Chapter S1

PERMUTATIONS AND COMBINATIONS

In this chapter, students will be able to:

- apply the addition and multiplication principles for counting;
- distinguish between permutations and combinations;
- understand that ${}^nP_r$ is the number of permutations of r objects taken from n distinct objects; and that in the special case when $n = r$, the number of permutations of n distinct objects arranged in a straight line is $n!$;
- understand that ${}^nC_r$ or $\binom{n}{r}$ is the number of combinations of r objects from n distinct objects;
- interpret the relation ${}^nP_r = {}^nC_r\, r!$ (i.e. permutation is selection followed by arrangement) and use the formula: ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \dfrac{n!}{r!(n-r)!}$;
- find the number of permutations of n distinct objects arranged in a circle;
- find the number of permutations of n objects which are not all distinct, e.g. if m of the objects are identical and the rest are distinct, the number of permutations is $\dfrac{n!}{m!}$;
- use ${}^nP_r$, $n!$ and ${}^nC_r$ to solve counting problems, including cases involving repetition and restriction.

1.1 Fundamental Principles of Counting

The Addition Principle:

Assume that there are n_1 ways for the event E_1 to occur,
n_2 ways for the event E_2 to occur,
$\vdots$
n_k ways for the event E_k to occur,

where $k \geq 1$.

If E_1, E_2, … , E_k are **mutually exclusive** (i.e. they cannot occur at the same time), then the number of ways for at least one of the events to occur is $n_1 + n_2 + \cdots + n_k$.

EXAMPLE 1

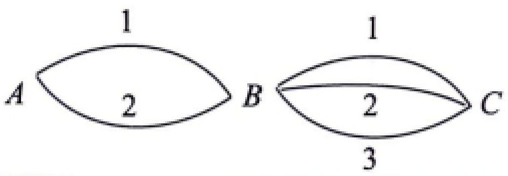
There is 1 way to go to Town B from Town A by land, 2 ways by air and 3 ways by sea. How many ways are there to go to Town B from Town A?

SOLUTION 1	LOGIC
By the Addition Principle, there are $1+2+3=6$ ways to travel to Town B from Town A.	You can travel to Town B ***either*** by land (1 way) ***OR*** by air (2 ways) ***OR*** by sea (3 ways)

The Multiplication Principle:
Suppose that an event E can be split into k events $E_1, E_2, \ldots, E_k$ in **ordered stages.**
If there are n_1 ways for the event E_1 to occur,
n_2 ways for the event E_2 to occur,
$\vdots$
and n_k ways for the event E_k to occur,
then the number of ways for the event E to occur is $n_1 \times n_2 \times \cdots \times n_k$.

EXAMPLE 2

Suppose there are 2 routes to go from Town A to Town B and 3 routes to go from Town B to Town C, how many possible routes are there to go from Town A to Town C, assuming that all routes must pass through B?



SOLUTION 2	LOGIC
By the Multiplication Principle, there are 2x3=6 possible routes to go from Town A to Town C.	You can go from Town A to Town C via Town B by taking one of the following routes: $A \xrightarrow{1} B \xrightarrow{1} C$; $A \xrightarrow{1} B \xrightarrow{2} C$; $A \xrightarrow{1} B \xrightarrow{3} C$; $A \xrightarrow{2} B \xrightarrow{1} C$; $A \xrightarrow{2} B \xrightarrow{2} C$; $A \xrightarrow{2} B \xrightarrow{3} C$. There are 2 ways you can go from Town A to Town B *AND* 3 ways to go from Town B to Town C.

1.2 Combinations

A combination of a set of objects is a selection of one or more of the objects where order of selection **does not matter**. For example, the 3-letter arrangement of ABC and ACB is the same combination.

Suppose we have 4 cards P, Q, R and S in a bag and we draw out 2 of them (it makes no difference to draw them out either one at a time or all at once). We will have a set of 2 cards, say PQ which we will call a *combination* (of 2 cards). Notice that we are *not concerned* with the *order* in which they were selected, but just which 2 cards were selected. How many different combinations can be drawn? There are a total of 6 combinations, namely PQ, PR, PS, QR, QS and RS. Thus, the number of combinations of 2 cards drawn from 4 is given by ${}^4C_2 = 6$.

Some other examples where order does not matter:

1. Select 4 students from a class of 30 to help Mrs. Wong to carry mathematics files.
2. Draw 3 balls from a bag containing 7 differently coloured balls.
3. Obtain a hand of 4 spades from an ordinary pack of 52 playing cards.

1.2.1 Combination (selection) of r objects taken from n distinct objects

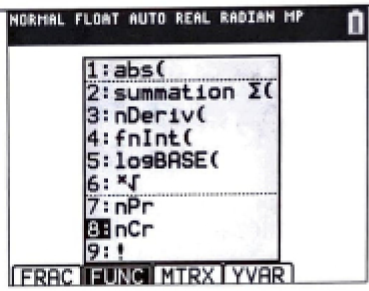
Theorem 1
The combination (or selection) of r objects taken from n distinct objects, **without replacement**, is given by ${}^{n}C_r = \dfrac{n!}{r!(n-r)!}$, where $0 \le r \le n$, $n, r \in \mathbb{Z}^+$.

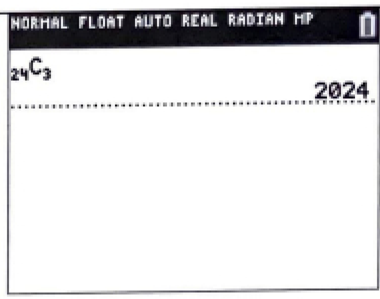
We can use the G.C. to find . For example, ${}^{24}C_3$

Press ALPHA WINDOW and select option 8: **'8:nCr'**.

Using the cursor keys, key in the correct values into the respective blanks.

Using G.C., ${}^{24}C_3 = 2024$.





EXAMPLE 3
How many ways can a committee of 5 be chosen from a class of 20 students
(i) if there is no restriction,
(ii) if the oldest has to be included?

SOLUTION 3	**LOGIC**
(i) Number of choosing a committee of 5 students (no restriction) = ${}^{20}C_5 = 15504$	The order of selection **does not matter**. Hence it is just choosing 5 from 20.
(ii) Number of ways of choosing a committee of 5 students (include the oldest) $= 1 \times {}^{19}C_4 = 3876$	As the oldest student has to be chosen, we choose the oldest student first ***AND*** choose 4 more students from the remaining 19 students (order of selection is not important).

