape of a trapezium *ABCE*. The point *D* on *CE* is such that *ABCD* is a rectangle. It is given that *AB* = *y* cm, BC = 4*x* cm and DE = 3*x* cm (see diagram). The area of the card is S cm^2. Given that the perimeter of the card is 20 cm,

(i) find an expression for S in terms of x, [3]

(ii) find the maximum value of S, justifying that this value is a maximum. [3]

(i)

By Pythagoras Theorem,

length of EA = $\sqrt{(3x)^2+(4x)^2}=5x$

$$\begin{aligned}20&=3x+2y+4x+5x\\&=12x+2y\\y&=10-6x\end{aligned}$$

$$\begin{aligned}S&=\frac{1}{2}(3x)(4x)+4xy\\&=6x^2+4xy\\&=6x^2+4x(10-6x)\\&=40x-18x^2\end{aligned}$$

$$\frac{dS}{dx}=40-36x$$

When S is maximum, $\frac{dS}{dx}=0$

Therefore,

$$40-36x=0$$

$$x=\frac{10}{9}$$

Using second derivative test,

$\frac{d^2S}{dx^2} = -36 < 0$ for all *x*.

When $x = \frac{10}{9}$, $\frac{d^2S}{dx^2} = -36 < 0$

Therefore, *S* is max when $x = \frac{10}{9}$.

$$\text{Max value of } S = 40\left(\frac{10}{9}\right) - 18\left(\frac{10}{9}\right)^2$$
$$= \frac{200}{9} = 22\frac{2}{9}\ \text{cm}^2$$

4. The curve C has equation $y = x^3 - ax^2 + 3x + 6$, where *a* is a constant.

(i) Find, in terms of *a*, the gradient of the normal to *C* at the point *P* where *x* = 1. [3]

[This part is not in syllabus]

The normal at P passes through the point (-5, 3).

(ii) Show that *a* satisfies the equation $a^2 - 10a + 24 = 0$ and hence find two possible values of *a*. [5]

(iii) For the smaller value of *a*, find the coordinates of the point of intersection of the normal at P and the line *y* = *x*. [2]

i) $y = x^3 - ax^2 + 3x + 6$

$\frac{dy}{dx} = 3x^2 - 2ax + 3$

At *P*, $x = 1$

$\frac{dy}{dx} = 3 - 2a + 3 = 6 - 2a$

Therefore, the gradient of the normal at $P = -\frac{1}{6-2a}$

ii)

When $x = 1$, $y = 1 - a + 3 + 6 = 10 - a$

At *P*, $x = -5$, $y = 3$

The normal passes through (1, 10 – *a*) and (-5, 3).

$$-\frac{1}{6-2a}=\frac{(10-a)-3}{1-(-5)}$$
$$-\frac{1}{6-2a}=\frac{7-a}{6}$$

$$(-7+a)(6-2a)=6$$
$$2a^2-20a+48=0$$
$$a^2-10a+24=0 \text{ (shown)}$$

$$a^2-10a+24=0$$
$$a=\frac{10\pm\sqrt{10^2-4(1)(24)}}{2}$$
$$a=4 \text{ or } a=6$$

iii) Smaller value of $a=4$,

Eqn of normal is :

Equation of the normal at *P*

$$y-3=-\frac{1}{(6-2a)}(x-(-5))$$
$$:\ y-3=-\frac{1}{(6-2\times4)}(x-(-5))$$
$$y=\frac{1}{2}x+\frac{11}{2}--------(1)$$
$$y=x-----(2)$$

Solving (1) and (2), *x* = 11 , *y* = 11

The point of intersection of the normal at *P* and the line *y* = *x* is (11, 11)

5 (i) By taking logarithms, find the exact root of the equation

$$e^{2-2x}=2e^{-2x}$$ [3]

(ii) Use differentiation to show that the curve *C* with equation $y=e^{2-2x}-2e^{-x}$ has a stationary point at ($2,-e^{-2}$). [3]

(iii) Sketch *C*, stating the exact value of the x-coordinate of its point of intersection with the *x*-axis.[2]

