National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 2)

This tutorial set is for the following sections from the notes:
§5 Real Vector Spaces
§6 Span, Linear Independence, Basis and Dimension
§7 Row Space, Column Space and Null Space

Basic Mastery Questions

1 Determine whether each of the following is a vector space. Justify your answer.

(i) $S=\{ax+b : ax+b=0 \text{ has no solution}\}$

(ii) $T=\{\mathbf{u}\in\mathbb{R}^3 : \mathbf{a}\times\mathbf{u}=\mathbf{0}\}$ where $\mathbf{a}$ is a fixed nonzero vector in $\mathbb{R}^3$,

(iii) $U=\left\{y : x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=0\right\}$.

(iv) Let V be the set of all functions $\mathrm{f}:\mathbb{R}\to\mathbb{R}$. We define addition and scalar multiplication on V as follows: For $\mathrm{f},\mathrm{g}\in V$ and $k\in\mathbb{R}$, $(\mathrm{f}+\mathrm{g})(x)=\mathrm{f}(x)+\mathrm{g}(x)$, $(k\mathrm{f})(x)=k\mathrm{f}(x)$.

(v) $\mathbb{R}^3$ with addition $\oplus$ and scalar multiplication $\otimes$ defined as

$$\mathbf{u}\oplus\mathbf{v}=\mathbf{0} \text{ for any } \mathbf{u},\mathbf{v}\in\mathbb{R}^3;$$

$$k\otimes\mathbf{u}=\mathbf{0} \text{ for any } \mathbf{u}\in\mathbb{R}^3,\ k\in\mathbb{R}.$$

(vi) $W=\left\{\text{all continuous functions } \mathrm{f}:[0,1]\to\mathbb{R} \text{ such that } \int_0^1 \mathrm{f}(x)\,\mathrm{d}x=0\right\}$.

2 Show that each of the following subsets of $\mathbb{R}^3$ is a vector space. Find also a basis for each of these subspaces.

(i) $U=\{(x,y,z) : x=t, y=-t, z=2t, t\in\mathbb{R}\}$.

(ii) $W=\{(x,y,z) : x-2y+3z=0\}$.

3 Show that each of the following subsets of $\mathbf{P}_2$ is a vector space. Find also a basis for each of these subspaces.

(i) $S=\{a+bx+2ax^2 : a,b\in\mathbb{R}\}$.

(ii) $T=\{\mathrm{f}(x)\in\mathbf{P}_2 : \mathrm{f}(x)=0 \text{ has two real roots 2 and } -1\}$.

4 Determine by observation whether each of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

(i) $\left\{\left(a,-3a,\sqrt{2}a\right):a\in\mathbb{R}\right\}$,

(ii) $\left\{(x,y,z):x^2+y^2+z^2=0\right\}$,

(iii) $\left\{(a,b,c):2a-b+5c=1\right\}$,

(iv) $\left\{(x,y,z):2x=y\right\}$,

(v) $\left\{(a,1,1):a\in\mathbb{R}\right\}$,

(vi) $\left\{(x,y,z):x=y=z\right\}$,

(vii) $\left\{(a,b,c):|a|+|b|+|c|<0\right\}$,

(viii) $\left\{(a,b,c):|a|=|b|+|c|\right\}$.

5 Prove that the set of vectors $\left\{(1,0,-1),(0,2,0),(1,1,1)\right\}$ is both a linearly independent and a spanning set of $\mathbb{R}^3$.

Determine, with reasons, whether each of the following sets of vectors forms a basis for $\mathbb{R}^3$ or not:

(i) $\left\{(1,0,-1),(0,2,0),(1,1,1)\right\}$,

(ii) $\left\{(1,0,-1),(0,2,0),(0,0,0)\right\}$,

(iii) $\left\{(1,0,-1),(0,2,0),(1,1,1),(0,1,0)\right\}$,

(iv) $\left\{(0,2,0),(1,1,1)\right\}$.

(1977 A Level / FM / Nov / P2)

6 **(a)** Let V and W be subspaces of the linear space $\mathbb{R}^n$. Show that the set of vectors of the form $\mathbf{v}+\mathbf{w}$, where $\mathbf{v}\in V$ and $\mathbf{w}\in W$, is a subspace of $\mathbb{R}^n$.

