Oscillations (Topic 10)

Content

1. Mathematical Pre-requisite
2. Free Oscillation and Simple Harmonic Motion
3. Energy in Simple Harmonic Motion
4. Damped Oscillation
5. Forced Oscillation and Resonance

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations.

(b) investigate the motion of an oscillator using experimental and graphical methods.

(c) Show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms both frequency and angular frequency.

(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.

(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\, \omega\sqrt{(x_0^2 - x^2)}$

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.

(h) describe the interchange between kinetic and potential energy during simple harmonic motion.

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in applications such as a car suspension system.

(j) describe practical examples of forced oscillations and resonance.

(k) describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.

(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

Important Note: For this topic your calculator needs to be in **radian mode**. Otherwise there will be calculation errors.

1. Mathematical Pre-requisite

In this chapter, you will use the sine (sinusoidal) graph frequently and hence, you need to know how to sketch the sine graph given an equation of the form $x = x_0 \sin\theta$ as shown in Fig. 1A below:

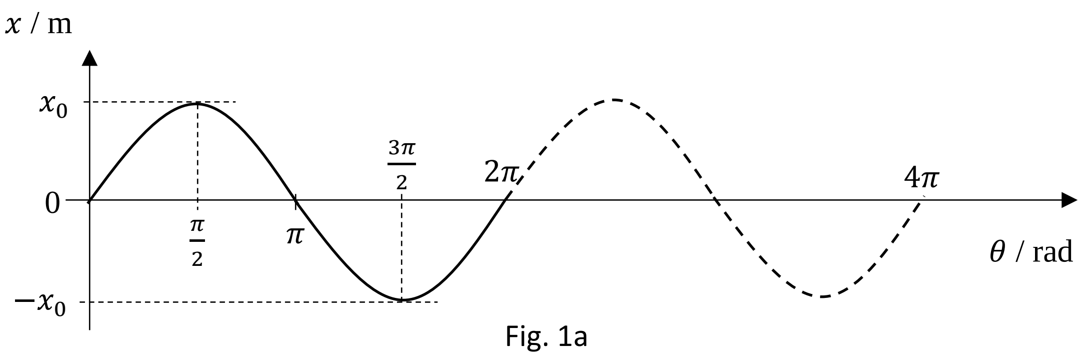


Fig. 1a

Notice that the sine graph repeats itself after every 2π radians or 360 degrees. The repeated pattern is shown in dotted line in Fig. 1a above.

Also, the displacement (x), of an object executing simple harmonic motion (for example, a pendulum) can be represented by an equation of the form: $x = x_0 \sin(\omega t)$. The term (ωt) has units of radians (i.e. it is dimensionless). The graph of x against t is shown in Fig. 1B below:

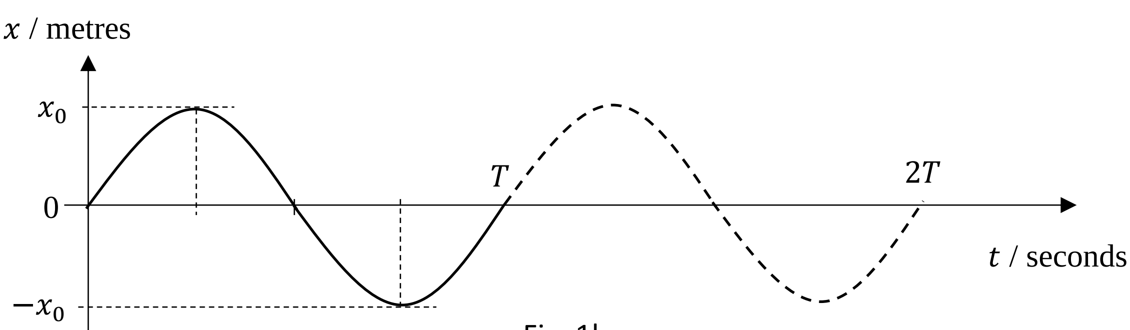


Fig. 1b

Notice that the graph in Fig. 1b repeats itself after every T seconds where T represents the period of oscillation. We can use this fact to convert θ in radian to t in second, or vice versa, using direct proportion as follows:

$$2\pi \rightarrow T$$

$$\theta \rightarrow t$$

By proportion: $\frac{\theta}{2\pi} = \frac{t}{T}$ hence, $\theta = \left(\frac{2\pi}{T}\right) t = \omega t$ where $\omega = \frac{2\pi}{T}$

Replacing θ with (ωt) in the equation $x = x_0 \sin\theta$, we get $x = x_0 \sin(\omega t)$

The equation $x = x_0 \sin(\theta - \phi) = x_0 \sin(\omega t - \phi)$ represents a translation (or shifting) of the sine graph along the positive θ (or positive t) direction by ϕ radians as shown in Fig. 1c below:

