Raffles Institution
H2 Mathematics
Solution for 2014 A-Level Paper 2

Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
	$x=3t^2$, $y=6t$	
(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=6t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=6 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{t}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=0.4 \Rightarrow \frac{1}{t}=0.4 \Rightarrow t=2.5$	
(ii)	Equation of tangent at point $P(3p^2, 6p)$ on C: $y-6p=\frac{1}{p}\left(x-3p^2\right)$ When $x=0$, $y-6p=\frac{1}{p}\left(-3p^2\right) \Rightarrow y=-3p+6p=3p$ Coordinates of D are $(0, 3p)$ Mid-point of PD is $\left(\frac{3p^2}{2}, \frac{9p}{2}\right)$ $x=\frac{3p^2}{2}$, $y=\frac{9p}{2} \Rightarrow x=\frac{3}{2}\left(\frac{2}{9}y\right)^2=\frac{2}{27}y^2$ $\therefore$ Cartesian equation is $x=\frac{2}{27}y^2$	

Question 2

No.	Suggested Solution	Remarks for Student
	$\frac{9x^2+x-13}{(2x-5)\left(x^2+9\right)}=\frac{A}{2x-5}+\frac{Bx+C}{x^2+9}$ $9x^2+x-13=A\left(x^2+9\right)+(Bx+C)(2x-5)$ $x=\frac{5}{2}$: $9\left(\frac{5}{2}\right)^2+\frac{5}{2}-13=A\left(\left(\frac{5}{2}\right)^2+9\right) \Rightarrow A=3$	

	$x=0: \quad -13=3(9)+C(-5)\Rightarrow C=8$ $x=1: \quad 9+1-13=3(1+9)+(B+8)(2-5)\Rightarrow B=3$ $\int_0^2 \frac{9x^2+x-13}{(2x-5)(x^2+9)}\,\mathrm{d}x=\int_0^2 \frac{3}{2x-5}+\frac{3x+8}{x^2+9}\,\mathrm{d}x$ $=\frac{3}{2}\left[\ln\lvert 2x-5\rvert\right]_0^2+\int_0^2 \frac{3x}{x^2+9}\,\mathrm{d}x+\int_0^2 \frac{8}{x^2+9}\,\mathrm{d}x$ $=-\frac{3}{2}\ln 5+\frac{3}{2}\left[\ln\left(x^2+9\right)\right]_0^2+\frac{8}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^2$ $=-\frac{3}{2}\ln 5+\frac{3}{2}\ln 13-\frac{3}{2}\ln 9+\frac{8}{3}\tan^{-1}\frac{2}{3}$ $=\frac{3}{2}\ln\frac{13}{45}+\frac{8}{3}\tan^{-1}\frac{2}{3}$ $\therefore a=\frac{3}{2}, b=\frac{13}{45}, c=\frac{8}{3}, d=\frac{2}{3}$	You may use the GC to verify your answers.

Question 3

No.	Suggested Solution	Remarks for Student
(i) **(a)**	Distance run by athelete who completes first 10 stages $2[4+8+12+\ldots+40]$ $=2\left[\frac{10}{2}\left(2(4)+(10-1)(4)\right)\right]$ $=10(8+36)=440\text{m}$	
(b)	Distance run by athelete who completes first n stages $=2\left[\frac{n}{2}\left(2(4)+(n-1)(4)\right)\right]$ $=n(4n+4)$ For $n(4n+4)\geq 5000$, consider the following <table><tr><td>n</td><td>$n(4n+4)$</td></tr><tr><td>34</td><td>4760 < 5000</td></tr><tr><td>35</td><td>5040 > 5000</td></tr></table>Least n is 35.	

(ii)

	$OA_1 = 4$
$A_1A_2 = 4$	$OA_2 = 8 = 4(2)$
$A_2A_3 = 8 = 4(2)$	$OA_3 = 16 = 4(2)^2$
$A_3A_4 = 16 = 4(2)^2$	$OA_4 = 32 = 4(2)^3$
$\vdots$	$\vdots$
$A_{n-1}A_n = 4(2)^{n-2}$	$OA_n = 4(2)^{n-1}$

Required distance

$$= 2\left(4+8+16+\ldots+4(2)^{n-1}\right)$$

$$= 2(4)\left(\frac{2^n-1}{2-1}\right)$$

$$= 8\left(2^n-1\right)$$

Let B be the point where he completes 10km exactly. Need n such that

$$8\left(2^n-1\right) \geq 10000$$

$$\Rightarrow 2^n \geq 1+\frac{10000}{8}$$

$$\Rightarrow n \geq 10.289$$

$\Rightarrow$Athlete ran a total of 10km after completing the 10th stage but not the 11th stage.

