5. Sequences and Series

1 MJC/2008Promo/6

(a) The sum to infinity of a geometric progression is $\frac{9}{2}$ and the second term of the progression is -2. Find the common ratio. **[3]**

(b) An arithmetic progression has n terms and a common difference d, where $d > 0$. Prove that the difference between the sum of the last k terms and the sum of the first k terms is $(n-k)kd$. **[4]**

2 HCI/2010Promo/6

(a) An arithmetic progression has first term a and common difference d. The sum of the first 3 terms is equal to the sum of the next 6 terms. Find d in terms of a. **[2]**

(b) The sum of the first n terms of a sequence is given by

$(a-2)^{-n} - 1$, where a is a constant.

(i) Show that the sequence is a geometric progression, and state its common ratio in terms of a. **[3]**

(ii) Find the set of values of a for which the sum to infinity of the sequence exists. **[3]**

3 SAJC/2010Promo/5(a)

Given that $S_n = \frac{2^{n+1}}{n!} - 1$, find

(i) T_n, giving your answer in a single fraction. **[2]**

(ii) the exact value of $T_4 + T_5 + ... + T_8$. **[2]**

4 JJC/2009Promo/3

(a) In a geometric progression, the first term is 2009 and its common ratio is $-\frac{5}{7}$.

(i) Find the least value n such that $|U_n| < \frac{1}{2009}$, where U_n denotes the nth term of the progression. **[3]**

(ii) Find, correct to 2 decimal places, the sum of all the negative terms of the progression. **[3]**

(b) 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5………., k…… is a sequence where the number k appears k times successively ($k = 1, 2, 3, 4, ...$). Find the 1000th term of the sequence. **[4]**

5 HCI/2009Promo/8

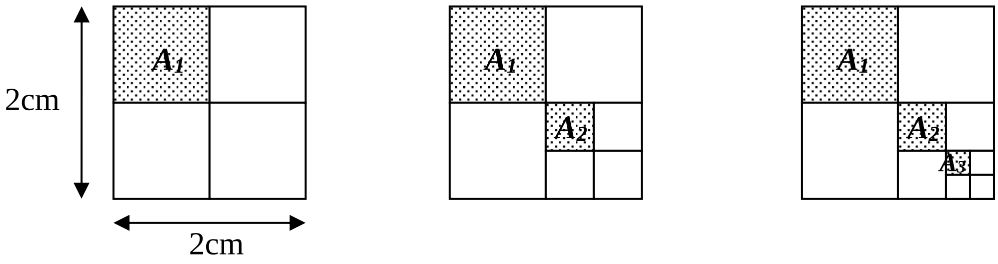


Diagram 1 **Diagram 2** **Diagram 3**

(i) Show that the areas of the shaded squares $A_1, A_2, \ldots, A_n$ in the n^{th} diagram form a geometric progression. **[2]**

(ii) Show that the total area of the shaded squares in the n^{th} diagram S_n is $\frac{4}{3}\left(1-\frac{1}{4^n}\right)$. **[2]**

(iii) Let S be the total shaded area in the n^{th} diagram as $n \to \infty$. Find the value of S. **[2]**

(iv) Find the least value of n for which the difference between S_n and S is less than 1% of S. **[3]**

6 RVHS/2009Promo/7

Each year in June approximately 10% of the trees die out and in December, the workers plant 100 new trees. At the end of December 2000 there are 1200 trees in the plantation.

(i) Find the number of living trees at the end of December 2002. **[3]**

(ii) Consider 2001 as the first year. Show that the number of living trees at the end of December in the nth year is given by $\left(\frac{9}{10}\right)^n (1200) + 1000(1-0.9^n)$. **[3]**

What happens to the number of living trees in the plantation for large values of n? **[2]**

7 MJC/2015Promo/11

(a) The sum of the first n terms of a series is $n(4n-1)$. Obtain an expression for the nth term of the series. Hence prove that the series is arithmetic. **[3]**

(b) A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The second, fifth and seventh terms of the arithmetic series are three consecutive terms of the geometric series.

(i) Show that the geometric series is convergent. **[4]**

(ii) It is given that the first term of the geometric series is the same as the first term of the arithmetic series. Let S be the sum to infinity of all the even-numbered terms of the geometric series and A_n be the sum of the first n terms of the arithmetic series. Given $a > 0$, find the least value of n such that $S + A_n < 0$. **[4]**

8 DHS/2010Promo/8

A special robotic pen is programmed to draw a pattern consisting of squares in increasing sizes with no overlapping of lines. The squares have lengths which follow an arithmetic progression and the first three squares have lengths 4 cm, 6 cm and 8 cm respectively (see Figure (a)).