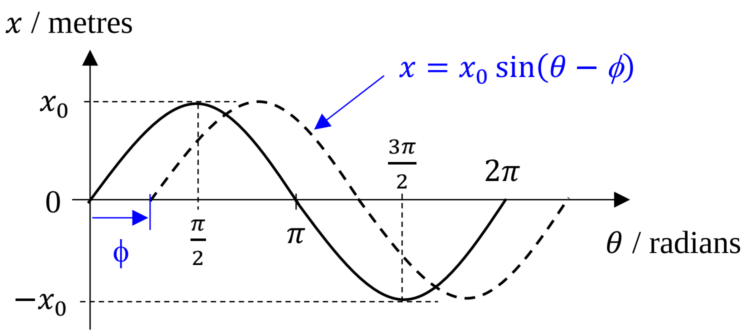


Fig. 1c

Hence if ϕ = 90°, then you will end up with a negative cosine function:

$$x = x_0 \sin\left(\theta - \frac{\pi}{2}\right) = x_0\left[\sin\theta\cos\left(\frac{\pi}{2}\right) - \cos\theta\sin\left(\frac{\pi}{2}\right)\right] = -x_0\cos\theta$$

since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$

You also need to know the following relations:

$\frac{d}{dt}[\sin(\omega t)] = \omega\cos(\omega t)$ i.e. differentiating a sine function gives a positive cosine function

$\frac{d}{dt}[\cos(\omega t)] = -\omega\sin(\omega t)$ i.e. differentiating a cosine function gives a negative sine function

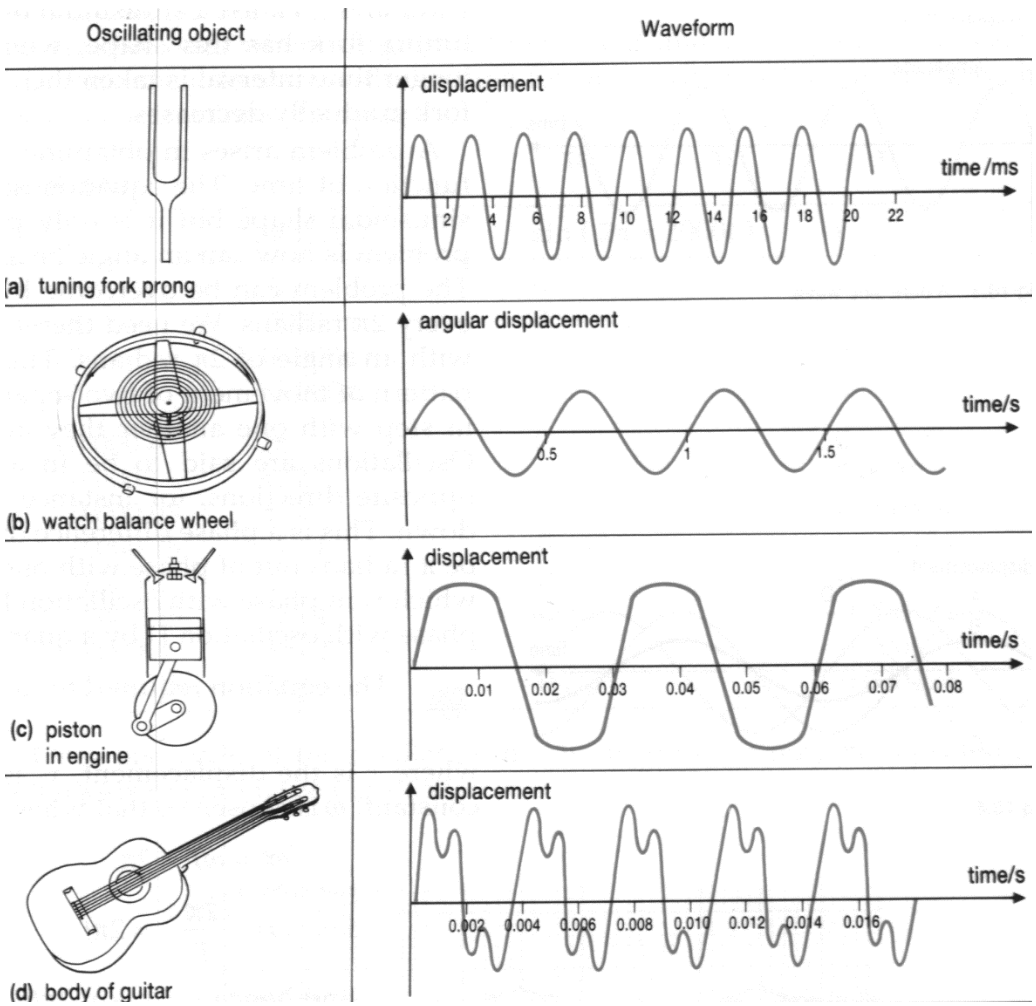
2. Free Oscillation and Simple Harmonic Motion

An object is said to be undergoing free oscillations when the only external force acting on it is the restoring force.