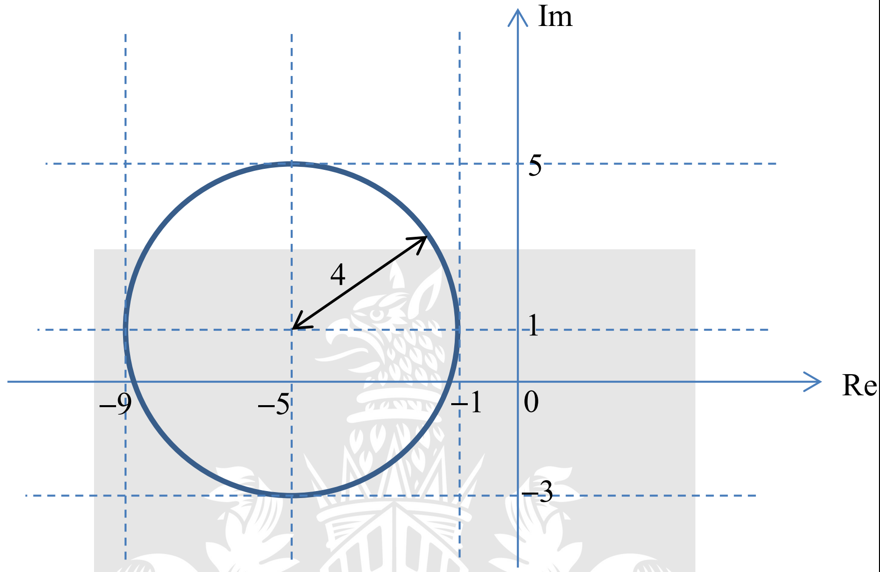
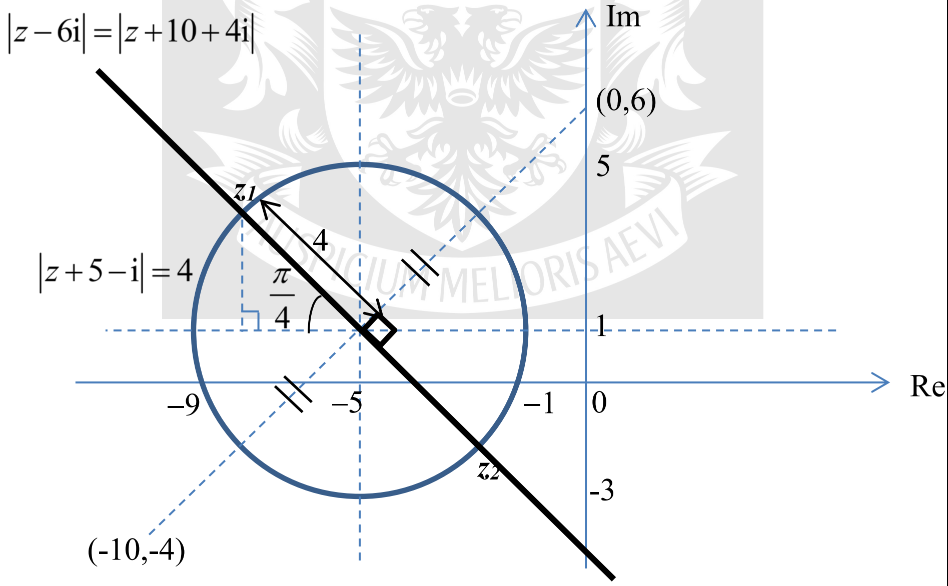
Distance covered between O and B = 10 000 − $8\left(2^{10}-1\right)$

= 1816

$OA_{11} = 4(2)^{11-1} = 4096$

Since $OA_{11} > 1816$, athlete is running away from O towards A_{11} when he reaches B and his distance from O is 1816 m.