EXAMPLE 4
How many hands of 4 cards can be dealt from an ordinary pack of 52 playing cards if
(i) there are no restrictions,
(ii) all four are spades,
(iii) exactly two are spades,
(iv) one is a heart, one a spade and two are clubs,
(v) all cards are of different suits,
(vi) at least one is a picture card.

These are the 4 suits:
13 spades, 13 hearts
13 clubs, 13 diamonds
Picture cards: J, Q, K

SOLUTION 4	**LOGIC**
(i) Number of ways of choosing 4 cards without restrictions = ${}^{52}C_4 = 270725$	The order of selection does not matter. We only need to choose 4 from 52.
(ii) Number of ways where all 4 are spades $= {}^{13}C_4$ ~~715~~ $\times {}^{39}C_0 = 715$	Since there are 13 spades in a pack of 52 cards we need to choose 4 spades from 13 spade cards i.e. ${}^{13}C_4$ ways.

EXAMPLE 17 (2011/TJC/II/Q6b)

Six people sit at a round table with eight identical chairs. Find the number of seating arrangements if

(i) there are no restrictions,

(ii) the seats are numbered.

SOLUTION 17	**LOGIC**
(i) Number of ways $= \frac{(8-1)!}{2!} = 2520$	Step 1: We treat the 2 empty seats to be 'taken' and calculate the circular arrangement for a round table of 7 units. Step 2: The 2 empty seats are treated as repeated objects in the circular arrangement and divided by 2!
(ii) Number of ways $= 2520 \times 8 = 20160$	Step 1: Arrange the 6 people around the round table with no restrictions. Step 2: Label the seats. i.e. 8 ways.

SELF-REVIEW 9

Hillary Tan has 9 cousins. In how many ways can she invite some or all of them to her birthday party? At the birthday party, Hillary sets up a round table of 10 seats with a different mathematics puzzle at each seat. Four of her cousins are from the Siow family, three are from the Ding family and two are from the Dong family and all her cousins turn up for the party. Find the number of ways that they can be seated with Hillary if families with the same surname are seated together but members of the Ding and Dong families are not adjacent to each other. [511, 5760]

Chapter S2

Probability

In this chapter, you will learn to

- understand that the probability of an event measures how likely the event will occur;
- construct a table of possible outcomes to calculate probabilities and understand that the total probability of all the possible outcomes is equal to 1;
- calculate probabilities using the addition and multiplication principles;
- calculate probabilities using permutations and combinations;
- understand the meaning of mutually exclusive events, determine if two events A and B are mutually exclusive and calculate probabilities when A and B are mutually exclusive;
- understand the meaning of independent events, determine if two events A and B are independent and calculate probabilities when A and B are independent;
- understand the meaning of conditional probability and use the result $\mathrm{P}(A \mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ to compute conditional probability;
- use a Venn diagram to interpret and calculate probabilities such as $\mathrm{P}(A')$, $\mathrm{P}(A\cup B)$, $\mathrm{P}(A\cap B)$ and $\mathrm{P}(A \mid B)$;
- construct a tree diagram and use it to interpret and calculate probabilities such as probabilities of combined events and conditional probabilities.

Probability is the likelihood of an outcome or event. Some examples of probabilities

1. The probability of getting a one when a fair die is tossed is $\frac{1}{6}$.
2. The probability (or odds) of winning the first prize in the Singapore Sweep is 1 in 3,200,000.

2.1 Basic Concepts

2.1.1 Random Experiments, Sample Space and Events

In our everyday life, we encounter **'random experiments'** very frequently. The word **'experiment'** or **'trial'** can be thought of as an action that can be repeated under the same set of conditions. The adjective **'random'** implies that the **outcome** of the experiment cannot be predicted with certainty.

The set of all possible outcomes of an experiment is called the **sample space** and is denoted by S.

A subset A of the sample space S of a random experiment is known as an **event**, i.e. an event consists of one or more possible outcomes.

Random experiment	Sample Space, S	An event, A
1. Toss an unbiased coin twice and the outcome is observed.	$S=\{\text{HH,HT,TH,TT}\}$	'a head is obtained' $=\{\text{HH,TH,HT}\}$
2. Draw a card at random from a pack of 52 playing cards and note its suit.	$S=\{\clubsuit,\underbrace{\diamondsuit,\heartsuit}_{\text{red}},\spadesuit\}$	'a black suit is obtained' $=\{\clubsuit,\spadesuit\}$
3. Observe Bernard's result at a game of chess.	$S=\{\text{win, lose, draw}\}$	'Bernard wins the game' $=\{\text{win}\}$

2.1.2 DEFINITION OF PROBABILITY

The probability of an event measures how likely the event will occur. The classical interpretation of the probability of an event is the relative frequency or ratio of the number of occurrences that correspond to the event to all possible outcomes given that all the outcomes have an equal chance of occurring.

> If the sample space S contains a **finite number** of **equally likely outcomes**, then the probability that an event A will occur is given by
>
> $$\mathrm{P}(A)=\frac{\mathrm{n}(A)}{\mathrm{n}(S)}\quad\left(\text{i.e. }\frac{\text{number of outcomes corresponding to event } A}{\text{total number of possible outcomes in the sample space S}}\right)$$

Using the example earlier

	Sample Space, S	An event, A	$\mathrm{P}(A)=\dfrac{\mathrm{n}(A)}{\mathrm{n}(S)}$	Remarks
1.	$S=\{\text{HH,HT,TH,TT}\}$	'a head is obtained' $=\{\text{HH,HT,TH}\}$	$\frac{3}{4}$	Coin is unbiased means that the outcomes are *equally likely*.
2.	$S=\{\clubsuit,\underbrace{\diamondsuit,\heartsuit}_{\text{red}},\spadesuit\}$	'a black suit is obtained' $=\{\clubsuit,\spadesuit\}$	$\frac{1}{2}$	The card is drawn at random. This ensures that the outcomes are *equally likely*.
3.	$S=\{\text{win, lose, draw}\}$	'Bernard wins the game' $=\{\text{ win }\}$	Not enough information to evaluate it	We do not know if it is *equally likely* for Bernard to win, lose or draw the game.

IMPORTANT NOTES:

- For any sample space S, $\mathrm{P}(S)=1$ (i.e. the sum of the probability of all possible outcomes is 1).
- For any event A of S, $0\le \mathrm{P}(A)\le 1$.
- If A is an **impossible event** (or **null event**), then $\mathrm{P}(A)=0$. A can also be represented by the empty set ϕ.
- $\mathrm{P}(A)=1$ means that A is a **sure event** and $A\equiv S$.