A restoring force is a resultant force that acts in opposite direction to the displacement of the object from an equilibrium position (where the resultant force is zero).

Simple harmonic motion is the simplest type of free oscillation.

2.1 Examples of oscillations



Other examples : https://www.youtube.com/watch?v=VKtEzKcg6_s

2.2 The Meaning of Some Terms used in SHM

- **Period ($\boldsymbol{T}$):** The period $\boldsymbol{T}$ is the time taken for one complete oscillation.

- **Frequency ($\boldsymbol{f}$):** The number of oscillations per unit time is called frequency, f. Frequency is equal to the inverse of the period, $f = \frac{1}{T}$. The SI units of frequency is Hz (hertz) where 1 Hz = 1 cycle (or oscillation) per second.

- **Angular frequency ($\boldsymbol{\omega}$):** Angular frequency, ω is defined as the product of the frequency and 2π. $\omega = 2\pi f = \frac{2\pi}{T}$. Units of angular frequency is rad s^{-1}.

- **Equilibrium position:** The position at which no net force acts on the oscillating object is called the **equilibrium position**.

- **Displacement ($\boldsymbol{x}$):** Displacement is the distance the object has moved from its equilibrium position in a stated direction. It is a vector quantity and it measured with respect from the equilibrium position.

- **Amplitude ($\boldsymbol{x_0}$):** Amplitude is the magnitude of the maximum displacement from equilibrium position. It is a scalar quantity.

  For example, the displacement, x of an object in simple harmonic motion can be represented by the equation: $x = x_0 \sin(\omega t + \phi) = 5 \sin\left(\omega t + \frac{\pi}{2}\right)$

  The amplitude is $x_0 = 5$ units

- **Phase:** The term $(\omega t \pm \phi)$ is called the <u>phase</u>. The phase denotes the <u>state of oscillation</u> at a particular time t and it has units of radian. In general, the state of motion/oscillation at time t means the displacement, velocity and acceleration of the object at time t.

  The term $\pm\phi$ inside the bracket represents the <u>initial phase</u> of the oscillating object which is the state of oscillation at $t = 0$. The initial phase (ie the value of $\pm\phi$) depends on the object's position at $t = 0$ and whether it is moving up or moving down at that instant.

- **Phase difference:** The term <u>phase difference</u> is used to compare the difference in the state of oscillation between two oscillators or the difference in the state of oscillation at two instances in time for one oscillator. Phase difference is a measure of how much one oscillation is out of step with another. Phase difference is expressed in units of radian.

  If the two oscillations are exactly in step with one another they are said to be <u>in phase</u> with one another or the phase difference is zero radian (or integer multiples of 2π radian).

  If two oscillators are <u>always</u> <u>moving in opposite directions</u>, then they are said to be in <u>antiphase</u> or the phase difference is π, 3π, 5π, 7π, or any odd integer multiples of π radian or a phase difference of half a cycle, 1.5 cycles, 2.5 cycles, 3.5 cycles, etc.

2.3 Simple Harmonic Motion (SHM)

Definition

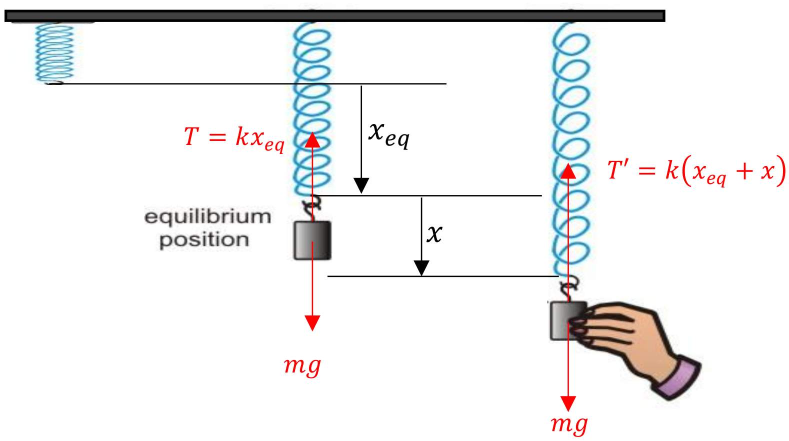
The motion of an object is simple harmonic if its acceleration is directly proportional to its displacement from its equilibrium position and its acceleration is always opposite in direction to its displacement.

Note:

To show that the motion of an object is simple harmonic, we have to use Newton's 2nd law to show that $a \propto -x$ where a is the acceleration and x is the displacement of the object from equilibrium position. The proportionality constant is ω^2.