Question 4

No.	Suggested Solution	Remarks for Student
(a)(i)	$\lvert z+5-\mathrm{i}\rvert = 4 \Rightarrow \lvert z-(-5+\mathrm{i})\rvert = 4$ 	
(ii)	$\lvert z-6\mathrm{i}\rvert = \lvert z+10+4\mathrm{i}\rvert \Rightarrow \lvert z-6\mathrm{i}\rvert = \lvert z-(-10-4\mathrm{i})\rvert$  $z_1 = \left(-5-4\cos\frac{\pi}{4}\right)+\mathrm{i}\left(1+4\sin\frac{\pi}{4}\right) = \left(-5-2\sqrt{2}\right)+\mathrm{i}\left(1+2\sqrt{2}\right)$ $z_2 = \left(-5+4\cos\frac{\pi}{4}\right)+\mathrm{i}\left(1-4\sin\frac{\pi}{4}\right) = \left(-5+2\sqrt{2}\right)+\mathrm{i}\left(1-2\sqrt{2}\right)$	

(b)(i)	$w=\sqrt{3}-\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$ $\lvert w\rvert=2$ and $\arg(w)=-\dfrac{\pi}{6}$ $\Rightarrow \lvert w^6\rvert=\lvert w\rvert^6=2^6$ and $\arg(w^6)=6\arg(w)+2\pi=-\pi+2\pi=\pi$ $\Rightarrow w^6=64\mathrm{e}^{\mathrm{i}\pi}$ Or $w^6=\left(2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)^6=2^6\mathrm{e}^{\mathrm{i}(-\pi)}=64\mathrm{e}^{\mathrm{i}\pi}$	
(ii)	For $\dfrac{w^n}{w*}$ to be real, $\arg\left(\dfrac{w^n}{w*}\right)=0$ or π $\Rightarrow -\dfrac{n\pi}{6}-\left(\dfrac{\pi}{6}\right)=-\dfrac{(n+1)\pi}{6}=0,\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$ 3 smallest positive whole number values of n are $-\dfrac{(n+1)\pi}{6}=-\pi,-2\pi,-3\pi\Rightarrow n=5,11,17$	

Section B: Statistics

Question 5

No.	Suggested Solution	Remarks for Student
(i)	Re-order the list of customers by alphabetical order of their names. We need a sample size of $10000\times 0.05=500$ Note that $10000/500=20$ Thus, first customer is selected by randomly generating a number from 1 to 20 inclusive, say 4. Then every 20^{th} customer from the 4^{th} customer will be included in the sample. In other words, the sample of 500 will consist of the 4^{th}, 24^{th}, 44^{th}, … 9984^{th} customers.	
(ii)	Advantage: customers of different first names (which has relation with religion and race as well) have equal chance of being represented. Disadvantage: Any customer who is not able to make it for the survey would result in re-sampling which may be time	

	consuming.	

Question 6

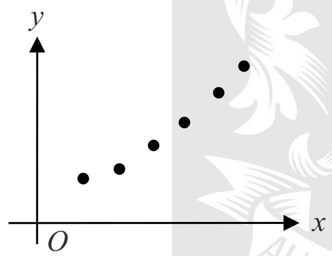
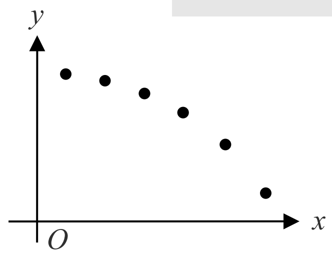
No.	Suggested Solution	Remarks for Student
	Available are 3 goalkeepers, 8 defenders, 5 midfielders, 6 attackers. Total:22 Team consists of 1 goalkeeper, 4 defenders, 2 midfielders, 4 attackers. Total:11	
(i)	${}^{3}C_{1}\,{}^{8}C_{4}\,{}^{5}C_{2}\,{}^{6}C_{4}=31500$ different teams can be formed.	
(ii)	<u>Case 1: Brother(midfield) not in but Brother(attacker) in</u> ${}^{3}C_{1}\,{}^{8}C_{4}\,{}^{4}C_{2}\,{}^{5}C_{3}=12600$ possible teams <u>Case 2: Brother(attacker) not in but Brother(midfield) in</u> ${}^{3}C_{1}\,{}^{8}C_{4}\,{}^{4}C_{1}\,{}^{5}C_{4}=4200$ possible teams $\therefore$ Total number of possible teams is 12 600 + 4200 = 16800	
(iii)	Let A be the midfielder who can play as either midfielder or defender. After the brothers left, there are 3 goalkeepers and 5 attackers. <u>Case 1: A serves as a possible midfielder</u> There are 4 midfielders and 8 defenders ${}^{3}C_{1}\,{}^{8}C_{4}\,{}^{4}C_{2}\,{}^{5}C_{4}=6300$ possible teams <u>Case 2: A serves as a possible defender</u> There are 3 midfielders and 9 defenders ${}^{3}C_{1}\,{}^{9}C_{4}\,{}^{3}C_{2}\,{}^{5}C_{4}=5670$ possible teams Number of possible teams without A = ${}^{3}C_{1}\,{}^{8}C_{4}\,{}^{3}C_{2}\,{}^{5}C_{4}=3150$ Total: 6300 + 5670 – 3150 = 8820	Note that teams formed without A is counted twice in Cases 1 and 2.