EXAMPLE 1:

A fair die is thrown and the number was observed. Write down the sample space and the set that corresponds to the following events:

(i) a multiple of 3 is obtained.

(ii) the number is less than 7.

(iii) the number is a factor of 6.

Hence state the probability of each of the event.

SOLUTION 1:

Sample space, $S = \{1, 2, 3, 4, 5, 6\}$

Let A be the event that 'the number obtained is a multiple of 3'. $A = \{3, 6\}$.

Let B be the event that 'the number obtained is less than 7'. $B = \{1, 2, 3, 4, 5, 6\}$.

Let C be the event that 'the number obtained is a factor of 6' $C = \{1, 2, 3, 6\}$.

(i) $P(A) = \frac{2}{6} = \frac{1}{3}$

(ii) $P(B) = \frac{6}{6} = 1$ (Event B is a sure event)

(iii) $P(C) = \frac{4}{6} = \frac{2}{3}$

REMARK 1: Do you know that a die need not be six-sided only and the numbering need not be in running order? Can you have a two-sided die?

REPRESENTING A SAMPLE SPACE USING A TABLE OF OUTCOMES

EXAMPLE 2:

Two unbiased dice are thrown and the outcome is observed. One with 5 faces numbered 1, 1, 1, 2 and 4. The other with 6 faces numbered 2, 4, 6, 6, 6 and 6. Find the probability that

(i) the sum of their scores is 7.

(ii) the sum of their scores is even.

(iii) the sum of their scores is more than 4.

SOLUTION 2:

Let D_1 and D_2 denote the numbers shown on the first and second die respectively.

D_1 \ D_2	2	4	6	6	6	6
1	3	5	7	7	7	7
1	3	5	7	7	7	7
1	3	5	7	7	7	7
2	4	6	8	8	8	8
4	6	8	10	10	10	10

Total number of possible outcomes = 30.

(i) The sum of scores is 7 when (1, 6) (twelve occurrences).

P(sum of scores is 7) = $\frac{12}{30} = \frac{2}{5}$

(ii) There are <u>12 ways</u> to have the sum of scores being even.

P(sum of scores is even) = $\frac{12}{30} = \frac{2}{5}$ Contrary to some's believe the probability is ½. Their argument is that the outcome is either odd or even. This example shows that this kind of argument doesn't work!

(iii) P(sum of scores > 4) =

SELF-REVIEW 1:

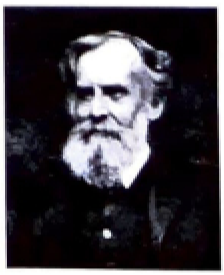
A standard fair die is thrown two times. Calculate the probability that the total score is 7. $[\frac{1}{6}]$

SOLUTION:

D_1 \ D_2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

REPRESENTING A SAMPLE SPACE USING A VENN DIAGRAM

Venn diagrams derive their name from John Venn, who popularized their use in 1880 with his paper entitled, "On the Diagrammatic and Mechanical Representation of Propositions and Reasonings". He developed George Boole's symbolic logic and is best known for Venn diagrams, which pictorially represent the relations between sets. Venn diagrams were a respected, if not ubiquitous, learning tool for all ages and grade levels.

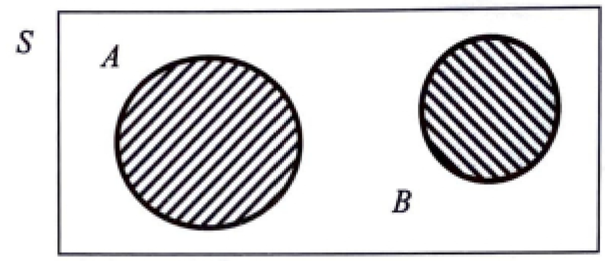


2.1.6 Mutually Exclusive Events

Two events, A and B are **mutually exclusive** if the occurrence of either **excludes** the possibility of the other event. That is, either event A occurs or event B occurs but **not both.**

For **two mutually exclusive events A and B** (that do not overlap as illustrated in the diagram below),

$$\mathrm{P}(A\cap B)=0 \text{ and } \mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B).$$



$P(A\cap B)=0$ ~~$P(A'\cap C')=0$~~ ~~$P(X\cup Y)$~~

Example 7: [2018/2/7 Modified] $P(A\cup B)=P(A)+P(C)$

The two mutually exclusive events A and C are such that $\mathrm{P}(A)=a$ and $\mathrm{P}(C)=c$.

Find an expression for $\mathrm{P}(A'\cap C')$. Draw a Venn diagram to illustrate the case when A' and C' are also mutually exclusive events.

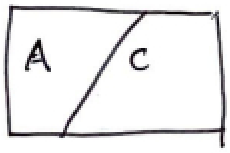
$P(A'\cap C')=P[(A\cup C)'] =$
$= 1-P(A\cup C)$
$1-P(A)-P(C)$

Solution 7:

$P(A'\cap C') = 1-P(A\cup C)$ since A and B are mutually exclusive
$= 1-(P(A)+P(C)-P(A\cap B))$
$= 1-a-c.$ ~~since $P(A\cap C)=0$~~

$P(A'\cap C')=0$

$P(A'\cup C')=P(A')+P(C')$



$1-P(A)-P(C)=0$

$1=P(A)+P(C)$

Example 8:

A number is chosen at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

Let A be the event a prime number is selected,
B be the event that an even number is selected and
C be the event that an odd number is selected.

(i) Find $\mathrm{P}(A\cap B)$. Are A and B mutually exclusive? Hence, find $\mathrm{P}(A\cup B)$.

(ii) Find $\mathrm{P}(A\cap C)$. Are A and C mutually exclusive?

SOLUTION 8:

(i) $A \cap B = \varnothing$. (no prime numbers are even numbers in this set)

$\mathrm{P}(A \cap B) = 0$.

A and B are mutually exclusive since $\mathrm{P}(A \cap B) = 0$

$A = \{3,5,7,11\}$ $\quad$ $B = \{4,6,8,10,12\}$

Since A and B are mutually exclusive events,

$\mathrm{P}(A \cup B)$

$= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= \mathrm{P}(A) + \mathrm{P}(B)$ since A and B are mutually exclusive $\mathrm{P}(A \cap B) = 0$

$= \dfrac{4}{10} + \dfrac{5}{10}$

$= \dfrac{9}{10}$

(ii) $A \cap C = \{3,5,7,11\}$

A and C are not mutually exclusive since $\mathrm{P}(A \cap C) \neq 0$

EXAMPLE 9 : [2016/H1/I/7]

The events A and B are such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(B) = 0.25$ and $\mathrm{P}(A \cap B) = 0.05$.

