CHIJ ST. THERESA'S CONVENT
PRELIMINARY EXAMINATION 2024
SECONDARY 4 EXPRESS / 5 NORMAL (ACADEMIC)

CANDIDATE NAME	

CLASS		INDEX NUMBER	

ADDITIONAL MATHEMATICS **4049/2**

Paper 2 **23 Aug 2024**
2 hours 15 minutes

Candidates answer on the Question Paper as well as on the graph paper provided.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 The equation of a curve is $y = 5\sin^2\left(x - \frac{\pi}{6}\right)$, where $0 \le x \le \frac{\pi}{2}$.

(a) Given that y is decreasing at a rate of 0.3 units per second, find the rate of change of x at $x = \frac{5\pi}{12}$. [3]

$y = 5\sin^2\left(x - \frac{\pi}{6}\right)$	
$\frac{dy}{dx} = 10\sin\left(x - \frac{\pi}{6}\right)\cos\left(x - \frac{\pi}{6}\right)$	**A1 –** Correct derivative
At $x = \frac{5\pi}{12}$,	
$\frac{dy}{dx} = 10\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 5$	
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$	**M1 –** Correct use of connected rates of change with substitution
$-0.3 = 5 \times \frac{dx}{dt}$	
$\frac{dx}{dt} = -0.06$ units/s	**A1**

(b) The normal to the curve at $x = \frac{5\pi}{12}$ intersects the vertical axis at $(0, k)$.
Find the exact value of k. [3]

Gradient of normal $= -\frac{1}{5}$	**FTA1 –** Correct gradient of normal
At $x = \frac{5\pi}{12}$, $y = 5\sin^2\left(\frac{\pi}{4}\right) = \frac{5}{2}$	**M1 –** for finding value of y
$\therefore y - \frac{5}{2} = -\frac{1}{5}\left(x - \frac{5\pi}{12}\right)$	
$y = -\frac{1}{5}x + \frac{5}{2} + \frac{\pi}{12}$	
At horizontal axis, $x = 0$ $\therefore k = \frac{5}{2} + \frac{\pi}{12}$	**A1**

2 **(a)** Find the values of *x* and *y* which satisfy the equations

$$8^x - 2^{-y} = 0,$$
$$\left(\sqrt{125^x}\right)^y = \frac{1}{\sqrt{5}}.$$

[3]

$8^x - 2^{-y} = 0$

$2^{3x} = 2^{-y}$

$y = -3x$

and

$\left(\sqrt{125^x}\right)^y = \frac{1}{\sqrt{5}}$

$5^{\frac{3xy}{2}} = 5^{-\frac{1}{2}}$

$\frac{3xy}{2} = -\frac{1}{2}$

M1 – Correct simplification of equations

$3xy = -1$

Hence, solving simultaneous equations

M1 – Eliminating one variable using substitution

$3x(-3x) = -1$

$9x = 1$

$x = \frac{1}{3} \quad or \quad -\frac{1}{3}$

$y = -1 \quad or \quad 1$

A1

When $x = \frac{1}{3}, \quad y = -1$, $x = -\frac{1}{3}, \quad y = 1$

(b) Show that the equation $3\left(2^{x+2}\right)-1=35\left(2^{-x}\right)$ has only one solution and find its value correct to 2 significant figures. [5]