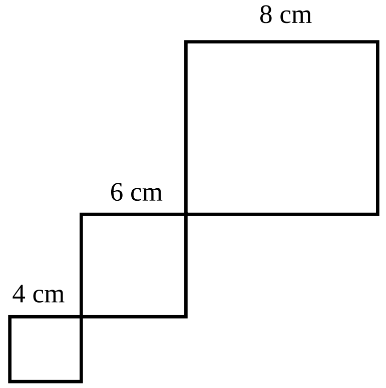


Figure (a)

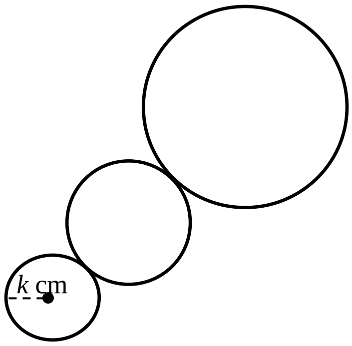


Figure (b)

(i) If the pattern in Figure (a) continues until the 30^{th} square is completed, calculate the total perimeter of the squares formed. **[2]**

(ii) If the amount of ink in one pen can only draw up to a maximum of 10 000 cm, find the length of the largest complete square drawn when the ink runs out. **[4]**

(iii) The robotic pen is reprogrammed to draw three circles as shown in Figure (b) which has the same total area as the three squares in Figure (a).

The smallest circle of radius k cm has the same area as the smallest square of length 4 cm.

Given that the radius of each circle follows a geometric progression, determine the common ratio R. **[3]**

9 PJC/2013Promo/14(a)

Mr Lee invests $\$A$ at the beginning of each year for n years, where n is a positive integer. Compound interest accumulates at the rate of R % annually and is calculated at the end of each year.

Show that the total value of Mr Lee's investment at the end of n years is given by

$$\$\,A\left(1+\frac{100}{R}\right)\left[\left(1+\frac{R}{100}\right)^n-1\right].$$ **[3]**

Suppose that $R = 8$.

(i) Mr Lee requires his investment to be worth \$50,000 at the end of 10 years. Find correct to the nearest dollar, the value of A required to achieve this. **[2]**

(ii) His wife decides that they can afford to invest \$1000 per year. In how many years will the value of their investment reach \$50,000? **[2]**

10 SAJC/2009Promo/13

(a) The cost of Sonia's new car was \$$P$. She accepted an interest-free loan of \$$P$, which she agreed to repay by monthly instalments. The first instalment was \$1200. The instalments were increased by \$50 per month so the second and third instalments were \$1250 and \$1300 respectively.

Given that the loan was repaid in k instalments, and that the final instalment was \$3250,

(i) find the value of k and P. **[3]**

The value of Sonia's car at the end of the first year was \$72000. After the first year, the value of the car depreciated, each month, by 2% of its value at the start of that month.

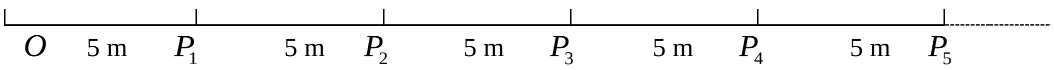
(ii) Calculate, to the nearest dollar, the value of Sonia's car at the end of the third year. **[2]**

(iii) Given that the value of the car depreciated to less than 50% of the cost of the new car by the n^{th} month. Find the value of n. **[3]**

(b) Given that the terms of the sequence $a_1, a_2, a_3, \ldots, a_n$ are in arithmetic progression and $b_r = \left(\frac{1}{3}\right)^{a_r}$, for $r = 1, 2, 3, \ldots, n$, show that the sequence $b_1, b_2, b_3, \ldots, b_n$ is geometric. Given that $b_2 = 27$ and the common difference of the arithmetic progression is 2, find an expression for a_r. **[4]**

11 SAJC/2015/Promo/10

(a) In a training exercise, athletes run from a starting point O to and from a series of points, $P_1, P_2, P_3, \ldots$ increasingly far away in a straight line. The distances between adjacent points are all 5 m. (Refer to the diagram below.)