(i) Draw a Venn diagram to represent this situation, showing the probability in each of the four regions.

(ii) Find the probability that

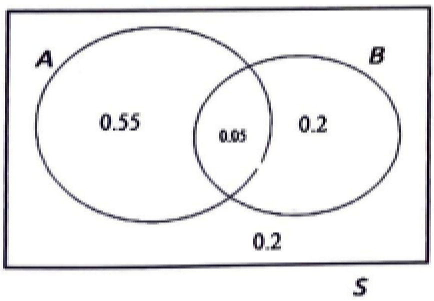
(a) At least one of A and B occurs,

(b) Exactly one of A and B occurs.

b)

SOLUTION 9 :

(i)



b) $P(A \cup B) - P(A \cap B)$

$= 0.8 - 0.05$

$= 0.75$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.25 - 0.05$

$= 0.8$

SELF-REVIEW 3:

Analysis of the results of a certain group of students who had taken Block Test in both Mathematics and Physics produced the following information 75% of the students passes Mathematics, 70% passed in Physics and 40% failed at least one of these subjects.

(i) Find the probability that the student taken at random passing exactly one of the two subjects.

(ii) Find the probability that the student passing both subjects. Hence, determine if the events of passing mathematics and that of physics are mutually exclusive. [0.15, 0.6]

REMARK: Why is the probability in Self-Review 3(ii) not equals to probability of passing Mathematics multiplied by the probability of passing Physics? This has to do with the concept of independence of events which we will discuss in the next section.

2.1.7 INDEPENDENT EVENTS

Two events A and B are **independent** if the occurrence of one does not affect the occurrence of the other. Some examples of independent events are:

- Landing on head after tossing a coin **AND** rolling a 5 on a 6-sided die.
- Choosing a King from a deck of cards, **replacing** it, **AND** then choosing an Ace as the second card.
- Rolling a 4 on a 6-sided die, **AND** then rolling a 1 on a second roll of the die.

Result 1:

If events A and B are independent, then

$$\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$$

This is the multiplication law for independent events.

Result 2:

If we can show that $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$, then we say event A and B are independent.

EXAMPLE 10:

Select one card randomly from a standard deck of 52. Let R be the event that the card is red, and let Q be the event that the card is queen. Are the events R and Q independent?

SOLUTION 10:

$\mathrm{P}(R)=\frac{26}{52}=\frac{1}{2}$, $\mathrm{P}(Q)=\frac{4}{52}=\frac{1}{13}$

The event $R\cap Q$ is selection of a red queen. Since there are two red queens in the deck,

$$P(R\cap Q)=\frac{2}{52}=\frac{1}{26}$$

Since $\mathrm{P}(R)\times\mathrm{P}(Q)=\frac{1}{2}\times\frac{1}{13}=\frac{1}{26}=\mathrm{P}(R\cap Q)$, R and Q are independent.

EXAMPLE 11:

In Example 2, A is the event that the sum of scores is divisible by 5 and B is the event that the sum of score is prime. Determine if A and B are independent events.

SOLUTION 11:

$$\mathrm{P}(A)=\frac{7}{30}$$

$$\mathrm{P}(B)=\frac{18}{30}=\frac{3}{5}$$

$$\mathrm{P}(A\cap B)=\frac{3}{30}=\frac{1}{10}$$

$$\mathrm{P}(A)\mathrm{P}(B)=\frac{7}{30}\left(\frac{3}{5}\right)=\frac{7}{50}\neq\frac{1}{10}=P(A\cap B)$$

Since $\mathrm{P}(A\cap B)\neq\mathrm{P}(A)\mathrm{P}(B)$, A and B are not independent.

The two previous examples illustrate the common ways in which independence is asserted, we apply the formula. We will revisit the concept of independent events again in Section 8.2 Conditional Probability.

$\frac{4}{5} = \frac{19}{30} + \frac{2}{5} - P(A \cap B)$

SELF-REVIEW 4:

Events A and B are such that $\mathrm{P}(A)=\frac{19}{30}$, $\mathrm{P}(B)=\frac{2}{5}$ and $\mathrm{P}(A\cup B)=\frac{4}{5}$, find $\mathrm{P}(A\cap B)$. Justify with reason if events A and B are **(i)** mutually exclusive, **(ii)** independent. [$\frac{7}{30}$; (i) Not (ii) Not]

2.1.8 SEQUENTIAL EVENTS

Sequential events are events that occur one after another in succession. The **tree diagram** is particularly useful in handling sequential events where the probability of one event occurring after another needs to be computed.

REPRESENTING A SAMPLE SPACE OF SEQUENTIAL EVENTS USING A TREE DIAGRAM

EXAMPLE 12:

A bag contains 8 white counters and 3 black counters. Two counters are drawn from the bag, one after the other. Find the probability of drawing one white and one black counter, in any order, if the first counter is **not** replaced.

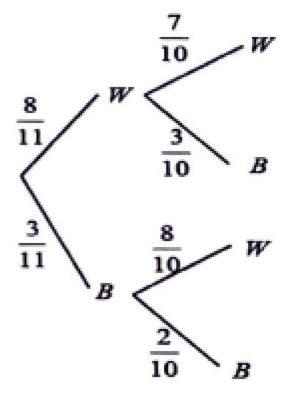
SOLUTION 12:

P(drawing one white and one black counter)

$P(WB) + P(BW)$

$= \left(\frac{8}{11}\right)\left(\frac{7}{10}\right) + \left(\frac{3}{11}\right)\left(\frac{8}{10}\right)$

$= \frac{24}{55}$



EXAMPLE 13:

Two bags each contains 7 balls which are indistinguishable apart from their colour. The first bag contains 4 red and 3 yellow balls and second, 3 red and 4 yellow balls. A ball is chosen at random from the first bag and placed in the second, then after thoroughly mixing, a ball is taken from the second bag and placed in the first. If a ball is now taken from the first bag, what is the probability that it is red?