$3\left(2^{x+2}\right)-1=35\left(2^{-x}\right)$

$12\left(2^{x}\right)-1=\dfrac{35}{2^{x}}$

let $y=2^{x}$

$12y-1=\dfrac{35}{y}$

M1 – Correct substitution to obtain a quadratic equation

$12y^{2}-y-35=0$

$(4y-7)(3y+5)=0$

M1 – Factorisation

$y=\dfrac{7}{4} \quad or \quad y=-\dfrac{5}{3}$

$2^{x}=\dfrac{7}{4} \quad or \quad 2^{x}=-\dfrac{5}{3}\,(rej)$

A1

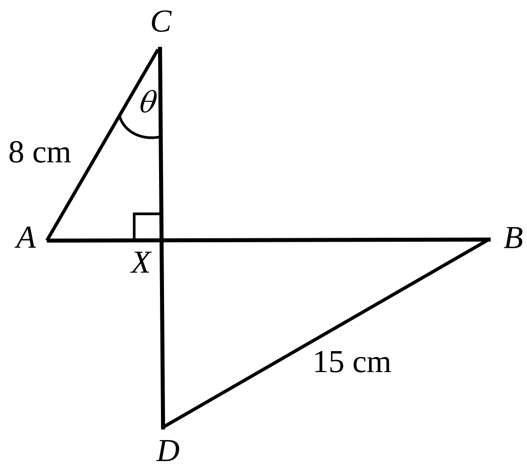
$x\lg 2=\lg\dfrac{7}{4}$

M1 – Using logarithm to solve

$x=\dfrac{\lg\frac{7}{4}}{\lg 2}=0.81$

A1 – Answer to 2s.f.

3



The diagram shows two perpendicular lines *AB* and *CD* which intersect at *X*. The points *A*, *B*, *C* and *D* lie on the circumference of a circle. *AC* = 8 cm, *BD* = 15 cm, and angle *ACD* equals to $\theta°$.

(a) Show that the length of *AB* is $8\sin\theta + 15\cos\theta$. [2]

As *A*, *B*, *C* and *D* lie on the circumference of a circle,

Angle *ABD* = $\theta°$

$AX = 8\sin\theta$

$XB = 15\cos\theta$ **M1**

$\therefore AB = AX + XB = 8\sin\theta + 15\cos\theta \quad (\textit{shown})$ **AG1**

(b) Express *AB* in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0° \le \alpha \le 90°$. [4]

$R = \sqrt{8^2 + 15^2} = 17$ **M1, A1**

$\alpha = \tan^{-1}\frac{15}{8} = 61.9°$ **M1**

$\therefore 8\sin\theta + 15\cos\theta = 17\sin(\theta + 61.9°)$ **A1**

(c) Find the value(s) of θ if AB = 16 cm. [3]

$17\sin\left(\theta + 61.9275°\right) = 16$

$\sin\left(\theta + 61.9275°\right) = \frac{16}{17}$

$\alpha = \sin^{-1}\frac{16}{17} = 70.250°$ **M1 –** Finding basic angle

$\theta + 61.9275° = 70.250° \quad or \quad 180° - 70.250°$

$\theta = 8.3226° \quad or \quad 47.8225°$

$\theta = 8.3° \quad or \quad 47.8°$ **A1, A1**

4 A calculator must not be used in this question.

It is given that $\frac{\cos(A+B)}{\cos(A-B)}=\frac{1}{3}$.

(a) Show that $\tan A\tan B=\frac{1}{2}$. [3]

$\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B+\sin A\sin B}=\frac{1}{3}$	**M1 –** Correct use of addition formulae
$3\cos A\cos B-3\sin A\sin B=\cos A\cos B+\sin A\sin B$	
$2\cos A\cos B=4\sin A\sin B$	**M1 –** Cross multiplying and simplifying
$\frac{\sin A\sin B}{\cos A\cos B}=\frac{2}{4}$	**M1 –** Dividing by cos*A*cos*B* to obtain expression
$\tan A\tan B=\frac{1}{2}$	

(b) If $\tan A=2+\sqrt{3}$, find an expression for $\tan B$, in the form $a+b\sqrt{3}$, where a and b are constants. [3]

$\left(2+\sqrt{3}\right)\tan B=\frac{1}{2}$	
$\tan B=\frac{1}{2\left(2+\sqrt{3}\right)}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$	**M1 –** Rationalising denominator
$=\frac{2-\sqrt{3}}{2(4-3)}$	**M1 –** Correct use of difference of two squares to simplify denominator
$=\frac{2-\sqrt{3}}{2}$	
$=1-\frac{1}{2}\sqrt{3}$	**A1**

(c) Hence, express $\sec^2 B$ in the form $c + d\sqrt{3}$, where c and d are constants. [3]

$\tan^2 B$

$= \left(1 - \frac{1}{2}\sqrt{3}\right)^2$ **FTB1 –** Correct squaring of tan B

$= 1 - \sqrt{3} + \frac{3}{4} = \frac{7}{4} - \sqrt{3}$

$\sec^2 B = 1 + \tan^2 B = \frac{11}{4} - \sqrt{3}$ **M1 A1**

5 The equation of a circle is $x^2 + y^2 - 6x + 16y + 48 = 0$.

(a) Find the radius and coordinates of the centre of the circle. [4]