Question 7

No.	Suggested Solution	Remarks for Student
(i)	Let X be the number of 6's out of 10 rolls of the fair die. $X\sim \mathrm{B}\left(10,\frac{1}{6}\right)$ $\mathrm{P}\left(X=3\right)\approx 0.155$	
(ii)	Let Y be the number of 6's out of 60 rolls of the fair die.	

	$Y \sim \mathrm{B}\left(60, \frac{1}{6}\right)$ Since $n = 60$ is large, and $np = 10 > 5$ and $n(1-p) = 50 > 5$, $Y \sim \mathrm{N}\left(10, \frac{25}{3}\right)$ approximately. $\mathrm{P}(5 \le Y \le 8) = \mathrm{P}(4.5 \le Y \le 8.5)$ by continuity correction ≈ 0.273	Be sure to check the conditions and state the approximate distribution used.
(iii)	Let W be the number of 6's out of 60 rolls of the biased die. $W \sim \mathrm{B}\left(60, \frac{1}{15}\right)$ Since $n = 60$ is large, and $p = \frac{1}{15}$ is small such that $np = 4 < 5$, $W \sim \mathrm{Po}(4)$ approximately. $\mathrm{P}(5 \le W \le 8) = \mathrm{P}(W \le 8) - \mathrm{P}(Y \le 4)$ ≈ 0.350	Be sure to check the conditions and state the approximate distribution used.

Question 8

No.	Suggested Solution	Remarks for Student
(a)(i)		
(ii)		
(b)(i)	Product moment correlation coefficient between m and P $= -0.9470$ (4dp) Product moment correlation coefficient between $\ln m$ and P $= -0.9749$ (4dp)	

(ii)	–0.9749 is nearer to –1 compared to –0.9470. The scatter diagram of P on m also suggest a non-linear relationship between the 2 variables. Thus $P = c\ \ln m + d$ is the better model. $P = 195693.5593 - 33659.72805 \ln m$ $P = 196000 - 33700 \ln m$ (3 s.f.)	
(iii)	From GC, estimated price is \$64015.93 ≈ \$64000 (3 s.f.)	

Question 9

No.	Suggested Solution	Remarks for Student
(i)	Null hypothesis, $\mathrm{H}_0 : \mu = 4.3$ Alternative hypothesis, $\mathrm{H}_1 : \mu < 4.3$ μ is the population mean number of minutes the bus is late.	
(ii)	Perform a one-tailed test at 10% significance level. Under H_0, $$T = \frac{\overline{X} - 4.3}{S / \sqrt{n}} \sim t(n-1),$$ From sample, $n = 10$, with $\overline{x} = \overline{t}$ and $s = \sqrt{\frac{10}{9}k^2} = \sqrt{\frac{10}{9}(3.2)}$ Given that null hypothesis is not rejected, p-value > 0.1, that is, $$\mathrm{P}\left(T < \frac{\overline{t} - 4.3}{\frac{\sqrt{\frac{10}{9}(3.2)}}{\sqrt{10}}} \right) > 0.1$$ $$\mathrm{P}\left(T < \frac{3(\overline{t} - 4.3)}{\sqrt{3.2}} \right) > 0.1$$ $$\frac{3(\overline{t} - 4.3)}{\sqrt{3.2}} > -1.38303$$ $$\overline{t} > 3.4753 \approx 3.48$$ $\therefore$ Set of values of $\overline{t}$ is $(3.48, \infty)$.	
(iii)	Perform a one-tailed test at 10% significance level. Under H_0, $$T = \frac{\overline{X} - 4.3}{S / \sqrt{n}} \sim t(n-1),$$ From sample, $n = 10$, with $\overline{x} = 4.0$ and $s = \sqrt{\frac{10}{9}k^2}$	

	Given that null hypothesis is rejected, p-value < 0.1, that is, $\mathrm{P}\left(T < \dfrac{4.0-4.3}{\dfrac{\sqrt{\frac{10}{9}\left(k^2\right)}}{\sqrt{10}}}\right) < 0.1$ $\mathrm{P}\left(T < \dfrac{-0.9}{k}\right) < 0.1$ $\dfrac{-0.9}{k} < -1.38303$ $k < 0.65075$ $0 < k^2 < 0.42347, \text{ that is, } 0 < k^2 < 0.423$ $\therefore$ Set of values of k^2 is (0,0.423).	