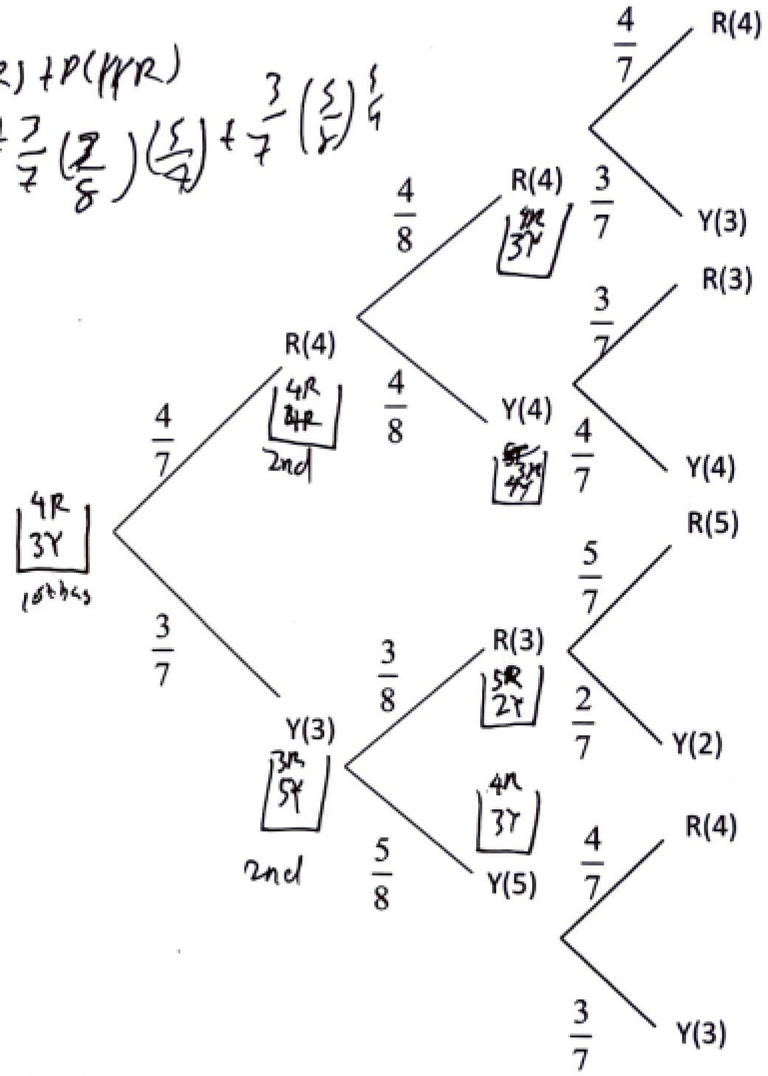
SOLUTION 13:

P(red ball drawn finally)

$= P(RRR) + P(RYR) + P(YRR) + P(YYR)$

$= \frac{4}{7}\left(\frac{4}{8}\right)\left(\frac{4}{7}\right) + \frac{4}{7}\left(\frac{4}{8}\right)\left(\frac{3}{7}\right) + \frac{3}{7}\left(\frac{3}{8}\right)\left(\frac{5}{7}\right) + \frac{3}{7}\left(\frac{5}{8}\right)\frac{4}{7}$

$\frac{31}{56}$



SELF-REVIEW 5:

A manufacturer carried out a survey of the defects in their soft toys. It is found that the probability of a toy having poor stitching is 0.03 and that a toy with poor stitching has a probability 0.7 of splitting open. A toy without poor stitching has a probability 0.02 of splitting open.

(i) Draw a tree diagram to represent this information.

(ii) Find the probability that a randomly chosen toy splitting open.

(iii) Find the probability that a randomly chosen toy has exactly one of the two defects, poor stitching or splitting open.

The manufacturer also finds that the fabric can become faded with a probability 0.05 and that this defect is independent of poor stitching or splitting open. A soft toy is chosen at random.

(iv) Find the probability that the soft toy has none of these 3 defects.

(v) Find the probability that the soft toy has exactly one of these 3 defects.

[0.0404, 0.0284, 0.903, 0.0745]

2.2 CONDITIONAL PROBABILITY

Conditional probability is the probability obtained with the additional information that some other event has already occurred. The conditional probability of an event B occurring given that event A has already occurred is denoted by $\mathrm{P}(B \mid A)$, which read as " the (conditional) probability of B given A ".

SOME EXAMPLES :

Mary claims :"I seem to be a very bad judge of the weather. I drag my umbrella with me three-quarters of the time and yet, although it only rains on a quarter of the days when I take it, it rains on two-thirds of the days when I do not take it. So, on a rainy day, I always seem to get wet." How true is Mary's statement?

George said, "Mathematics examinations are all a matter of luck really. If I happen to get the first question out, then it is more likely that I will finish the second; if I get the second right, it is more likely that I will finish the third, and so on. But if I cannot do the first question, I get in a panic and then I probably find that I cannot do the second or the third or the fourth."

EXAMPLE 14:

Consider the experiment of an unbiased die numbered '1' to '6' on its six faces is tossed.

(i) Find the probability that an even number is obtained.

(ii) Given that a number less than 6 is obtained, find the probability that it is an even number.

(iii) Given that an even number is obtained, find the probability that the number is less than 6.

SOLUTION 14:

Let S be the sample space i.e. $S = \{1, 2, 3, 4, 5, 6\}$,

X be the event that an even number is obtained i.e. $X = \{2, 4, 6\}$,

Y be the event that the number obtained is less than '6', i.e. $Y = \{1, 2, 3, 4, 5\}$.

(i) Without any prior information of the outcome,

$$\text{P(an even number is obtained)} = \mathrm{P}(X) = \frac{\mathrm{n}(X)}{\mathrm{n}(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) Since we know that event Y has occurred, the number '6' cannot occur.

Hence we now have a **reduced sample space** given by $Y = \{1, 2, 3, 4, 5\}$.

Based on this smaller sample space,

P(an even number is obtaned | the number obtained is less than '6')

$$\cancel{= \frac{n(\{2,4\})}{n\{1,2,3}} = \frac{n(X \cap Y)}{n(Y)} = \frac{2}{5}$$

$$P(X|Y)\ \frac{n(\{2,4\})}{n(\{1,2,3,4,5\})}$$

iii) P(number obtained is less than 6 given that the number obtaned is even)

$$= P(Y|X) = \frac{n(\{2,4\})}{n(\{2,4,6\})} = \frac{2}{3}$$

$$= \frac{n(Y \cap X)}{n(X)}$$

2.2.1 Definition of Conditional Probability

EXAMPLE 14 illustrated that for two events X and Y, the conditional probability of X given Y is simply the proportion of outcomes in X that are in Y, i.e. $\mathrm{P}(X \mid Y)=\dfrac{\mathrm{n}(X \cap Y)}{\mathrm{n}(Y)}$.