$\left(x^2 - 6x\right) + \left(y^2 + 16y\right) + 48 = 0$	**M1 –** Use of completing the square
$(x-3)^2 - 9 + (y+8)^2 - 64 + 48 = 0$	**A1 –** for correct $(x-3)^2$
$(x-3)^2 + (y+8)^2 = 25 = 5^2$	**A1 –** for correct $(y+8)^2$
Radius = 5 units	
Coordinates of center = $(3,-8)$	**FTA1**
OR	
$2g = -6$ and $2f = 16$	
$-g = 3$ and $-f = -8$	
$\therefore C(3,-8)$	**B1, B1**
$r = \sqrt{3^2 + (-8)^2 - 48} = 5$	**M1, A1**

(b) The point $A(0,-4)$ lies on the circle.
Given that *AB* is a diameter of the circle, find the coordinates of *B*. [2]

Let $B(x,y)$. As centre is the midpoint of a diameter	
$3 = \dfrac{x+(0)}{2}$	
$x = 6$	**M1 –** Using midpoint or proportions
$-8 = \dfrac{y+(-4)}{2}$	
$y = -12$	
$\therefore B(6,-12)$	**A1**

(c) A line with equation $y = mx$, where $m > 0$, does not intersect the circle. A is the point on the circle closest to the line.

(i) Find the value of m. [2]

Equation of line passing through A and centre of circle:

$gradient = \dfrac{-4-(-8)}{0-3} = \dfrac{4}{-3}$ — **M1 –** for finding equation of line passing through the centre

$y+4 = -\dfrac{4}{3}x$

$y = -\dfrac{4}{3}x - 4$

$\therefore m = \dfrac{3}{4}$ — **A1**

(ii) Hence, find the coordinates of the point on the line that is closest to the circle. [2]

$\dfrac{3}{4}x = -\dfrac{4}{3}x - 4$ — **M1** – Equating the lines to solve

$x = -\dfrac{48}{25}$

$y = \dfrac{3}{4}\left(-\dfrac{48}{25}\right) = -\dfrac{36}{25}$

$\therefore \left(-\dfrac{48}{25}, -\dfrac{36}{25}\right)$ — **A1**

6 It is given that $f(x) = 4x^p + qx^2 - 3x + 1$, where p and q are constants, has a factor of $x-1$ and leaves a remainder of -33 when divided by $x+2$.

(a) Find the values of p and q. [3]

Using factor theorem,	
$0 = 4(1)^p + q - 3 + 1$	
$0 = 4 + q - 3 + 1$	**M1** – Correct use of factor theorem
$q = -2$	
Using remainder theorem,	**M1** – Correct use of remainder theorem
$-33 = 4(-2)^p + 4(-2) + 6 + 1$	
$-32 = 4(-2)^p$	
$-8 = (-2)^p$	**A1**
$p = 3$	

(b) Using the values of p and q found in part **(a)**, solve the equation $f(x) = 0$ completely, leaving non-integer roots in their simplest surd form. [4]

Using factor theorem,	
$4x^3 - 2x^2 - 3x + 1 = 0$	**M1** – Correct Factorisation
$(x-1)(4x^2 + 2x - 1) = 0$	**M1** – Correct use of quadratic formula
$4x^2 + 2x - 1 = 0$	
$x = \dfrac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$	**A1, A1**
$\therefore x = 1$ *or* $= \dfrac{-2 \pm \sqrt{20}}{8} = \dfrac{-2 \pm 2\sqrt{5}}{8}$	
$= \dfrac{-1+\sqrt{5}}{4}$ *or* $\dfrac{-1-\sqrt{5}}{4}$	

Name: ________________________________ Class: ________ (   )

7 **(a)** Prove the identity $\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}=-2\cot 2\theta$. [3]

$LHS=\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}$

$=\frac{-\left(2\cos^2\theta-1\right)}{\frac{1}{2}\left(2\sin\theta\cos\theta\right)}$