In the exercise, athletes start at O and run stage 1 from O to P_1 and back to O, then stage 2 from O to P_2 and back to O, and so on.

Write down an expression for the distance run by an athlete who completes n stages of the exercise. Hence find the least number of stages that the athlete needs to complete in order to run at least 6 km. **[4]**

(b) Each time a ball falls vertically onto a horizontal surface, it rebounds to two-thirds of the height from which it fell. The ball is initially dropped from a point 12 m above the surface. Show that the total distance the ball has travelled just before it touches the surface for the nth time is $\left[60 - 72\left(\frac{2}{3}\right)^n\right]$ m. Hence state the total distance it has travelled before coming to rest. **[4]**

12 RI/2010Promo/6

An increasing arithmetic progression whose n^{th} term is denoted by u_n is such that u_1, u_4 and u_8 are consecutive terms of a geometric progression and

$$u_{10} + u_{12} + u_{14} + ... + u_{38} + u_{40} = 1056 .$$

Find the first term and common difference of the arithmetic progression and show that $u_{108} = 232$. **[8]**

13 TJC/2010Promo/4

The progression H is formed by adding corresponding terms of an arithmetic progression with common difference −2 and a convergent geometric progression with first term 4.

Given that the second term of the progression H is −4 and the sum of the first 3 terms of H is also −4, find the first term of H, leaving your answer in exact form. **[6]**

14 RI/2014Promo/6

The annual wage of a certain occupation offered by Company P begins with an initial amount of \$9500 and increases by \$400 every year for 14 years till it reaches \$15100 and remains constant at \$15100 thereafter.

Let $p(n) = a_1 + a_2 + a_3 + \ldots + a_n$, where a_i denotes the annual wage in the i^{th} year offered by Company P for i = 1, 2, 3, …, n.

(i) Show that $p(n) = \begin{cases} 200n^2 + 9300n & \text{for } n \leq 15, \\ 15100n + A & \text{for } n > 15, \end{cases}$

where A is a constant to be determined. **[4]**

(ii) The annual wage of the same occupation offered by Company Q begins with the same initial amount of \$9500 but it increases at a fixed rate of r % every year.

Sam has been offered a job for the same occupation at both Company P and Q. Find the value of r, correct to 2 decimal places, such that the total annual wages earned by Sam if he works for Company P for 15 years is equivalent to the total annual wages earned if he works for Company Q for 12 years. **[4]**

15 IJC/2010Promo/12

The positive even integers, starting at 2, are grouped into sets containing 1, 2, 4, 8, … integers, as indicated below, so that the number of integers in each set after the first is twice the number of integers in the previous set.

$$\{\,2\,\},\ \ \{\,4,\,6\,\},\ \ \{\,8,10,12,14\,\},\ \ \{\,16,18,\,20,\,22,\,24,\,26,\,28,\,30\,\},\ \ldots$$

(i) Write down expressions, in terms of r, for

(a) the number of integers in the rth set, **[1]**

(b) the first integer in the rth set, **[1]**

(c) the last integer in the rth set. **[1]**

(ii) The sum of all the integers in the 50$^{\text{th}}$ set is denoted by T. Show that T may be expressed as $T = 2^{48}(3\times 2^{50}-2)$. **[2]**

(iii) Find S_n in terms of n, where S_n is the sum of all the terms from the first set to the nth set. **[4]**

(iv) Hence find the least value of n such that $S_n >$ 100 000. **[2]**

16 CJC/2012Promo/6 amended

It is given that $\displaystyle\sum_{r=1}^{N}\frac{2r+1}{r^2(r+1)^2} = 1-\frac{1}{\left(N+1\right)^2}$.

(i) Deduce the value of $\displaystyle\sum_{r=2}^{\infty}\frac{2r+1}{r^2(r+1)^2}$. **[2]**

(ii) Hence, find $\displaystyle\sum_{r=0}^{N-2}\frac{2r+3}{(r+1)^2(r+2)^2}$ in terms of N. **[2]**

17 ACJC/2020Promo/10 amended

It is given that

$$\sum_{r=1}^{n}\frac{2r+3}{r(r+1)(r+2)} = \frac{7}{4}-\frac{3}{2(n+1)}-\frac{1}{2(n+2)}.$$

(i) Explain why $\displaystyle\sum_{r=1}^{\infty}\frac{2r+3}{r(r+1)(r+2)}$ converges, and state the convergence limit. **[2]**