If we divide both the numerator and denominator by $\mathrm{n}(S)$, the number of elements in the sample space S, we get a more useful result $\mathrm{P}(X \mid Y)=\dfrac{\frac{\mathrm{n}(X \cap Y)}{\mathrm{n}(S)}}{\frac{\mathrm{n}(Y)}{\mathrm{n}(S)}}=\dfrac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)}$.

$P(X/Y) = P(X) = \dfrac{P(X \cap Y)}{P(Y)}$ when X and Y are independent

$= \dfrac{P(X)P(Y)}{P(Y)}$

> If X and Y are two events such that $\mathrm{P}(Y) > 0$, then the probability that X occurs given that Y has already occurred is given by the conditional probability $\mathrm{P}(X \mid Y)$ with
>
> $$\mathrm{P}(X \mid Y)=\frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)}$$

NOTE:

(i) In general, $\mathrm{P}(X \mid Y) \neq \mathrm{P}(Y \mid X)$ (illustrated in Example 12) 14

(ii) $\mathrm{P}(X \mid Y)=\dfrac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)} \quad \Rightarrow \quad \mathrm{P}(X \cap Y)=\mathrm{P}(X \mid Y)\mathrm{P}(Y)$

$\mathrm{P}(Y \mid X)=\dfrac{\mathrm{P}(X \cap Y)}{\mathrm{P}(X)} \quad \Rightarrow \quad \mathrm{P}(X \cap Y)=\mathrm{P}(Y \mid X)\mathrm{P}(X)$

Thus, $\mathrm{P}(X \mid Y)\mathrm{P}(Y)=\mathrm{P}(Y \mid X)\mathrm{P}(X)$

THINKZONE

When would $\mathrm{P}(X \mid Y)=\mathrm{P}(Y \mid X)$?

EXAMPLE 15: [J86/2/6 modified]

A game is played with an ordinary six-sided die. A player throws this die, and if the result is 2, 3, 4 or 5, that result is the player's score. If the result is 1 or 6, the player throws the die a second time and the sum of the two numbers resulting from both throws is the player's score. Events A and B are defined as follows:

A : the player's score is 5, 6, 7, 8 or 9;
B : the player has two throws.

Show that $\mathrm{P}(A)=\dfrac{1}{3}$.

Find (i) $\mathrm{P}(A \cap B)$, (ii) $\mathrm{P}(A \cup B)$, (iii) $\mathrm{P}(A \mid B)$, (iv) $\mathrm{P}(B \mid A)$ and (v) $\mathrm{P}(B \mid A')$.

SOLUTION 15:

$P(A)$

$= P(\text{score} = 5, 6, 7, 8, 9)$

$= P(D_1 = 5) + P(D_1 = 1, D_2 = 4) + P(D_1 = 1, D_2 = 5) + P(D_1 = 1, D_2 = 6)$

$+ P(D_1 = 6, D_2 = 1) + P(D_1 = 6, D_2 = 2) + P(D_1 = 6, D_1 = 3)$

$= \frac{1}{6} + \frac{1}{6}\left(\frac{1}{6}\right) \times 6$

$= \frac{1}{3}$ (shown).

(i) $P(A \cap B)$

$= P(\text{score} = 5, 6, 7, 8, 9)$ and two throws

$= p(A) - p(D_1 = 5) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

(ii) $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{2}{6} - \frac{1}{6}$

$= \frac{1}{2}$

(iii) $P(A | B)$

$= \frac{P(A \cap B)}{P(B)}$

$= \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$

(iv)

(v)

$$P(B|A') = \frac{P(B \cap A')}{P(A')}$$

$$= \frac{P(\text{two throws and score is not } 5, 6, 7, 8 \text{ or } 9)}{P(\text{score is not } 5, 6, 7, 8, \text{ or } 9)}$$

$$= \frac{P(D_1 = 1, D_2 = 1, 2, 3) + P(D_1 = 6, D_2 = 4, 5, 6)}{1 - P(A)}$$

$$= \frac{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right) + \frac{1}{6}\left(\frac{3}{6}\right)}{1 - \frac{1}{3}} = \frac{1}{4}$$

Self-Review 6:

Two tetrahedral dice, with faces labelled 1, 2, 3 and 4 are thrown and the number on each which each lands is noted. The 'score' is the sum of these two numbers. Find the probability that (i) the score is even given that at least one die lands on a 3, (ii) at least one die lands on a 3 given that the score is even.

[$\frac{3}{7}$, $\frac{3}{8}$, Not independent]

Example 16:

A bag contains 20 balls, 10 of which are red, 8 of which are white and 2 of which are blue. The balls are indistinguishable apart from the colour. Two balls are drawn in succession, without replacement. Using a tree diagram or otherwise, find the probability that

(i) the first ball is blue and the second ball is white,
(ii) the second ball is white given that the first ball is blue,
(iii) one ball is blue and the other is white,
(iv) the second ball drawn is red,
(v) the second ball drawn is red if it is known that the first ball drawn is not blue.

Solution 16:

(i) P(the 1st ball is blue and the 2nd ball is white)

$= P(BW)$

$= \frac{2}{20} \times \frac{8}{19} = \frac{4}{95}$

(ii) P(2nd ball is white | 1st is blue)

$$= \frac{P(\text{2nd ball is white and 1st is blue})}{P(\text{1st is blue})}$$

$$= \frac{\frac{2}{20} \times \frac{8}{19}}{\frac{2}{20}}$$

$$= \frac{8}{19}$$

Or P(2nd ball is white | 1st ball is blue)

$$= \frac{8}{19}$$

> The conditional probability in (ii) can be found on one of the branches of the tree diagram when you draw a tree diagram!

(iii) P(one ball is blue and the other white)

$$= \mathrm{P}(BW) + \mathrm{P}(WB)$$

$$= \frac{2}{20}\times\frac{8}{19}+\frac{8}{20}\times\frac{2}{19}=\frac{8}{95}$$

(iv) P(second ball is red)

$$= P(RR) + P(WR) + P(BR)$$

(v) $$= \frac{10}{20}\times\frac{9}{19} + \frac{8}{20}\times\frac{10}{19} + \frac{2}{20}\times\frac{10}{19} = \frac{1}{2}$$

P(second ball drawn is red | the first ball drawn is not blue)

$$= \frac{P(R_2 \cap B_1')}{P(B_1')} = \frac{P(R_1R_2) + P(W_1R_2)}{1 - P(B_1)} = \frac{\frac{10}{20}\left(\frac{9}{19}\right) + \frac{8}{20}\left(\frac{10}{19}\right)}{1-\frac{2}{20}} = \frac{85}{171}$$

SELF-REVIEW 7:

Three children, Sean, Ryan and Justin, have equal plots in a circular patch of garden. The boundaries are marked out by pebbles. Sean has 80 red and 20 white flowers in his patch, Ryan has 30 red and 40 white flowers and Justin has 10 red and 60 white flowers. Their sister, Michelle, wants to pick a flower for her teacher.