$=\frac{-\cos 2\theta}{\frac{1}{2}\sin 2\theta}$

$=\frac{-2\cos 2\theta}{\sin 2\theta}$

$=-2\cot 2\theta = RHS \quad (shown)$

M1 – Correct use of double angle formulae for $\sin 2\theta$

M1 – Correct use of double angle formulae for $\cos 2\theta$

M1 – Correct use of $\cot\theta=\frac{\cos\theta}{\sin\theta}$

(b) Hence, solve the equation $\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}+\tan 2\theta+1=0$, for $0°\le\theta\le 90°$. [5]

$-2\cot 2\theta+\tan 2\theta+1=0$

$\frac{-2}{\tan 2\theta}+\tan 2\theta+1=0$

$\left(\tan 2\theta\right)^2+\tan 2\theta-2=0$

$\left(\tan 2\theta+2\right)\left(\tan 2\theta-1\right)=0$

$\tan 2\theta=-2$	*or*	$\tan 2\theta=1$
Basic Angle α	*or*	Basic Angle α
$=\tan^{-1}2$	*or*	$=\tan^{-1}1$
$=63.435°$	*or*	$=45°$
$2\theta=116.565°, 296.565°$	*or*	$2\theta=45°, 225°$
$\theta=58.3°, 148.3°(rej)$	*or*	$\theta=22.5°, 112.5°(rej)$

$\therefore \theta=22.5°, 58.3°.$

M1 – Correctly forming quadratic equation

M1 – Correctly solving quadratic equation

M1 – Correctly finding basic angle for $\tan 2\theta=-2$

A1, A1

8 The blood alcohol concentration, C mg/L, in a person t minutes after he consumes a bottle of wine can be modelled by the formula

$$C = 1250\left(e^{kt} - e^{-0.1t} \right).$$

(a) In Singapore, a driver can be charged with drink driving if he drives when his blood alcohol concentration exceeds 800mg/L. When Jonathan consumes a bottle of wine, his blood alcohol concentration will only fall to 800mg/L after three hours.

Show that $k = -0.0025$ when corrected to 2 significant figures, and find his blood alcohol concentration after 1 hour. [4]

$800 = 1250\left(e^{180k} - e^{-0.1(180)}\right)$	
$\frac{800}{1250} = e^{180k} - e^{-18}$	**M1** – Separating constants and variable
$e^{180k} = \frac{800}{1250} + e^{-18}$	**M1 –** Correctly using logarithms to solve for k
$180k = \ln\left(\frac{800}{1250} + e^{-18}\right)$	
$k = -0.0024793$	**AG1**
When $t = 60$,	
$C = 1250\left(e^{-0.0024793(60)} - e^{-0.1(60)}\right)$	
$= 1074.123$	
$= 1070$ mg/L	**A1**

Using $k = -0.0025$,

(b) Find the rate of change of Jonathan's blood alcohol concentration after 1 hour. [2]

$\frac{dC}{dt} = 1250\left(-0.0025e^{-0.0025t} + 0.1e^{-0.1t}\right)$	**B1**
When $t = 60$,	
$\frac{dC}{dt} = 1250\left(-0.0025e^{-0.0025(60)} + 0.1e^{-0.1(60)}\right)$	
$= -2.3799$	
$= -2.38$ mg/Lmin	**B1**

(c) After consumption of alcohol, the blood alcohol concentration will rise to a peak before decreasing slowly over time. Explain why the blood alcohol concentration found in part **(a)** is not the peak level, and find the peak blood alcohol concentration level. [5]

When t = 60, $\frac{dC}{dt} = -2.38 \neq 0$ Hence blood alcohol concentration is not at peak.	**B1**
$\frac{dC}{dt} = 1250\left(-0.0025e^{-0.0025t} + 0.1e^{-0.1t}\right) = 0$	**M1** – equating to 0 to solve
$-0.0025e^{-0.0025t} + 0.1e^{-0.1t} = 0$	
$0.0025e^{-0.0025t} = 0.1e^{-0.1t}$	
$\frac{e^{-0.0025t}}{e^{-0.1t}} = \frac{0.1}{0.0025}$	**M1** – Correctly solving using logarithms
$e^{0.0975t} = 40$	
$0.0975t = \ln 40$	
$t = 37.835$ mins	**A1**
$\Rightarrow C = 1250\left(e^{-0.0025(37.835)} - e^{-0.1(37.835)}\right)$	
$= 1108.756$ $= 1110$ mg/L	**A1**

9 **(a)** The equation of a curve is $y = \ln\sqrt{\dfrac{x^2+1}{2x+1}}$.