(ii) Find the least value of n such that $\displaystyle\sum_{r=1}^{n}\frac{2r+3}{r(r+1)(r+2)} > \frac{8}{5}$. **[2]**

(iii) Evaluate

$$\frac{9}{3\times4\times5}+\frac{11}{4\times5\times6}+\frac{13}{5\times6\times7}+\ldots\ldots+\frac{2N+1}{N\left(N^2-1\right)},$$

leaving your answer in terms of N. **[3]**

18 IJC/2009Promo/8

A sequence $u_1, u_2, u_3, ..., u_n, ...$ is such that $u_1 = 3$ and

$$u_{n+1} = \frac{3}{4}u_n + 4, \quad \text{for } n \geq 1.$$

(i) Express u_n in the form $16 + A\left(\frac{3}{4}\right)^{n-1}$, where A is a constant to be determined. **[4]**

(ii) Determine whether the sequence $u_1, u_2, u_3, ..., u_n, ...$ converges. **[2]**

19 TJC/2009Promo/10 amended

A recurrence relation is defined by

$$x_{n+1} = \frac{x_n^2 + 6x_n}{x_n^2 + x_n + 1},$$

where $n = 1, 2, 3, \ldots$

(i) If x_n converges to α when n is large, find the exact values of α. **[3]**

(ii) If $x_1 = 0.5$, obtain the first 4 terms of the sequence $\{x_n\}$. What can you say about the sequence $\{x_n\}$ when n is large? **[2]**

20 MJC/2010Promo/10 amended

It is given that $\sum_{r=3}^{n} \frac{1}{r(r-2)} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right)$.

(i) Give a reason why the series in part **(i)** is convergent and state the sum to infinity. **[2]**

(ii) By completing the square for the expression $r(r-2)$, show that for $r \geq 3$,

$$\frac{1}{(r-1)^2} < \frac{1}{r(r-2)}.$$

Hence show that $\sum_{r=3}^{\infty} \frac{1}{r^2} < \frac{1}{2}$. **[4]**

21 AJC/2010Promo/12

A sequence of positive numbers $\{u_n\}$ is given by the recurrence relation

$$u_{n+1} = \frac{4(1+u_n)}{4+u_n} \text{ and } u_1 = 1.$$

(i) Given the sequence $\{u_n\}$ converges to l as $n \to \infty$, find the value of l. **[2]**

(ii) If $u_n < l$, show that $u_{n+1} > u_n$ for all $n \in \mathbb{Z}^+$. **[3]**

Answers

1	(a) $r=-\frac{1}{3}$
2	(a) $d=-\frac{a}{10}$ (b) (i) $\frac{1}{a-2}$ (ii) $\{a\in\mathbb{R}: a>3 \text{ or } a<1\}$
3	(i) $\frac{2^n(2-n)}{n!}$ (ii) $-\frac{836}{315}$
4	(a) (i) 47 (ii) -2929.79 (b) 45
5	(iii) $\frac{4}{3}$ (iv) least $n=4$
6	(i) 1162 (ii) approaches 1000
7	(a) $T_n=8n-5$ (b)(ii) least $n=23$
8	(i) 3960 cm (ii) 98 cm (iii) R = 1.43
9	(i) $3196 (ii) 21 years
10	(a) (i) k = 42; P = 93450 (ii) $44336 (iii) n = 34 (b) $a_r=2r-7$
11	(a) Least number of stages is 35 (b) 60 m
12	$a=18, d=2$
13	$4\sqrt{2}-2$
14	(i) -42000 (ii) $r=8.39$ (2 dp).
15	(i) (a) No. of integers r^{th} set $=2^{r-1}$ (b) First integer in r^{th} set $=2^r$ (c) Last integer in r^{th} set $=2^{r+1}-2$ (iii) $S_n=2^n(2^n-1)$ (iv) Least n is 9
16	(i) $\frac{1}{4}$ (ii) $1-\frac{1}{N^2}$
17	(i) $\frac{7}{4}$ (ii) n = 13 (iii) $\frac{5}{8}-\frac{3}{2N}-\frac{1}{2(N+1)}$
18	(i) -13 (ii)
19	(i) 0 or $\pm\sqrt{5}$ (ii) 0.5; $\frac{13}{7}$; 2.31; 2.22; $\sqrt{5}$
20	(i) $\frac{3}{4}$
21	(i) 2