(iv) Use your calculator to find the area of the region bounded by *C*, the *x*-axis and the lines x = 0 and *x* = 1. [1]

i) $e^{2-2x} = 2e^{-x}$

$$\ln\left(e^{2-2x}\right) = \ln\left(2e^{-x}\right)$$
$$2-2x = \ln 2 + \ln e^{-x}$$
$$2-2x = \ln 2 - x$$
$$x = 2 - \ln 2$$

ii) $y = e^{2-2x} - 2e^{-x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2e^{2-2x} + 2e^{-x}$$

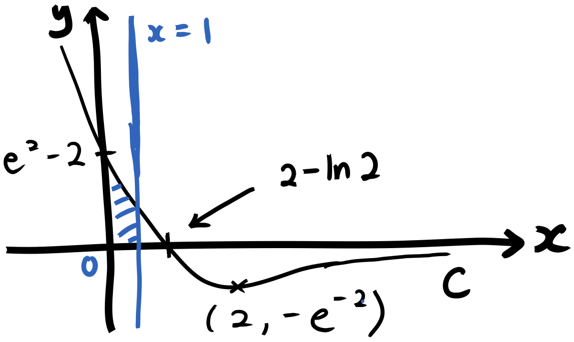
For a stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$-2e^{2-2x} + 2e^{-x} = 0$$
$$-2e^{-2x}\left(e^{2} - e^{x}\right) = 0$$
$$e^{-2x} = 0 \text{ (no solution) or } e^{2} - e^{x} = 0$$
$$\Rightarrow e^{2} = e^{x}$$
$$\Rightarrow x = 2$$
$$y = e^{2-2(2)} - 2e^{-2} = -e^{-2}$$
$$\text{Stat point is } (2, -e^{-2}).\text{(Shown)}$$

iii)
Intercepts :
When *x* = 0, *y* = $e^2 - 2$
When y = 0, $x = 2 - \ln 2$ (from (i))
Asymptotes:

As $x \to \infty, y = \frac{e^{2}}{e^{2x}} - \frac{2}{e^{x}} \to 0 - 0 = 0$

Stat point: $(2, -e^{-2})$ (from (ii)



iv)

Area of the region bounded by C = $\int_0^1 \left(e^{2-2x} - 2e^{-x}\right)dx = 1.93 \text{ units}^2 \text{(from GC)}$

Section B

[This part is not in syllabus]

6. Suky is organising a pop concert. She sells 5000 tickets at $\$X$ each, 10000 tickets at $\$Y$ each and 15000 tickets at $\$Z$ each. Suky wants to find out whether those who bought the tickets thought that the price they paid was good value for money. She decides to do this by choosing a stratified random sample of size 150.

(i) Describe how Suky might choose her sample. [3]

(ii) State one reason for using stratified random sampling in this context. [1]

i) Divide the 30000 ticket holders into three strata based on the price of the tickets – for $\$X$ each , $\$Y$ each and $\$Z$ each.

For \$X ticket holders, $\frac{5000}{30000} \times 150 = 25$

For \$Y ticket holders , $\frac{10000}{30000} \times 150 = 50$

For \$Z ticket holders, $\frac{15000}{30000} \times 150 = 75$

Within each stratum, use simple random sampling to choose her sample. For example, for the stratum \$X ticket holders, list them from 1 to 5000. Use a computer to generate 25 random numbers and select the ticket holders corresponding to the numbers generated. Repeat the process for the other two strata.

ii) Stratified sampling will ensure that the sample of ticket holders would be a good representative of the different groups of ticket holders with different ticket values. Suky can then analyse whether each group of ticket holders (for \$X each, for \$Y each and \$Z each) thought that the price they paid was good value for money.