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{x^2+1} - \dfrac{1}{2x+1}$. [4]

$y = \ln\sqrt{\dfrac{x^2+1}{2x+1}}$

$= \dfrac{1}{2}\left[\ln\left(x^2+1\right) - \ln\left(2x+1\right)\right]$

B1 – Simplifying Logarithms using power rule

B1 – Simplifying Logarithms using quotient rule

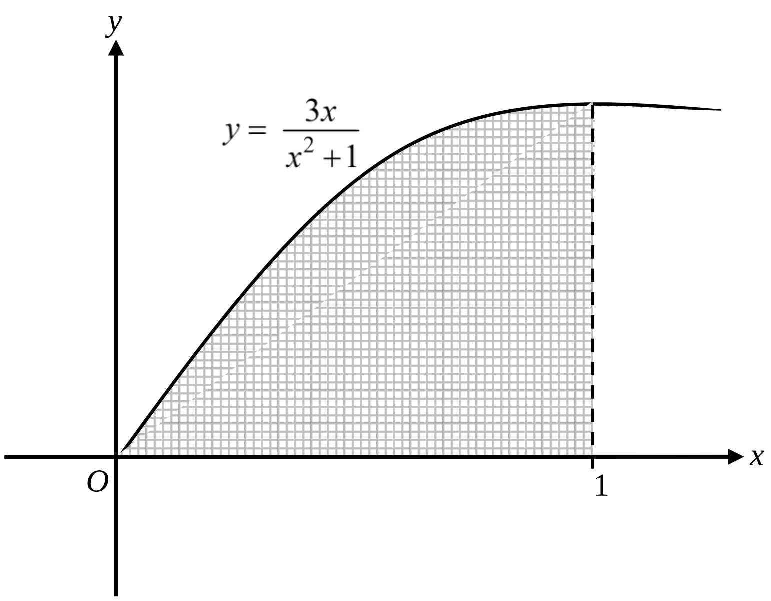
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1} - \dfrac{2}{2x+1}\right)$

$= \dfrac{x}{x^2+1} - \dfrac{1}{2x+1}$ $(shown)$

A2,1 – Correct derivatives

(b) The diagram below shows part of the graph of $y = \dfrac{3x}{x^2+1}$.

Using the result from part **(a)**, find the area of the shaded region bounded by the curve, the x- axis and the line $x = 1$. Express your answer in the form $a \ln b$, where a and b are constants. [6]



Continuation of working space for Question 9.

By reverse differentiation

$$\int \frac{x}{x^2+1} - \frac{1}{2x+1}\ \mathrm{d}x = \ln\sqrt{\frac{x^2+1}{2x+1}} + c$$

B1 – Correct identification of reverse differentiation

$$\int \frac{x}{x^2+1}\ \mathrm{d}x = \ln\sqrt{\frac{x^2+1}{2x+1}} + \frac{1}{2}\ln(2x+1) + c$$

B1 – $\frac{1}{2}\ln(2x+1)$

Area of shaded region

$$= \int_0^1 \frac{3x}{x^2+1}\ \mathrm{d}x$$

B1 – Correct definite integral

$$= 3\int_0^1 \frac{x}{x^2+1}\ \mathrm{d}x$$

$$= 3\left[\ln\sqrt{\frac{x^2+1}{2x+1}} + \frac{1}{2}\ln(2x+1)\right]_0^1$$

$$= 3\left[\left(\ln\sqrt{\frac{2}{3}} + \frac{1}{2}\ln 3\right) - 0\right]$$

M1 – substitution into a logarithmic function to find value

$$= 3\left(\frac{1}{2}\ln\frac{2}{3} + \frac{1}{2}\ln 3\right)$$

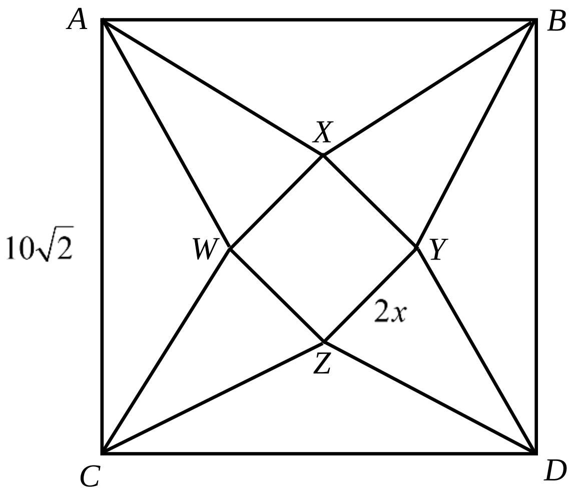
M1– combining into single fraction using logarithm laws