(b) For each of the following subsets of $\mathbb{R}^3$ determine whether or not it is a subspace, giving reasons for your answers. Find a basis for each subset which you consider to be a subspace.

(i) $\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:x_1=1\right\}$,

(ii) $\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:x_1=x_2=x_3\right\}$,

(iii) $\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:7x_1=x_2\right\}$,

(iv) $\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:{x_1}^2+{x_2}^2=0\right\}$.

(1978 A Level / FM / Jun / P2)

7 Let U and W be subspaces of a vector space V. Give a counterexample to show that $U \cup W$ need not be a subspace of V.

8 **(a)** Enlarge the following sets of linearly independent vectors to form bases.

(i) Enlarge $\{(1,2,1),(-1,2,1)\}$ to form a basis for $\mathbb{R}^3$.

(ii) Enlarge $\left\{\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ to form a basis for $\mathbf{M}_{2,2}(\mathbb{R})$.

(b) Reduce the following spanning sets to form bases.

(i) Reduce $\{1+x, 2+x^2, 2x+x^2, 1+x+x^2\}$ to form a basis for $\mathbf{P}_2$.

(ii) Reduce $\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ to form a basis for $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - 2y + z = 0\right\}$.

Practice Questions

9 Let $\mathbf{A}$ be a fixed 2×2 matrix, and let $W = \{\mathbf{X} \in \mathbf{M}_{2,2}(\mathbb{R}) : \mathbf{AX} = \mathbf{XA}\}$. Prove that W is a subspace of $\mathbf{M}_{2,2}(\mathbb{R})$.

10 **(a)** Explain a subspace of $\mathbb{R}^2$ can <u>only</u> be $\{\mathbf{0}\}$, a line through the origin, or $\mathbb{R}^2$.

(b) Explain a subspace of $\mathbb{R}^3$ can <u>only</u> be $\{\mathbf{0}\}$, a line through the origin, a plane through the origin, or $\mathbb{R}^3$.

11 Given that $\left\{\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \begin{pmatrix} b \\ 1 \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}\right\}$ is not a basis for $\mathbb{R}^3$, prove that $a^3 - 3ab + b^3 + 1 = 0$.

(1984 A Level / FM / Jun / P2)

12 Find a basis for the vector space spanned by the vectors

$$(1,2,-1),\ (3,-1,2),\ (2,-10,8),\ (7,-7,8).$$

What is the dimension of this space?

For what value (or values) of a does $(1,4,a)$ belong to the space?

(1976 A Level / FM / Jun / P2)

13 The vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ form a basis for a vector space V, and

$$\mathbf{a} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$$

is a vector in V, $\alpha_1, \alpha_2, \alpha_3$ being scalars. Obtain a necessary and sufficient condition that the vectors $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent.

A vector $\mathbf{x}$ in V is such that

$$\mathbf{x} = \lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3,$$

where scalars $\lambda_1, \lambda_2, \lambda_3$ are positive. If $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent and $\mathbf{x}$ is a linear combination of these vectors with nonnegative scalar coefficients, show that $\alpha_1 > 0$.

Show, by a counterexample, that if $\alpha_1 > 0$, then the coefficients expressing $\mathbf{x}$ as a linear combination of $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ need not be nonnegative.

(1975 A Level / FM / Nov / P2)

14 Let V and W be subspaces of $\mathbb{R}^n$. Show that $V \cap W$ is also a subspace of $\mathbb{R}^n$. Find a basis for $V \cap W$ in the following cases:

(i) $V = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0\right\}$, $W = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 = 2x_2\right\}$.

(ii) $V = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 = 0\right\}$, $W = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 7x_4 = 3x_2\right\}$.

(iii) $V = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{Ax} = \mathbf{0}, \text{ where } \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\right\}$,

$W = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{Bx} = \mathbf{0}, \text{ where } \mathbf{B} = \begin{pmatrix} 2 & 1 & -7 \\ -4 & -2 & 14 \end{pmatrix}\right\}$.

(1979 A Level / FM / Jun / P1)

15 The set $\mathbf{P}_2$ consists of all polynomials in x, of degrees less than or equal to 2, and having real coefficients, i.e. $\mathbf{P}_2 = \left\{ax^2 + bx + c : a, b, c \in \mathbb{R}\right\}$.

Show that , with usual operations of addition and multiplication by a real number, $\mathbf{P}_2$ is a linear (vector) space over $\mathbb{R}$, of dimension 3.