Example 2.1

The figure below shows a mass m attached to an ideal spring of spring constant k. When the mass is at rest, the extension of the spring is x_{eq}.



The mass is then pulled downwards by a small displacement x where $x \ll x_{eq}$ and released.

(a) Show that the subsequent motion of the mass is simple harmonic.

When mass is at equilibrium position: $kx_{eq} = mg$ ------------ (1)

When mass is displaced downwards from equilibrium position, from Newton's 2nd law:

$-T' + mg = ma$ T' points in opposite direction to $(x_{eq} + x)$ hence the –ve sign

$-k(x_{eq} + x) + mg = ma$

$-kx_{eq} - kx + mg = ma$

$-kx = ma$ since from (1), $kx_{eq} = mg$

$a = -\frac{k}{m}x$ ------------ (2)

Since k and m are constants, equation (2) shows that the acceleration, a of m is directly proportional to displacement x from equilibrium position and is directed in the opposite direction to displacement x due to the negative sign. Hence the motion of m is simple harmonic.

(b) Determine an expression for the period and frequency of oscillation.

Comparing equation (2) with the defining equation for s.h.m. ie. $a = -\omega^2 x$,

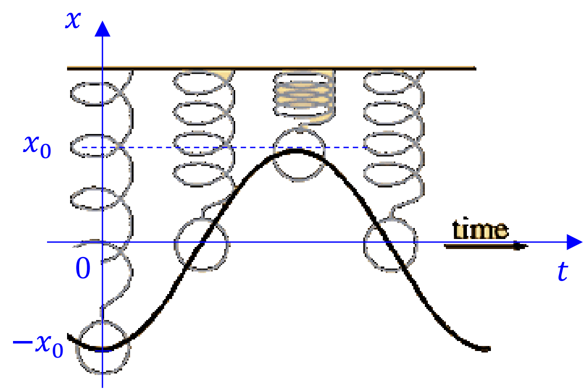
we have: $\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$

Period of oscillation, $T = 2\pi\sqrt{\frac{m}{k}}$

Frequency of oscillation, $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

(c) If the mass is released at the lowest position at $t = 0$. State an equation to describe the variation with time t of the displacement x of the mass.

Imagine the motion of the mass after it is released:



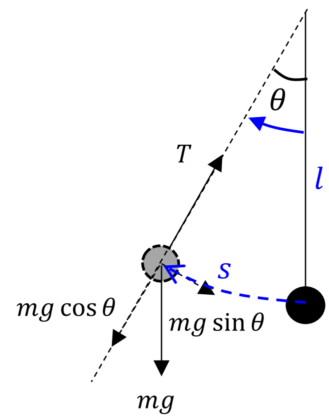
Taking displacement above the equilibrium position as positive:

$$x = -x_0\cos(\omega t) = -x_0\cos\left(t\sqrt{\frac{k}{m}}\right)$$

Example 2.2 (For illustration only. No need to memorise)

Show that the motion of a simple pendulum of length l is simple harmonic if the displacement is small and hence derive an expression for the period T of the pendulum in terms of l and any other constants.

If θ is sufficiently small, the arc s can be approximated to a horizontal line of displacement s. The restoring force is $mg\sin\theta \approx -mg\theta$ for small angular displacement θ. Note that the negative sign is needed because the direction of the restoring force is opposite to the angular displacement θ.



Hence, from Newton's 2nd law:

$$-mg\theta = ma \qquad \text{where } \theta \text{ in radians} \approx \frac{s}{l}$$

$$-mg\frac{s}{l} = ma$$

$$a = -\left(\frac{g}{l}\right)s \qquad \text{------------} \qquad (1)$$

Equation (1) shows that the acceleration of the bob is directly proportional to its displacement since g and l are constants and a is directed in the opposite direction to s from the negative sign. Hence the motion of the bob is simple harmonic.

To find the period, we have to compare equation (1) with the defining equation for s.h.m., $a = -\omega^2 x$, to get:

$$\omega^2 = \frac{g}{l} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{l}} = \frac{2\pi}{T} \quad \Rightarrow \quad \text{Period, } T = 2\pi\sqrt{\frac{l}{g}}$$

2.4 Equations of Motion in SHM

Just like in Kinematics (rectilinear motion), there is a set of equations of motion in s.h.m. where you can use to solve for the displacement, velocity and acceleration in terms of time. The equations are shown below in comparison with rectilinear motion.

Simple Harmonic Motion	Rectilinear Motion
$a = -\omega^2 x$	$a =$ constant
$x = x_0 \sin(\omega t - \phi)$	$s = ut + \frac{1}{2}at^2$
$v = \frac{dx}{dt} = \omega x_0 \cos(\omega t - \phi)$	$v = u + at$
$v = \pm\omega\sqrt{x_0^2 - x^2}$	$v^2 = u^2 + 2as$

Note that x_0 is the amplitude, ωx_0 the maximum magnitude of velocity and $\omega^2 x_0$ the maximum magnitude of acceleration.