7. A particular type of electronic device is being tested to determine for how long information stored in it is retained after power has been switched off. A random sample of 250 such devices is chosen and the time, T hours, for which information is retained is measured for each one. The results obtained are summarised as follows:

$\sum(t-75) = 305 \qquad \sum(t-75)^2 = 29555$

Find unbiased estimates of the population mean and variance. [3]

This type of device has previously been considered capable of retaining information for 75 hours, on

$$\bar{t} = \frac{\sum(t-75)}{250} + 75$$
$$= \frac{305}{250} + 75$$
$$= 76.22$$
$$s^2 = \frac{1}{250-1}\left(29555 - \frac{(305)^2}{250}\right)$$
$$= 117.2004016 \approx 117$$

Let X be the r.v. "retained time for the particular type of electronic device after which the device has been switched off" and μ be the population mean.

$H_0 : \mu = 75$

$H_1 : \mu > 75$

Under H_0, since $n = 250 > 50$ is large,

$$\bar{X} \sim N(75, \frac{117.2004}{250}) \text{ approximately by CLT}$$

p-value = 0.0374 > 0.025

Therefore, we do not reject H_0 and conclude that at 2.5% significance level, there is insufficient evidence that the retained time for the particular type of electronic device after which the device has been switched off is longer than 75 hours. Thus, the manufacturer's claim is not valid.

average, after power is switched off, but the manufacturers now claim that information is retained for longer than this. Test at the $2\frac{1}{2}\%$ significance level whether the claim is justified. [4]

8. A shop sells batteries in packs of 10. An advertiser claims that individual batteries each have a lifetime of at least 100 hours. The probability that an individual battery has a lifetime less than 100 hours is 0.2, independently of all other batteries.

(i) Find the probability that, in a randomly chosen pack of 10 batteries, each of the batteries satisfies the advertiser's claim. [1]

Customers are satisfied if at least 8 of the batteries in a pack have a lifetime of at least 100 hours.

(ii) Find the probability that a randomly chosen pack will satisfy customers. [3]

[This part is not in syllabus]

A customer buys a batch of 80 packs of these batteries.

(iii) Using a suitable approximation, estimate the probability that at least 75% of packs in the batch will satisfy the customer. State the mean and variance of the distribution that you use. [4]

i) Let *X* be the r.v. “no of batteries each have a lifetime of less than 100 hours out of 10 batteries”.

$X \sim \mathrm{B}(10, 0.8)$

$\mathrm{P}(X = 10) = 0.107$ (3 sig. fig.)

ii) P(a pack of 10 will satisfy customers)=

$$\mathrm{P}(X \geq 8) = 1 - \mathrm{P}(X \leq 7)$$
$$= 0.677799526 \approx 0.678$$

iii) Let *W* be the r.v. “no of packs of 10 batteries that will satisfy customers out of 80 packs”.

$W \sim \mathrm{B}(80, 0.67780)$

Since $n = 80 > 50, np = 54.2 > 5, nq = 25.8 > 5$

$W \sim \mathrm{N}(54.223, 17.471)$ approx.

75% of 80 packs = 60 packs

$\mathrm{P}(W \geq 60) \xrightarrow{c.c} P(T > 59.5) = 0.103$

[This part is not in syllabus for 2020]

9. The ages *x*, in years, and the heights *y*, in cm, for 10 boys are given in the following table.

Boy	*A*	*B*	C	D	E	F	G	H	I	J
x	8.2	10.1	6.6	13.5	6.8	11.4	7.8	6.9	12.8	7.5
y	123	135	119	141	112	151	122	116	141	123