For each of the following subsets of $\mathbf{P}_2$, determine whether or not it is a subspace, giving brief reasons for your answers. Give a basis for each subset which you consider to be a subspace.

(a) $\left\{\mathrm{f}(x) \in \mathbf{P}_2 : \mathrm{f}(0) = 0\right\}$,

(b) $\left\{\mathrm{f}(x) \in \mathbf{P}_2 : \mathrm{f}(0) = 1\right\}$,

(c) $\left\{\mathrm{f}(x) \in \mathbf{P}_2 : \mathrm{f}(1) = 0\right\}$,

(d) $\left\{\mathrm{f}(x) \in \mathbf{P}_2 : \mathrm{f}(-x) = \mathrm{f}(x) \text{ for all } x \in \mathbb{R}\right\}$.

(1980 A Level / FM / Jun / P1)

16 In each of the following cases, state whether, with usual operations, the given set forms a linear space over the field $\mathbb{R}$. For each of those which you consider to be a linear space, give a basis for the space, and for each of those which you consider not to be a linear space, justify your answer.

(a) $S_1 = \left\{ \text{solutions of } \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0 \right\}$.

(b) $S_2 = \left\{ \text{solutions of } \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x - 3 \right\}$.

(c) $S_3 = \{(x, y) : y = 2x + 1\}$.

(d) $S_4 = \{(x, y, z) : x + y + z = 0\}$.

(d) $S_5 = \{z : z = a + b\mathrm{i}, a, b \in \mathbb{R}\}$ (i.e. $S_5 = \{z \in \mathbb{C}\}$).

(1982 A Level / FM / Nov / P1)

17 S is a subspace of $\mathbb{R}^4$ spanned by vectors

$$\{(1,0,1,1), (5,0,2,2), (-2,0,1,1), (3,0,0,0)\},$$

and T is a subspace of $\mathbb{R}^4$ spanned by vectors

$$\{(1,-1,0,1), (0,0,0,0), (0,1,1,0), (0,3,3,0)\}.$$

Let $S + T$ denote the set of vectors of the form $\mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Find the bases for the subspaces S, T, $S \cap T$ and $S + T$.

Find also a basis of a subspace U of $\mathbb{R}^4$, which is such that $S \cap U = \{\mathbf{0}\}$ and $S + U = \mathbb{R}^4$.

(1979 A Level / FM / Nov / P1)

Application Problems

18 (Solution to a Higher-Order Linear Differential Equation)

Let V be the set of solutions of the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 .$$

(i) Show that V is a vector space by verifying the axioms.

(ii) State a set of 3 linearly independent vectors in V.

(iii) It can be assumed that $\dim(V) = 3$. Find the general solution to this differential equation.

(iv) Let W be the set solutions passing through $(\pi, 0)$. Explain whether W is a vector space.

(v) Deduce the general solution of the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = 6 .$$

19 (Relationship between Two Planes)

Consider the matrix equation $\mathbf{Au} = \mathbf{b}$ corresponds to the following system of two linear equations and three unknowns:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \end{aligned} \quad (*).$$

Let the rank of its coefficient matrix be r and the rank of its augmented matrix be q. It is assumed that neither row of the coefficient matrix contains only 0.

(i) Find all the possible values for the ordered pair (r, q).

(ii) Geometrically, each equation in (*) represents a plane in 3-dimensional space.

What can you say about the relationship between the two planes for different values of (r, q)?

20 (Relationship among Three Planes)

Consider the system of three linear equations and three unknowns:

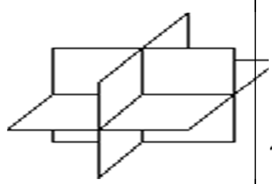
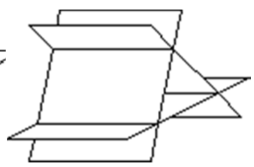
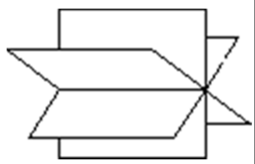
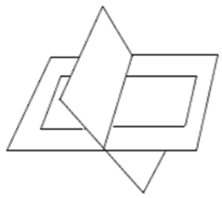
$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \quad (**) \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

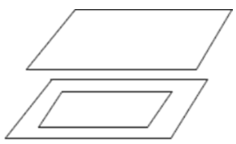
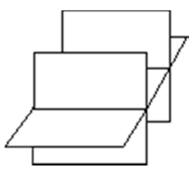
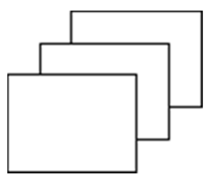
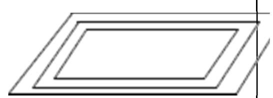
Let the rank of its coefficient matrix be r and the rank of its augmented matrix be q. It is assumed that no row in the coefficient matrix contains only 0's.