Example 2.3

The displacement *x* (in metres) of an oscillating object in simple harmonic motion is given by

$$x = 2.0\ sin 4.0\ t$$

(a) State the amplitude of the oscillations

$x_0 = 2.0\ m$

(b) State the angular frequency of the oscillations

$\omega = 4.0\ rad\ s^{-1}$

(c) Calculate the magnitude of the maximum acceleration of the object.

$a_0 = \omega^2 x_0 = 4.0^2 \times 2.0 = 32\ ms^{-2}$

(d) Derive an expression for the velocity of the object in terms of the time *t*

$v = \frac{dx}{dt} = 2.0 \times 4.0 \cos 4.0\ \ t$

$= 8.0 \cos 4.0\ t$ (in m s^{-1})

(e) Calculate the maximum velocity of the object and the first time it occurs after t = 0 s.

$8.0\ m\ s^{-1}$

At t = 0 s, the object is at the equilibrium position. The velocity is maximum at the equilibrium position. The maximum velocity will occur the next time the object is at the equilibrium position which is half a period.

$\omega = \frac{2\pi}{T}$

$T = \frac{2\pi}{4.0} = \frac{\pi}{2}$

The first time the object is at maximum velocity after t = 0 s is $\frac{\pi}{4} = 0.79\ s$

(f) Calculate the displacement at which the velocity of the object is half its maximum value.

$v = \pm\omega\sqrt{x_0^2 - x^2}$

$4.0 = \pm 4.0\sqrt{2.0^2 - x^2}$

$x = \pm 1.7\ m$

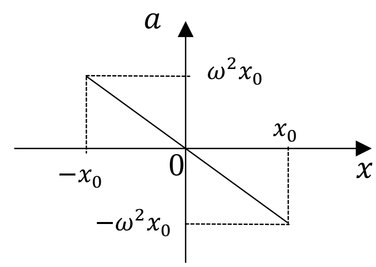
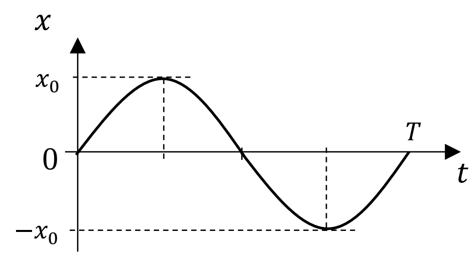
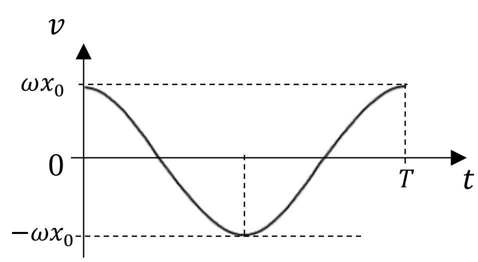
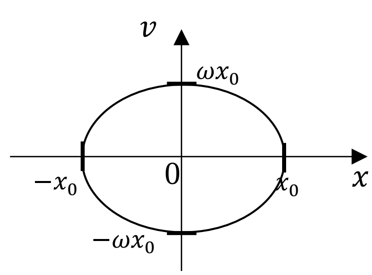
(g) Derive an expression for the acceleration of the object in terms of the time *t*

$a = \frac{dv}{dt} = 8.0 \times 4.0(-sin 4.0\ t) = -32.0 \sin 4.0\ t$

(Observe that the maximum acceleration is consistent with the value calculated in (c).)

2.5 Graphing the equations of motion.

You also need to know how to sketch the graphs for each equation of motion.

Equation	Graph
$a = -\omega^2 x$	
$x = x_0 \sin(\omega t)$	
$v = \frac{dx}{dt} = \omega x_0 \cos(\omega t)$	
$v = \pm\omega\sqrt{x_0^2 - x^2}$	

Explain the physical significance of the " $\pm$ " sign in the formula $v = \pm\omega\sqrt{x_0^2 - x^2}$

At any displacement x, the oscillator can be moving up or down (to or fro), hence the $\pm v$.

Let's study the relationship between acceleration, velocity and displacement by looking at the equations and graphs above.

Let's assume the oscillations start at equilibrium position at t = 0 s and the velocity is in the positive direction, i.e. $x = x_0 \sin(\omega t)$ and $v = \omega x_0 \cos(\omega t)$

At t = 0 s, the displacement is 0, i.e. equilibrium position. The velocity however is maximum at ωx_0 in the positive direction. The acceleration at equilibrium position is of course 0. Look at the table below to understand how acceleration, velocity and displacement changes.