(i) Find the probability that she picks a red flower.

(ii) If she picks a red flower, find the probability that it comes from Ryan's plot. $[\frac{16}{35}, \frac{5}{16}]$

2.2.2 INDEPENDENT EVENTS

From Section 2.1.7, we learnt that if two events X and Y are independent, then $\mathrm{P}(X \cap Y) = \mathrm{P}(X)\,\mathrm{P}(Y)$. Furthermore, recall that we define independent events as events that do not affect the occurrence of each other. Thus, $\mathrm{P}(X \mid Y) = \mathrm{P}(X)$ or $\mathrm{P}(Y \mid X) = \mathrm{P}(Y)$. We may use any of the 3 equations below to check for independence:

> X and Y are independent events $\Leftrightarrow \mathrm{P}(X \mid Y) = \mathrm{P}(X)$
> $\Leftrightarrow \mathrm{P}(Y \mid X) = \mathrm{P}(Y)$
> $\Leftrightarrow \mathrm{P}(X \cap Y) = \mathrm{P}(X) \times \mathrm{P}(Y)$

Moreover, if X and Y are independent events then

(i) X and Y' are independent ,

(ii) X' and Y are independent,

(iii) X' and Y' are independent.

We will prove (i), the rest will be left as exercises.

(i) Now $\mathrm{P}(X \cap Y)=\mathrm{P}(X)P(Y)$, we aim to show $\mathrm{P}(X \cap Y')=\mathrm{P}(X)\mathrm{P}(Y')$.

$$\begin{aligned}\mathrm{P}(X \cap Y') &= \mathrm{P}(X)-\mathrm{P}(X \cap Y)\\ &=\mathrm{P}(X)-\mathrm{P}(X)\mathrm{P}(Y)\\ &=\mathrm{P}(X)\left(1-\mathrm{P}(Y)\right)\\ &=\mathrm{P}(X)\mathrm{P}(Y')\end{aligned}$$

Thus, X and Y' are independent.

NOTE:

- In general, $\mathrm{P}(X \cap Y) \neq \mathrm{P}(X)\cdot\mathrm{P}(Y)$.
- For mutually exclusive events X and Y, $\mathrm{P}(X \mid Y)=0$ as $\mathrm{P}(X \cap Y)=0$.

2.3 PROBLEM SOLVING TECHNIQUES IN PROBABILITY

Probability problems come in various forms. It is difficult to categorise the methods to use for different problems. Nevertheless, here are some methods you can try:

(i) Table of Outcomes

(ii) Venn Diagrams

(iii) Tree Diagrams (particularly useful for sequential events)

(iv) Using counting principles (sometimes involving Permutations and Combinations)

$$\mathrm{P}(X)=\frac{\text{number of possible outcomes for } X \text{ to occur}}{\text{number of all possible outcomes for experiment}}$$

(v) Using the ideas in conditional probabilities, independent events and mutually exclusive events.

We shall now apply the methods mentioned above, wherever possible, to solve the following probability problems.

EXAMPLE 17: [2013/CJC/2/6]

For events A and B, it is given that $\mathrm{P}(B)=\frac{3}{5}$, $\mathrm{P}(A \cup B)=\frac{7}{8}$ and $\mathrm{P}(A' \cap B)=\frac{11}{36}$.

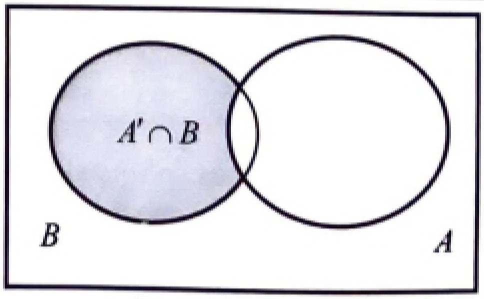
Find **(i)** $\mathrm{P}(A)$, **(ii)** $\mathrm{P}(B' \mid A)$.

A third event C, has $\mathrm{P}(C)=\frac{4}{7}$ and that A and C are independent.

(iii) Find $\mathrm{P}(A' \cap C)$.

SOLUTION 17:

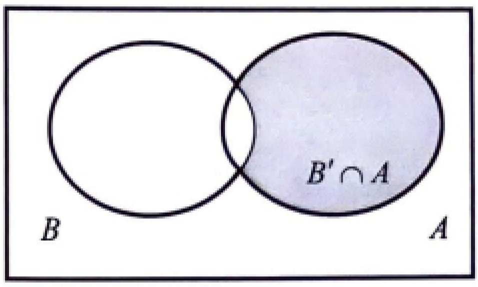
(i) $$P(B'|A)=\frac{P(B'\cap A)}{P(A)}=\frac{P(A\cup B)-P(B)}{41/72}=\frac{\frac{7}{8}-\frac{3}{5}}{41/72}=\frac{99}{205}$$



(ii) A and C are independent $\Rightarrow$ A' and C are independent

$P(A' \cap C) = P(A')P(C)$

$= \left(1-\frac{41}{72}\right)\left(\frac{4}{7}\right) = \frac{31}{126}$



(iii)

EXAMPLE 18: [J91/2/6 modified]

A choir has 7 sopranos, 6 altos, 3 tenors and 4 basses. The sopranos and altos are women and the tenors and basses are men. At a particular rehearsal, three members of the choir are chosen at random to make tea.

(i) Find the probability that three sopranos are chosen.

(ii) Find the probability that exactly one bass is chosen.

(iii) Find the probability that the chosen group contains exactly one tenor or exactly one bass (or both).

(iv) Find the probability that the chosen group contains exactly one tenor, given that it contains exactly one bass.