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

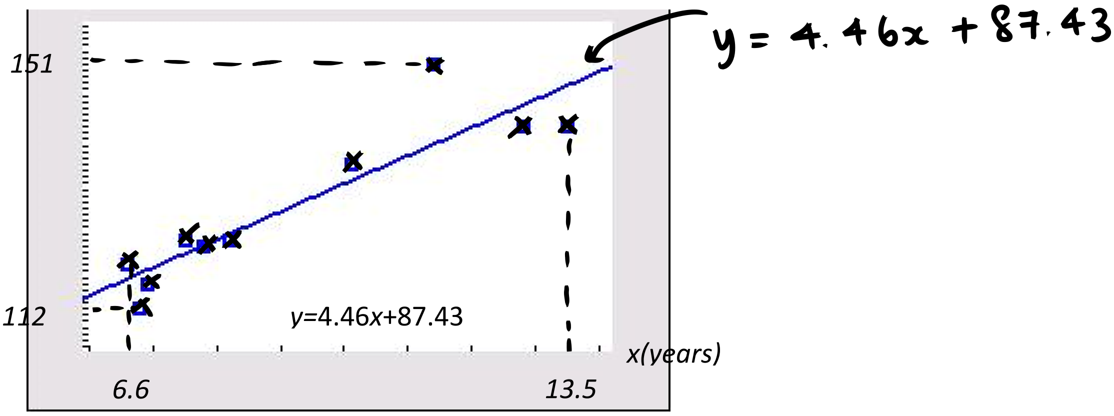
(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find the equation of the regression line of y on x, in the form y = mx + c, giving the values of m and c correct to 2 decimal places. Sketch this line on your scatter diagram. [2]

(iv) Use the equation of your regression line to calculate an estimate of the height of a boy whose age is 13.2 years and comment on the reliability of your estimate. [3]

(i)

y(cm)



ii) Product moment correlation coefficient , $r = 0.9032560806 \approx 0.903$

This value indicates that there is a strong positive linear correlation between the ages and the heights of the 10 boys. The height increases at a constant rate as the ages of the boys increases.

iii) The regression line of y on x is $y = 4.461675227x + 87.43105492 \approx 4.46x + 87.43$

(iv) For x = 13.2 , $y = 4.461675227(13.2) + 87.43105492 = 146.3251679 \approx 146$

The height of the boy whose age is 13.2 years is 146 cm.

Since r is close to 1 and the boy's age is within the data range of 6.6 to 13.5, the estimate is reliable.

10. A company producing barbecue sauce claims that the mass of a salt in a bottle of the sauce has a mean of 12g. The mass of salt is known to have a normal distribution with standard deviation 0.8 g. A random sample of 20 bottles is selected. The sample mean is *m* g. A test at the 5% significance level is carried out on this sample, and the company claim is accepted. Find the set of possible values of *m*. [5]

The company launches a new variety of the sauce and claims that the mean salt content per bottle has been reduced. The mass of salt in a random sample of 40 bottles of the new variety has a mean of 11.75 g. The mass of salt still has a normal distribution with standard deviation 0.8 g. Test the company's claim about the new variety of sauce, using a 5% significance level. [4]

Let X be the r.v. "the mass of salt in a bottle of the sauce" and μ be population mean.

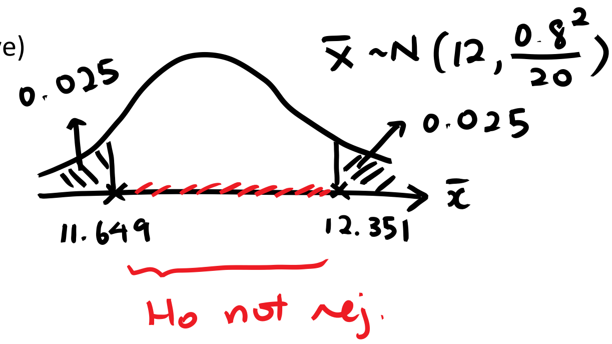
$$H_0 : \mu = 12$$
$$H_1 : \mu \neq 12$$

Under H_0, $\bar{X} \sim N(12, \frac{0.8^2}{20})$

$\bar{x} = m$

Since H_0 is not rejected at 5%,

Critical values : 11.649 and 12.351(refer to normal curve)



$11.7 < \bar{x} < 12.3$

$11.7 < m < 12.3$

Set of possible values of *m* : $\{m \in \mathbb{R} \mid 11.7 < \bar{x} < 12.3\}$

Let *Y* be the r.v. "the mass of salt in a bottle of the new variety of sauce" and μ be population mean.