Displacement $x = x_0 \sin(\omega t)$	0	x_0 Positive amplitude	0	$-x_0$ Negative amplitude
Velocity $v = \omega x_0 \cos(\omega t)$	ωx_0 Maximum in positive direction	0 Minimum	ωx_0 Maximum in negative direction	0
Acceleration $a = -\omega^2 x$	0	$-\omega^2 x_0$ Maximum in negative direction	0	$\omega^2 x_0$ Maximum in positive direction

The displacement, velocity and acceleration changes in this fashion through every cycle.

2.6 Relation between uniform circular motion and simple harmonic motion

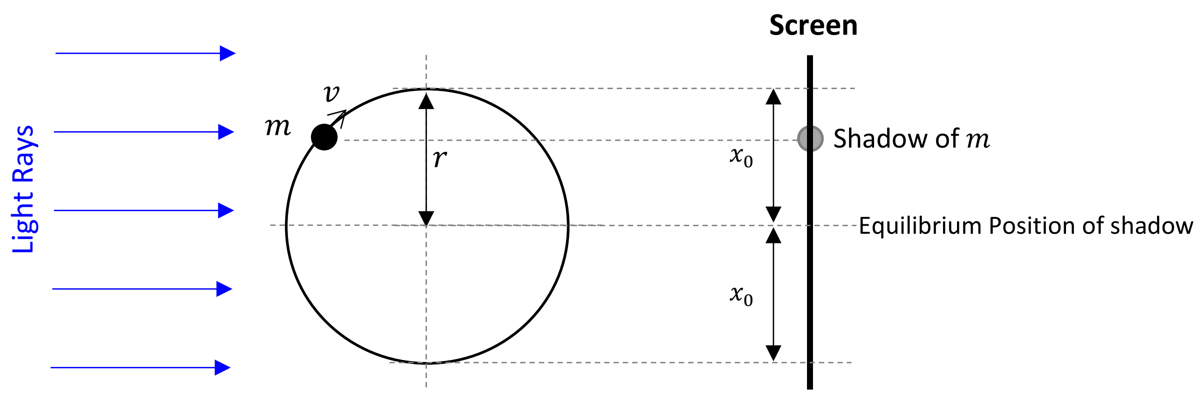
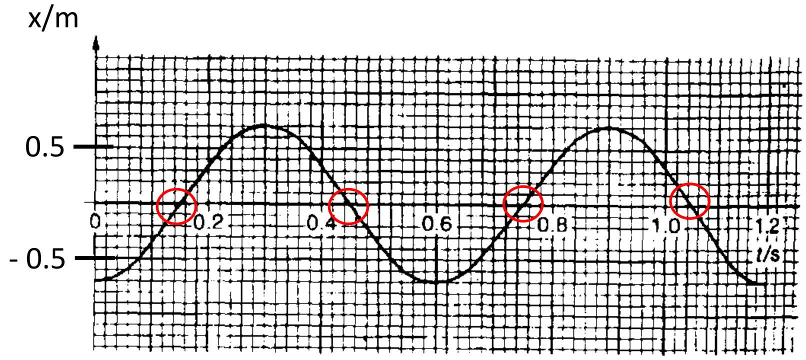


Fig 2.6a Fig 2.6b

Fig 2.6a shows a mass m moving in uniform circular motion with constant speed $v = r\omega$. The radius of the circular path is r. The angular speed is $\omega = \frac{2\pi}{T}$ where T is the period (time taken to travel one complete circle).

Fig 2.6b shows the shadow of m projected along a screen. The shadow moves along the screen in simple harmonic motion with amplitude $x_0 = r$ and angular frequency $\omega = \frac{2\pi}{T}$. The speed of the shadow varies between 0 (at the amplitude positions) and a maximum of $v = r\omega = \omega x_0$ at the equilibrium position.

Example 2.4



The graph above shows the displacement – time graph of an object oscillating in simple harmonic motion.

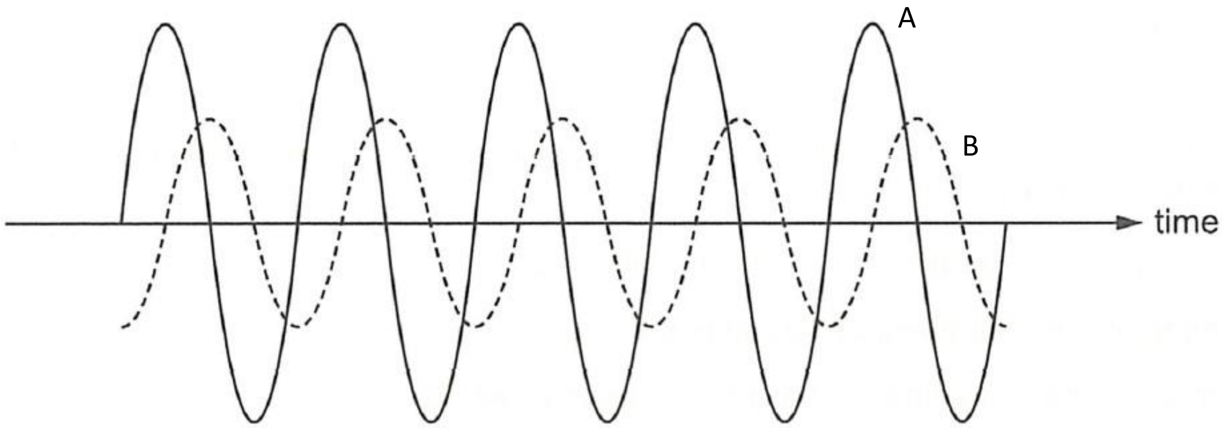
(a) State the amplitude of the oscillations.
0.70 m

