RAFFLES INSTITUTION

2023 YEAR 5 PROMOTION EXAMINATION

FURTHER MATHEMATICS

9649
September 2023
3 hours

Additional materials: List of Formulae (MF26)
Writing Paper

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 A sequence $u_0, u_1, u_2, \ldots$ is such that $u_n = 5^{An} + B\left(6^n\right)$ where A and B are constants and $n \geq 0$. Given that $u_1 = 79$ and $u_2 = 949$, find the values of A and B. [3]

~~Prove by induction that $10u_n - 2^n\left(3^{n+4}\right)$ is a multiple of 19 for all non-negative integers n.~~ ~~[5]~~

2 ~~The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where F_1 and F_2 are its foci. F_1 is at $(c, 0)$, F_2 is at $(-c, 0)$ where $a > c > 0$ and $b = \sqrt{a^2 - c^2}$.~~

(a) ~~Given that the point $P(x, y)$ lies on the ellipse, show that $PF_1 = \frac{a^2 - cx}{a}$ and hence $PF_1 + PF_2 = 2a$.~~ ~~[4]~~

(b) ~~Given that a point $Q(4, 3)$ is on the ellipse and $QF_1 \times QF_2 = k$, show that $k = a^2 - B + \frac{A}{a^2 - B}$ where A and B are constants to be determined.~~ ~~[4]~~

3 **(i)** Using the recurrence relation $x_{n+1} = \frac{1}{3}\ln(x_n + 1)$ with $x_1 = -0.8$, write down x_2, x_5 and x_8. Determine which root of the equation $3x - \ln(x+1) = 0$ does the sequence converge to. [2]

It is given that $\mathrm{f}(x) = 3x - \ln(x+1) - \varepsilon$ and ε is very close to zero. It is known, from graphical work, that the equation $\mathrm{f}(x) = 0$ has two single roots, $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

(ii) Explain why β is close to 0. [1]

(iii) The Newton-Raphson iterative formula can be written as $x_{n+1} = x_n - \mathrm{g}\left(x_n\right)$, where $\mathrm{g}(x) = \frac{\mathrm{f}(x)}{\mathrm{f}'(x)}$. Write down $\mathrm{g}(x)$ in this case. [1]

(iv) The Newton-Raphson method is used to find the value of β. Use two iterations of the Newton-Raphson method, with initial approximation $x_1 = 0$, to show that $\beta \approx \frac{\varepsilon}{2} - \frac{\varepsilon^2}{16}$, where terms in ε^3 and higher powers of ε have been ignored. [4]

4 A curve C is defined parametrically by

$$x = 3(\cos\theta + \theta\sin\theta) \quad \text{and} \quad y = 3(\sin\theta - \theta\cos\theta).$$

Find, giving your answers in terms of π.

(i) the exact length of the arc PQ, where P and Q are points on the curve when $\theta = 0$ and $\theta = \dfrac{\pi}{2}$ respectively, [4]

(ii) the exact area of the surface formed when the arc PQ is rotated completely about the x-axis. [5]

5 It is given that, for $n \geq 1$,

$$I_n = \int_0^a x^n e^{-x^2}\,dx.$$

(i) Find I_1 in terms of a. [1]

(ii) Show that $I_{n+2} = \dfrac{n+1}{2} I_n - \dfrac{1}{2}\left(e^{-a^2} a^{n+1}\right)$. [3]

(iii) Hence find I_5 in terms of a. [3]

(iv) Find the exact volume of solid of revolution formed when the region bounded between the graph of $y = x^4 e^{-x^2}$ and the positive x-axis is rotated through 2π radians about the y-axis. [3]
[You may assume that $a^n e^{-a^2} \to 0$ as $a \to \infty$ for any $n \in \mathbb{Z}^+$.]