$H_0 : \mu = 12$
$H_1 : \mu < 12$

Under H_0, $\bar{Y} \sim \text{N}(12, \frac{0.8^2}{40})$

$\bar{y} = 11.75$

p-value = 0.0241 < 0.05

Therefore H_0 is rejected. There is sufficient evidence to conclude that, at 5% significance level, the new variety of the sauce has reduced the mean salt content per bottle.

11. A pet shop sells two types of animal food. Type A is supplied by a manufacturer and sold in packets with the food content having a mean mass of 1 kg. The masses of the food content are normally distributed. It is known that 20% of the packets contain less than 990 g of food.

(i) Find the standard deviation of the distribution. [3]

Type B animal food is mixed by the shop owner from two ingredients P and Q. One packet contains 3 scoops of ingredient P and 2 scoops of ingredient Q. The masses, in grams, of the food in scoops of ingredients P and Q have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard Deviation
Ingredient P	240	10
Ingredient Q	145	8

(ii) Find the probability that a randomly selected packet of Type B has a mass of food less than 1kg. State the mean and variance of any distribution that you use. [4]

(iii) Find the probability that the mass of food in a randomly selected packet of Type B is more than the mass of food in a randomly selected packet of Type A. State the mean and variance of any distribution that you use. [4]

i) Let *A* be the r.v. "mass of the food content of Type A which is supplied by a manufacturer and sold in packets".

$$A \sim \mathrm{N}\left(1000, \sigma^2\right)$$

$$P\left(A < 990\right) = 0.2$$

$$P\left(Z < \frac{990-1000}{\sigma}\right) = 0.2$$

$$\frac{-10}{\sigma} = -0.8416212335$$

$$\sigma = 11.8818295 \approx 11.9$$

ii) Let *P* & *Q* be the r.v. "mass of ingredient P and ingredient Q" respectively.

$$P \sim \mathrm{N}\left(240, 10^2\right)$$

$$Q \sim \mathrm{N}\left(145, 8^2\right)$$

Let

$$B = P_1 + P_2 + P_3 + Q_1 + Q_2 \sim \mathrm{N}\left(3(240) + 2(145), 3\left(10^2\right) + 2\left(8^2\right)\right)$$

$$B \sim \mathrm{N}\left(1010, 428\right)$$

Mean = 1010

Variance = 428

$$P\left(B < 1000\right) = 0.3144171483 \approx 0.314$$

The probability that a randomly selected packet of Type B has a mass of food less than 1 kg is 0.314.

iii) $A \sim \mathrm{N}\left(1000, 141.1778723\right)$

$$B \sim \mathrm{N}\left(1010, 428\right)$$

$$B - A \sim \mathrm{N}\left(10, 569.1778723\right)$$

Mean = 10 Variance = 569.18

$$P\left(B > A\right) = P\left(B - A > 0\right)$$
$$= 0.6624490098 \approx 0.662$$

The probability that the mass of food in a randomly selected packet of Type B is more than the mass of food in a randomly selected packet of Type A is 0.662.

12. Jai is playing a game which involves throwing a fair six-sided die. If the result is a 3 , 4 , 5 or 6, his score is the number shown. If the result is a 1 or a 2, he throws the die a second time and score is the sum of the two numbers from his two throws.

(i) Draw a tree diagram to represent the possible outcomes. [3]

Events A and B are defined as follows:

Event A: Jai's score is 5 or 6,

Event B: Jai has two throws.

(ii) Show that $P(A) = \frac{4}{9}$. [2]