(i) Find all the possible values for the ordered pair (r, q).

(ii) Geometrically, the equations in (**) represents three planes p_1, p_2 and p_3 in 3-dimensional space.

For each of diagrams below, find the corresponding values of (r, q), and explain how you can further distinguish between two diagrams corresponding to the same pair of (r, q) algebraically.

The three planes intersect at exactly one point.	Each plane intersect the other two in a line each. The three planes form a prismatic surface.	The three planes intersect in a common line.	Two of the planes coincide and the other intersect them in a line.
 Diagram (I)	 Diagram (II)	 Diagram (III)	 Diagram (IV)

Two of the planes coincide and the other is parallel to them.	Two of the planes are parallel and the other intersects each in a line.	The three planes are parallel.	The three planes coincide.
 Diagram (V)	 Diagram (VI)	 Diagram (VII)	 Diagram (VIII)

Consider each of the planes as a set of points, what can you say about $p_1 \cap p_2 \cap p_3$ in each of the diagrams?

Numerical Answers

Basic Mastery Questions

1 **(i)** No **(ii)** Yes **(iii)** Yes **(iv)** Yes **(v)** No **(vi)** Yes.

2 **(i)** $\{(1,-1,2)\}$. **(ii)** $\{(2,1,0),(-3,0,1)\}$.

3 **(i)** $\{x, 1+2x^2\}$ **(ii)** $\{x^2-x-2\}$

4 **(i)** Yes **(ii)** Yes **(iii)** No **(iv)** Yes
(v) No **(vi)** Yes **(vii)** No **(viii)** No

5 **(i)** Yes **(ii)** No **(iii)** No **(iv)** No

6 **(b)** **(i)** No **(ii)** Yes **(iii)** Yes **(iv)** Yes

8 **(a)** **(i)** $\{(1,2,1),(-1,2,1),(0,1,0)\}$. **(ii)** $\left\{\begin{pmatrix}1&3\\1&0\end{pmatrix},\begin{pmatrix}1&3\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$.

(b) **(i)** $\{1+x, 2+x^2, 2x+x^2\}$. **(ii)** $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

Practice Questions

12 dimension is 2. $\{(1,2,-1),(3,-1,2)\}$.

14 **(i)** $\{(0,1,0,2)\}$, **(ii)** $\{(14,-7,0,-6),(0,0,1,0)\}$, **(iii)** $\left\{\begin{pmatrix}5\\-3\\1\end{pmatrix}\right\}$.

15 (Do not forget to prove that the dimension is 3)
(i) $\{x^2, x\}$, **(ii)** not a vector space, **(iii)** $\{x^2-1, x-1\}$, **(iv)** $\{x^2, 1\}$.

16 **(i)** $\{e^x, e^{2x}\}$. **(ii)(iii)** not a vector space. **(iv)** $\{(-1,1,0),(-1,0,1)\}$. **(v)** $\{1, \mathbf{i}\}$.

17 A basis for S is $\{(1,0,1,1),(5,0,2,2)\}$, a basis for T is $\{(1,-1,0,1),(0,1,1,0)\}$.
a basis for $S\cap T$ is $\{(1,0,1,1)\}$, a basis for $S+T$ is $\{(1,0,1,1),(5,0,2,2),(1,-1,0,1)\}$.
a basis for U is $\{(0,1,0,0),(0,0,1,0)\}$.

Application Problems

18 **(ii)** $\{1, \cos x, \sin x\}$. **(iii)** $y = C_1 + C_2\cos x + C_3\sin x$. **(v)** $y = 6x + C_1 + C_2\cos x + C_3\sin x$.

19 **(i)** $(2,2)$, $(1,2)$, $(1,1)$.

20 **(i)** $(3,3)$, $(2,3)$, $(2,2)$, $(1,2)$, $(1,1)$.