6 **(i)** The point represented by $5 + 5i$ is a vertex of an equilateral triangle. Given that the vertices all lie on the locus of $|z - 2 - i| = 5$, find the complex number represented by the vertex that lies below the real axis, giving your answer in exact cartesian form $x + iy$. [3]

(ii) The complex number z satisfies

$$|z - 2 - i| \leq 5 \text{ and } |z - 2 - i| \leq |z^* - 7 - 3i|.$$

On an Argand diagram, sketch the region in which the point representing z can lie, and show that the exact area of the region is $A\left(\pi - \cos^{-1}\dfrac{\sqrt{B}}{10} + \dfrac{\sqrt{C}}{100}\right)$ where A, B and C are integers to be determined. [7]

[Turn over

7 The polar equations of the curves C_1 and C_2 are respectively given by

$$r = \frac{3}{2}, \quad 0 \le \theta \le \frac{\pi}{2}, \qquad \text{and} \qquad r = 1 + \sin 3\theta, \quad 0 \le \theta \le \frac{\pi}{2}.$$

(a) Sketch C_1 and C_2 and find the polar coordinates of the points where the curves intersect. [4]

S is the finite region between C_1 and C_2, for which $r > \frac{3}{2}$ and $r < 1 + \sin 3\theta$.

(b) Obtain the perimeter of S, giving your answer correct to 3 decimal places. [3]

(c) Find exactly the area of S. [4]

8 **(a)** Use de Moivre's theorem to find an expression for $\sin 5\theta$ in terms of power of $\sin\theta$.

Hence find $\sin^2 \frac{\pi}{5}$ in the form $a + b\sqrt{5}$ where a and b are to be determined.

Deduce $\sin^2 \frac{2\pi}{5}$ in the form of $c + d\sqrt{5}$ where c and d are to be determined. [7]

(b) Use de Moivre's theorem to find an expression for $\cos^6\theta$ in terms of $\cos 6\theta$, $\cos 4\theta$, and $\cos 2\theta$. Hence find the exact value of $\int_0^{\frac{\pi}{4}} \cos^6\theta \, \mathrm{d}\theta$. [5]

9 **(i)** Lee deposits $\$30\,000$ into an investment account at the beginning of a year. At the end of each year, he is awarded a dividend of 6% of the amount in the account during that year, and the dividend is added to the account. He needs to pay a fixed amount of yearly management fee, which is deducted from his account at the end of each year. The initial deposit amount in the investment account grows by 70% after 10 years. Find the total amount of management fees Lee pays in the 10 years. [5]

(ii) On the same day, Roy deposits the same initial amount into an account with another investment bank. The amount in the account at the end of the nth year is denoted by $\$u_n$. The sequence $u_0, u_1, u_2, \ldots$ follows the recurrence relation

$$u_0 = 30\,000,$$
$$u_1 = 30\,000b,$$
$$u_n = bu_{n-1} - \frac{1}{50}u_{n-2}$$

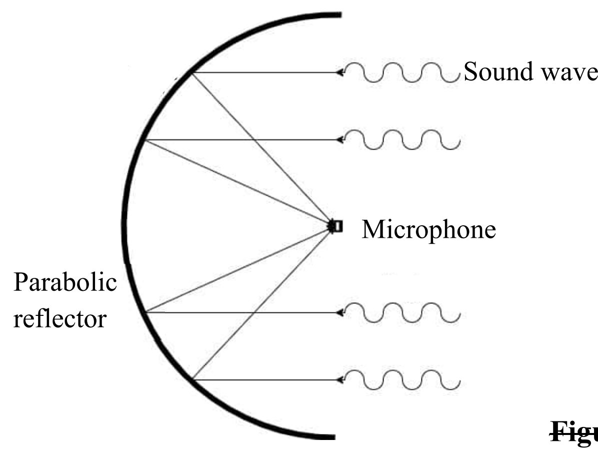
for b a positive constant and $n \ge 2$, $n \in \mathbb{Z}^+$.

(a) Find an expression for u_n in terms of b and n. [6]

(b) If Roy has the same amount in his account as Lee after 10 years, find the value of b correct to 5 significance figures. [1]

10 ~~Parabolic microphone dishes are used in American Football broadcasting to capture the sound of the players and the football. Parabolic microphone operators move along the sideline, typically following the line of scrimmage and point the long-ranging mic toward the action to capture the sound of the play.~~

~~As shown in **Figure 1**, the device consists of a parabolic reflector that collects incoming sound waves and focuses them onto a single point at which the microphone is positioned. The microphone then converts this into an electrical signal. The sound ends up being amplified because the sound energy from a large area is focused on a single point. In addition, the electrical signal from the microphone can also be amplified. The collected sound becomes part of the televised broadcast.~~