Question 10

No.	Suggested Solution	Remarks for Student
(i)(a)	Required probability = $\left(\frac{1}{10}\right)\left(\frac{2}{10}\right)\left(\frac{1}{10}\right) = \frac{1}{500}$	
(b)	Required probability = $1-\left(\frac{9}{10}\right)\left(\frac{8}{10}\right)\left(\frac{9}{10}\right) = \frac{44}{125}$	
(c)	$\mathrm{P}(\times\times+)+\mathrm{P}(\times+\times)+\mathrm{P}(+\times\times)$ $=\left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\left(\frac{2}{10}\right)+\left(\frac{3}{10}\right)\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)+\left(\frac{4}{10}\right)\left(\frac{1}{10}\right)\left(\frac{4}{10}\right)$ $=\frac{29}{500}$	
(ii)	$\mathrm{P}(\text{the other two symbols are + and o} \mid \text{exactly one is } \bigstar)$ $=\dfrac{\mathrm{P}(+\text{o}\bigstar,\ +\bigstar\text{o},\ \text{o}+\bigstar,\ \text{o}\bigstar+,\ \bigstar+\text{o},\ \bigstar\text{o}+)}{\mathrm{P}(\text{at least one }\bigstar)-\mathrm{P}(\bigstar\bigstar\bigstar)-\mathrm{P}(\text{exactly 2 }\bigstar\text{ are displayed})}$ $=\dfrac{\left(\frac{4}{10}\right)\left(\frac{4}{10}\right)\left(\frac{1}{10}\right)+\left(\frac{4}{10}\right)\left(\frac{2}{10}\right)\left(\frac{3}{10}\right)+\left(\frac{2}{10}\right)\left(\frac{3}{10}\right)\left(\frac{1}{10}\right)+\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)+\left(\frac{1}{10}\right)\left(\frac{3}{10}\right)\left(\frac{3}{10}\right)+\left(\frac{1}{10}\right)\left(\frac{4}{10}\right)\left(\frac{2}{10}\right)}{\frac{44}{125}-\frac{1}{500}-\left(\frac{1}{10}\right)\left(\frac{2}{10}\right)\left(\frac{9}{10}\right)-\left(\frac{1}{10}\right)\left(\frac{8}{10}\right)\left(\frac{1}{10}\right)-\left(\frac{9}{10}\right)\left(\frac{2}{10}\right)\left(\frac{1}{10}\right)}$	

	$$=\frac{\frac{1}{1000}(16+24+6+8+9+8)}{\frac{1}{1000}(352-2-18-8-18)}=\frac{71}{306}$$	

Question 11

No.	Suggested Solution	Remarks for Student
	Let X be the number of originals sold per week. $X \sim \text{Po}(2)$ Let Y be the number of prints sold per week. $Y \sim \text{Po}(11)$	
(i)(a)	$\text{P}(Y>8)=1-\text{P}(Y\le 8)\approx 0.768$	
(b)	$X+Y \sim \text{Po}(13)$ $\text{P}(X+Y<15)=\text{P}(X+Y\le 14)\approx 0.675$	
(ii)	Let W be the number of originals sold in n weeks. $W \sim \text{Po}(2n)$ $\text{P}(W<3)<0.01$ $\text{P}(W=0)+\text{P}(W=1)+\text{P}(W=2)<0.01$ $\text{e}^{-2n}+2n\text{e}^{-2n}+\frac{\text{e}^{-2n}(2n)^2}{2}<0.01$ $\text{e}^{-2n}\left(1+2n+2n^2\right)<0.01$ ...(1) <table><tr><td>n</td><td>$\text{e}^{-2n}\left(1+2n+2n^2\right)$</td></tr><tr><td>4</td><td>0.01375 >0.01</td></tr><tr><td>5</td><td>0.00277 < 0.01</td></tr></table>$\therefore$ least n is 5.	Use GC smartly. Though you are supposed to use (1), you can key in $\text{P}(W<3)<0.01$ instead of (1) which is more prone to typos.
(iii)	Let V be the number of prints sold in 52 weeks. $V \sim \text{Po}(572)$ Since 572 > 10, $V \sim \text{N}(572, 572)$ approximately $\text{P}(V>550)=\text{P}(V>550.5)$ by continuity correction ≈ 0.816	
(iv)	Regarding Poisson distribution: Number of paintings sold each week is unlikely to remain constant uniformly, as there are peak and lull periods for art/painting exhibition.	

	Regarding independence: Sales between originals and prints are unlikely to be independent. For example, a person organizing an art event may buy an original for display, and at the same time will need a number of prints for advertising purpose.	