SOLUTION 18:

7 sopranos, 6 altos $\rightarrow$ 13 women

3 tenors, 4 basses $\rightarrow$ 7 men

(i) $\mathrm{P}(3 \text{ sopranos}) = \frac{^{7}C_3}{^{20}C_3} = \frac{7}{228}$

> Alternatively, we can also solve part (i)
> P(3 sopranos) = $\frac{7}{20}\left(\frac{6}{19}\right)\left(\frac{5}{18}\right) = \frac{7}{228}$ and also adopt a similar approach for the other parts of the question but we choose not to since **<u>when ordering is not important, the combinatorics method is preferred</u>.**

(ii) P(exactly 1 bass) $= \frac{^{4}C_1 \times {}^{16}C_2}{^{20}C_3} = \frac{8}{19}$

(iii) P(exactly 1 tenor or exactly 1 bass)

> When the question is phrased in the form of P(*A* or *B* (or both)), it refers to $\mathrm{P}(A \cup B)$.

$= \mathrm{P}(\text{exactly 1 tenor}) + \mathrm{P}(\text{exactly 1 bass}) - \mathrm{P}(\text{exactly 1 tenor \& 1 bass})$

$= \frac{^{3}C_1 \times {}^{17}C_2}{^{20}C_3} + \frac{8}{19} - \frac{^{3}C_1 \times {}^{4}C_1 \times {}^{13}C_1}{^{20}C_1} = \frac{34}{95} + \frac{8}{19} - \frac{13}{95}$

$= \frac{61}{95}$

(iv) $\mathrm{P}(\text{exactly 1 tenor} \mid \text{exactly 1 bass})$

$$= \frac{\mathrm{P}(\text{exactly 1 tenor \& exactly 1 bass})}{\mathrm{P}(\text{exactly 1 bass})}$$

EXAMPLE 19: [2010/NYJC/2/10b]

A bag contains 10 red marbles and 90 blue marbles which are indistinguishable, apart from colour. Marbles are drawn from the bag singly and at random, with replacement. Valerie and Aazon took turns to draw marbles from the above mentioned bag. The colour is observed and the marble is replaced. The first person who draws a red marble wins the game and the game will stop.

(v) Given that Aazon starts first, find the probability that Valerie wins the game.

(vi) What is the least number of draws before we can have a 90% chance of having a winner?

SOLUTION 19:

(i) Let A be the event that "Aazon picks a red marble" and V be the event that "Valerie picks a red marble".

Probability required

$$= \mathrm{P}(A' \cap V) + \mathrm{P}((A' \cap V' \cap A' \cap V) + \mathrm{P}(A' \cap V' \cap A' \cap V' \cap A' \cap V) + \cdots$$

$$= \left[\left(\frac{9}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^5\left(\frac{1}{10}\right) + \cdots\right]$$

$$= \frac{\left(\frac{9}{10}\right)\left(\frac{1}{10}\right)}{1 - \left(\frac{9}{10}\right)^2}$$

$$= \frac{9}{19}$$

(ii) Let the least number of draws required be n, then

$$\frac{1}{10} + \left(\frac{9}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right) + \ldots \left(\frac{9}{10}\right)^{n-1}\left(\frac{1}{10}\right) > 0.9$$

$$\frac{(0.1)(1 - 0.9)^n}{1 - 0.9} > 0.9$$

$$0.9^n < 0.1$$

$$n > \frac{\ln 0.1}{\ln 0.9} = 21.854$$

Least $n = 22$

THINKZONE:

- How would you change the game to make it a fair game?

EXAMPLE 21: [2016/RI/2/8]

For events A and B, it is given that $\mathrm{P}(A)=\frac{5}{8}$ and $\mathrm{P}(B)=\frac{2}{3}$.

(i) Find the greatest and least possible values of $\mathrm{P}(A\cap B)$.

It is given in addition that $\mathrm{P}(A'|B')=\frac{3}{8}$.

(ii) Find $\mathrm{P}(A'\cap B')$. $\hookrightarrow$ $\frac{\mathrm{P}(A'\cap B')}{\mathrm{P}(B')}$

(iii) Find $\mathrm{P}(A\cup B)$.

(iv) Determine if A and B are independent events.

(v) Given another event C such that $\mathrm{P}(C)=\frac{3}{8}$, $\mathrm{P}(A\cap C)=\mathrm{P}(B\cap C)=\frac{1}{4}$ and $\mathrm{P}(A\cup B\cup C)=\frac{11}{12}$, find $\mathrm{P}(A\cap B\cap C)$.

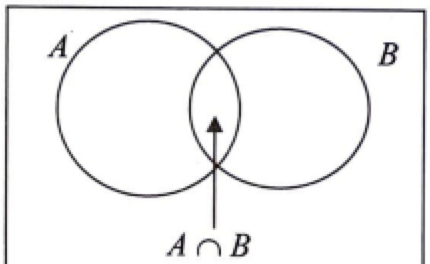
SOLUTION 21:

(i) $\mathrm{P}(A)+\mathrm{P}(B)-1\le \mathrm{P}(A\cap B)\le \min\{\mathrm{P}(A),\ \mathrm{P}(B)\}$

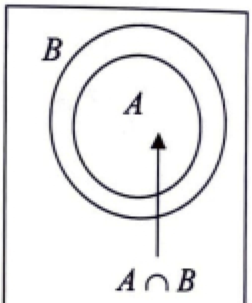
$P(A\cap B) = P(A)+P(B)-P(A\cup B)$

Greatest value of $P(A\cap B)$ is $5/8$

Least value of $P(A\cap B)=\frac{7}{24}$



Least value occurs when $\mathrm{P}(A\cup B)=1$



Greatest value occurs when A is a proper subset of B.

(i) $\mathrm{P}(A'\cap B')=\mathrm{P}(A'|B')\mathrm{P}(B')=\frac{3}{8}\times\frac{1}{3}=\frac{1}{8}$

(ii) $\mathrm{P}(A\cup B)=1-\mathrm{P}(A'\cap B')=\frac{7}{8}$

(iii)

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$$

$$\frac{7}{8}=\frac{5}{8}+\frac{2}{3}-\mathrm{P}(A\cap B)$$

$$\mathrm{P}(A\cap B)=\frac{5}{12}=\frac{5}{8}\times\frac{2}{3}=\mathrm{P}(A)\mathrm{P}(B)$$

Hence, A and B are independent events.

Alternatively,
since $\mathrm{P}(A'|B')=\frac{3}{8}=1-\frac{5}{8}=\mathrm{P}(A')$
A' and B' are independent events
$\Rightarrow$ A and B are independent events

(v) $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

$$\frac{11}{12}=\frac{5}{8}+\frac{2}{3}+\frac{3}{8}-\frac{5}{12}-\frac{1}{4}-\frac{1}{4}+P(A\cap B\cap C)$$

$$P(A\cap B\cap C)=\frac{1}{6}$$