~~**Figure 1**~~

~~**Figure 2** shows a cross-sectional view of the parabolic microphone dish, superimposed on a set of coordinate axes so that it has equation $x^2 = 4ay$, where a is a positive constant. The dish measures 32 cm in diameter at the top and 8 cm deep in the centre. Take 1 unit to represent 1 cm on both axes. The microphone is placed at the focus of the parabola which is denoted by M.~~

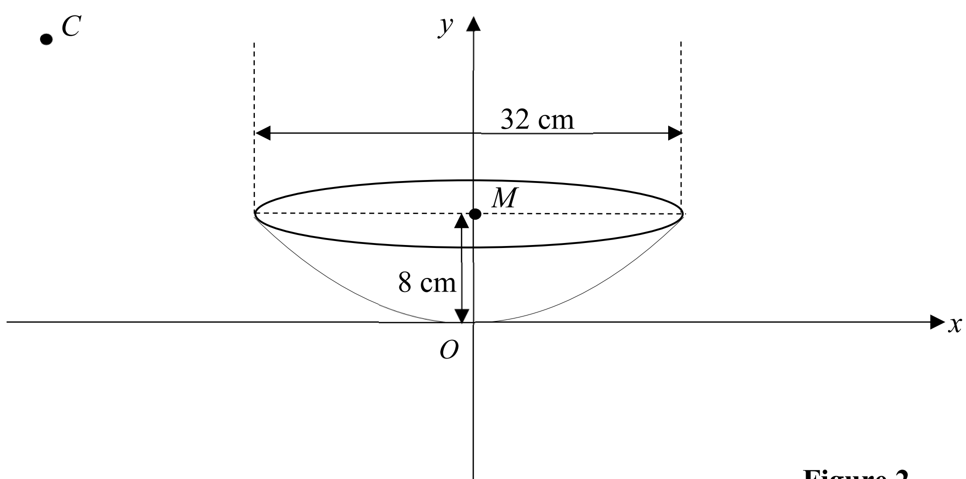


Figure 2

~~**[Turn over**~~

~~**(a)** Find the coordinates of focus, M and the equation of the directrix.~~ ~~[2]~~

~~**(b)** To capture the sound coming from the point $C(-300, 400)$, the parabolic dish is rotated anticlockwise about O. **Figure 3** shows a cross-sectional view of the rotated parabolic microphone dish. The vertex of the rotated parabola remains at O and OC is parallel to the axis of symmetry of the rotated parabola.~~

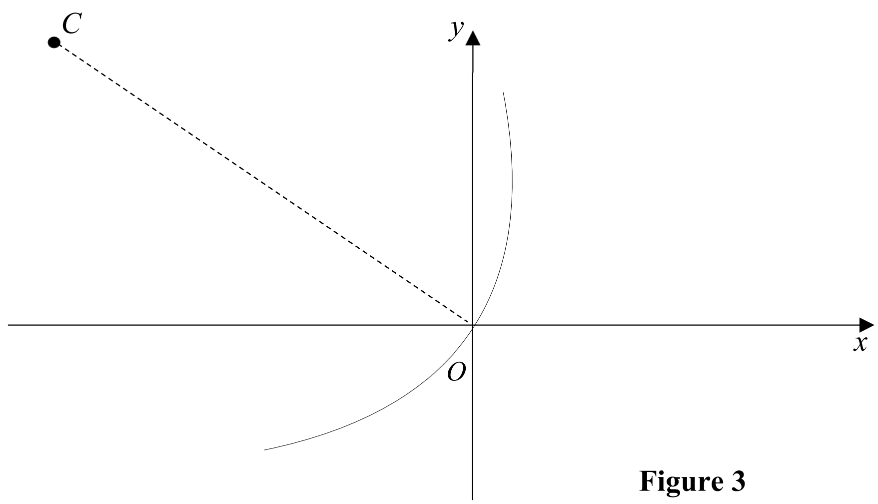


Figure 3

~~**(i)** By considering the polar equation of the parabola $x^2 = 4ay$ with the origin as the pole and the positive x-axis as the initial line, obtain the Cartesian equation of the rotated parabola in the form $x^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where B, C, D, E and F are constants to be determined.~~ ~~[5]~~

~~**(ii)** Find the coordinates of the focus and the equation of the directrix of the rotated parabola.~~ ~~[5]~~