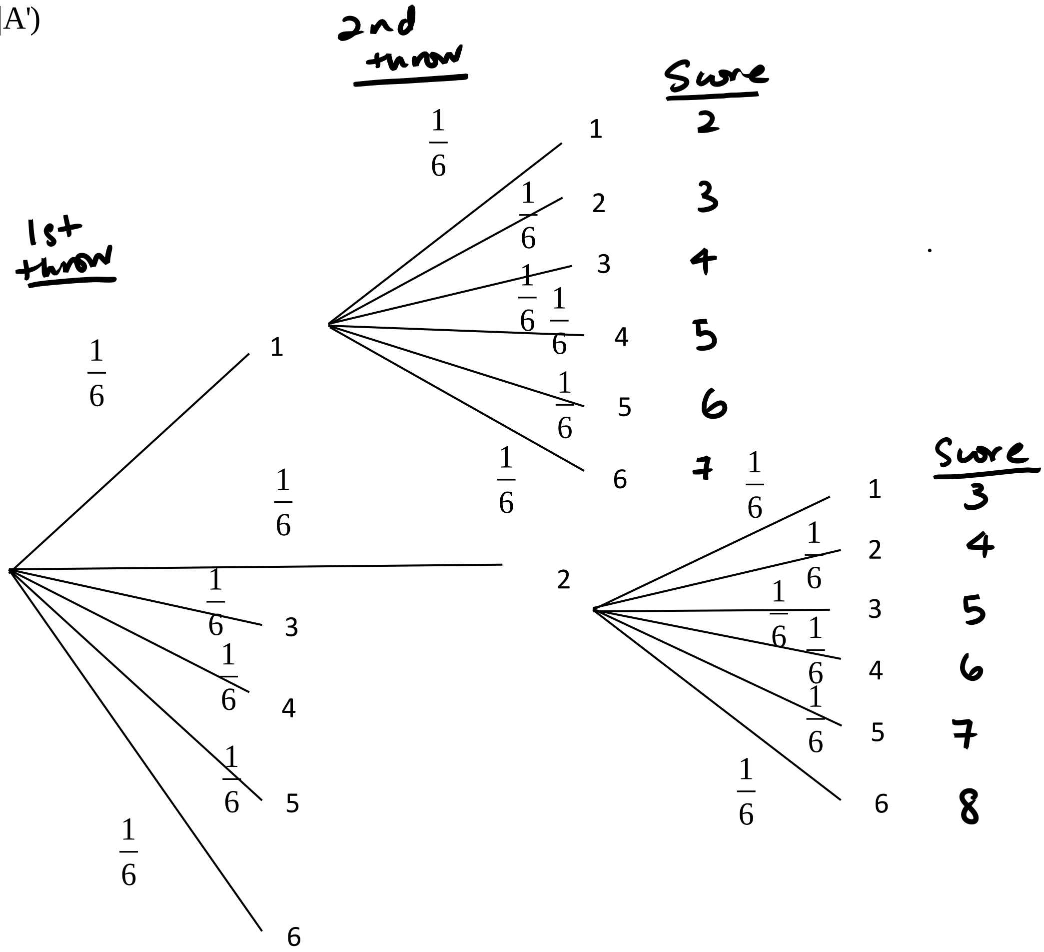
Find

(iii) $P(A \cap B)$ [1]

(iv) $P(A \cup B)$ [2]

(v) $P(B|A')$ [4]

i)



ii)

$$\mathrm{P(A)} = P(6) + P(1,4) + P(2,3) + P(6) + P(1,5) + P(2,4)$$

$$= \frac{1}{6}\times 2 + \frac{1}{6}\times\frac{1}{6}\times 2 + \frac{1}{6}\times\frac{1}{6}\times 2$$

$$= \frac{4}{9}\ \text{(Shown)}$$

iii)

$$\mathrm{P(A\cap B)} = P(1,4) + P(2,3) + P(1,5) + P(2,4)$$

$$= \left(\frac{1}{6}\times\frac{1}{6}\right)\times 4$$

$$= \frac{1}{9}$$

iv) $\mathrm{P(B)} = \frac{2}{6}$

$$\mathrm{P(A\cup B)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A\cap B)}$$

$$= \frac{4}{9} + \frac{2}{6} - \frac{1}{9} = \frac{2}{3}$$

v) $\mathrm{P(B|A')} = \dfrac{\mathrm{P(B\cap A')}}{\mathrm{P(A')}} = \dfrac{\mathrm{P(B)} - \mathrm{P(A\cap B)}}{1 - \mathrm{P(A)}}$

$$= \frac{\frac{2}{6} - \frac{1}{9}}{1 - \frac{4}{9}} = \frac{2}{5}$$