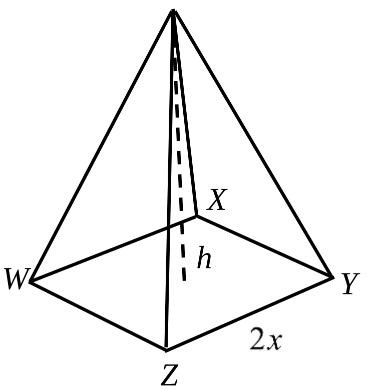
$$= \frac{3}{2}\ln\left(\frac{2}{3}\times 3\right)$$

$$= \frac{3}{2}\ln 2$$

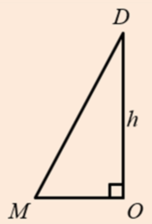
A1 – Simplified form

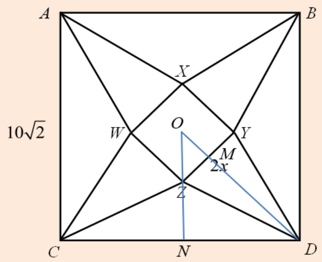
10 In the diagram below, $ABCD$ is a square paper with side $10\sqrt{2}$ cm. The net of a regular pyramid with square base $WXYZ$ was cut from the paper. AB is parallel to WY and the base of the pyramid has sides $2x$ cm.





(a) By expressing the perpendicular height, h cm, of the pyramid in terms of x, show that $V = \frac{8}{3}x^2\sqrt{25-5x}$. [4]





Let centre of square paper be O and midpoint of YZ and CD be M and N respectively.

$OM = \frac{1}{2}XY = x$

$ON = ND = \frac{1}{2}AC = 5\sqrt{2}$

$OD = \sqrt{\left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2} = \sqrt{100} = 10$ — **M1** – Finding OD or AD

$\Rightarrow MD = 10 - x$

$h = \sqrt{MD^2 - OM^2}$ — **M1** – Using Pythagoras Thm to find h

$= \sqrt{(10-x)^2 - x^2}$

$= \sqrt{100-20x}$ — **A1** – $h = \sqrt{100-20x}$

$V = \frac{1}{3}(2x)^2\sqrt{100-20x}$

$= \frac{1}{3}\left(4x^2\right)2\sqrt{25-5x}$ — **AG1**

$= \frac{8}{3}x^2\sqrt{25-5x}$ (*shown*)

(b) Given that x can vary, find the value of x for which the volume of the pyramid is stationary. [6]

$\frac{dV}{dx}=\frac{16}{3}x\sqrt{25-5x}+\frac{8}{3}x^2\left(\frac{-5}{2\sqrt{25-5x}}\right)$	**B1, B1 –** Correctly using product rule
At stationary point, $0=\frac{16}{3}x\sqrt{25-5x}+\frac{8}{3}x^2\left(\frac{-5}{2\sqrt{25-5x}}\right)$	**M1** – Finding stationary point by equating to 0
$\frac{16}{3}x\sqrt{25-5x}=\frac{8}{3}x^2\left(\frac{5}{2\sqrt{25-5x}}\right)$	**M1** – removing surds in denominator through cross multiplication or combining into a single fraction
$\frac{16}{3}x(25-5x)=\frac{20}{3}x^2$	**M1** – Correct simplification
$20x-5x^2=0$	
$5x(4-x)=0$	
$x=0\ (rej)\quad or\quad x=4$	**A1**

(c) Determine whether this value of x gives a maximum or minimum value for the volume of the pyramid. [2]

By first derivative test,

x	3.99	4	4.01
$\frac{dV}{dx}$	+	0	-
Shape	/	—	\

M1 – First derivative test

When x = 4, Volume is a **maximum**.

A1

~ ~ ~ **End of Paper** ~ ~ ~