(b) Calculate the angular frequency of the oscillations.
$T = 0.60\ s$
$\omega = \frac{2\pi}{T} = 10.5\ rad\ s^{-1}$

(c) Calculate the maximum velocity of the oscillations and circle the points on the graph at which the maximum velocity occurs.
$v_0 = \omega x_0 = 10.5 \times 0.70 = 7.3\ m\ s^{-1}$ (at the equilibrium positions)

(d) Calculate the maximum acceleration and state the time at which the acceleration is maximum.
$a_0 = \omega^2 x_0 = 10.5^2 \times 0.70 = 77\ m\ s^{-2}$
At the amplitude positions, t = 0, 0.3, 0.6, 0.9, 1.2 s

Example 2.5



The graphs above show the displacement-time graphs of two objects oscillating in simple harmonic motion with the same angular frequency. The amplitudes of the oscillations are different.

Determine the phase difference between the objects' oscillations.

The two oscillations are one quarter of a cycle out of phase.
Phase difference $= \frac{2\pi}{4} = \frac{\pi}{2}$ (More details on phase difference will be taught in Waves)

3. Energy in Simple Harmonic Motion

Consider the following equation of motion for a mass, m in s.h.m.: $v = \pm\omega\sqrt{x_0^2 - x^2}$

The kinetic energy of the mass is: $\frac{1}{2}mv^2 = \frac{1}{2}m\left[\pm\omega\sqrt{x_0^2 - x^2}\right]^2 = \frac{1}{2}m\omega^2x_0^2 - \frac{1}{2}m\omega^2x^2$

Rearranging the terms: $\frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2x_0^2$

From the above equation, we can deduce the following:

(1) Total energy of oscillation: $E_T = \frac{1}{2}m\omega^2x_0^2$

(*Consider that at the equilibrium position, velocity is maximum and potential energy is set to zero. Hence the at this point, the kinetic energy is equal to the total energy*)

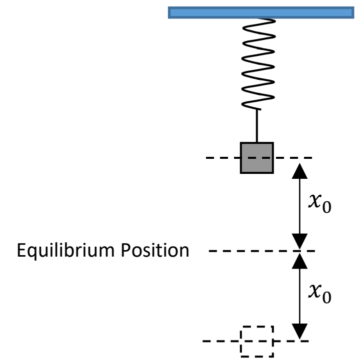
(2) Total potential energy of oscillation: $E_p = \frac{1}{2}m\omega^2x^2$

(*This can be observed from* $\frac{1}{2}m\omega^2x_0^2 - \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2x^2$ *which is the total energy minus the kinetic energy which must give the potential energy*)

(3) Kinetic energy of the mass: $E_k = \frac{1}{2}mv^2 = E_T - E_p$

Example 3.1

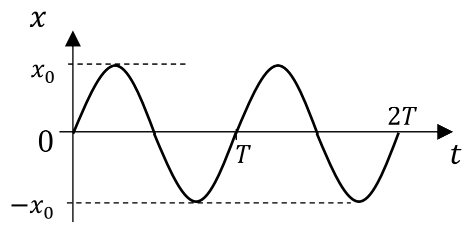
Consider a spring-mass system of mass m and spring constant k, oscillating vertically in simple harmonic motion with amplitude x_0 as shown below.



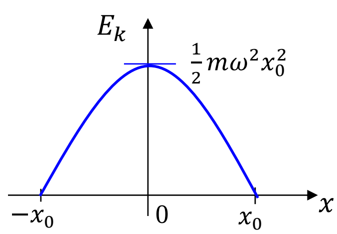
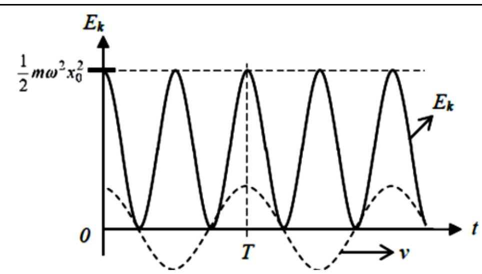
At $t = 0$, the mass is at the equilibrium position and moving upwards.

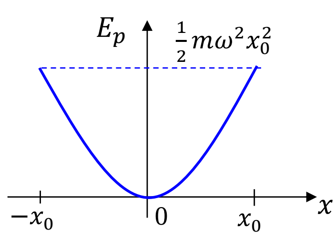
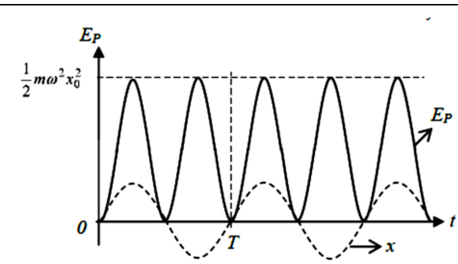
(a) State an equation for the displacement x of the mass as a function of time t and sketch the graph of x against t for two periods.

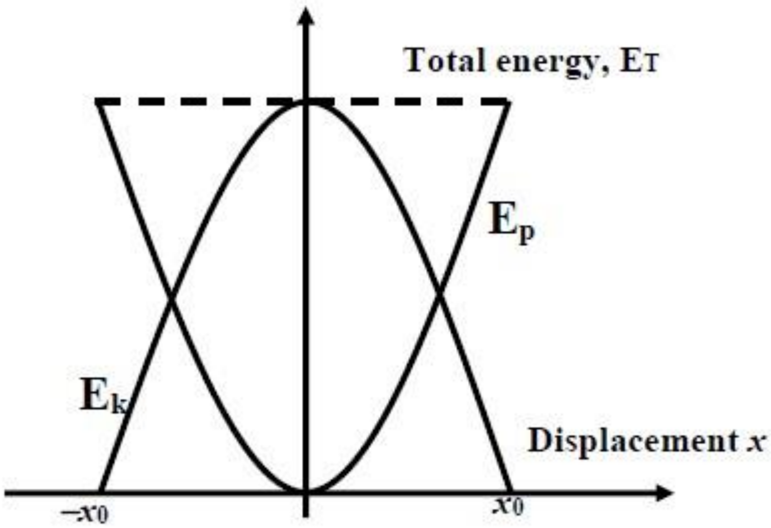
$x = x_0 \sin(\omega t)$



(b) Sketch the following energy graphs and state the equation relating the dependent variable to the independent variable shown on the graphs.

Kinetic energy, E_k as a function of displacement, x	Kinetic energy, E_k as a function of time, t for 2 cycles.
	
$E_k = \frac{1}{2}m\omega^2x_0^2 - \frac{1}{2}m\omega^2x^2$	$E_k = \frac{1}{2}m\omega^2x_0^2\cos^2(\omega t)$

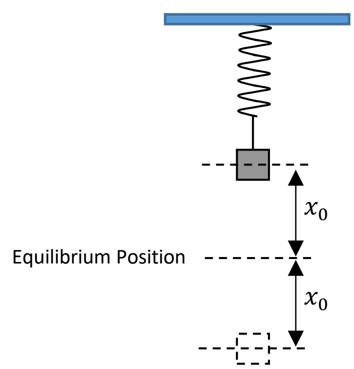
Potential energy, E_p as a function of displacement, x	Potential energy, E_p as a function of time, t for 2 cycles.
	
$E_p = \frac{1}{2}m\omega^2x^2$	$E_p = \frac{1}{2}m\omega^2x_0^2\sin^2(\omega t)$



Note that when you add the graphs of potential energy and kinetic energy, you will get the total energy which is constant.

Example 3.2

Consider the same spring-mass system mentioned in the previous example.



Suppose that, at $x = +x_0$, the extension of the spring is zero.

(a) State an equation relating the forces acting on mass, m at the equilibrium position. Express your answer in terms of the spring constant k, x_0, m and the free fall acceleration, g.

$mg = kx_0$

(b) Hence, show that the elastic potential energy stored in the spring when the mass is at the equilibrium position is given by

$$\text{Elastic potential energy} = \frac{1}{2}mgx_0$$

At the equilibrium position, the extension of the spring is $x = x_0$.

The elastic potential energy $= \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2 = \frac{1}{2}(kx_0)x_0 = \frac{1}{2}(mg)x_0$ since $mg = kx_0$ (shown)

(c) Complete the table below. Express your answer in terms of mgx_0 where applicable.

	Mass at the highest point of oscillation	Mass at the equilibrium position	Mass at the lowest point of oscillation
Kinetic energy	0	$\frac{1}{2}mgx_0$	0
Gravitational potential energy	$2mgx_0$	mgx_0	0
Elastic potential energy	0	$\frac{1}{2}mgx_0$	$2mgx_0$
Total energy	$2mgx_0$	$2mgx_0$	$2mgx_0$

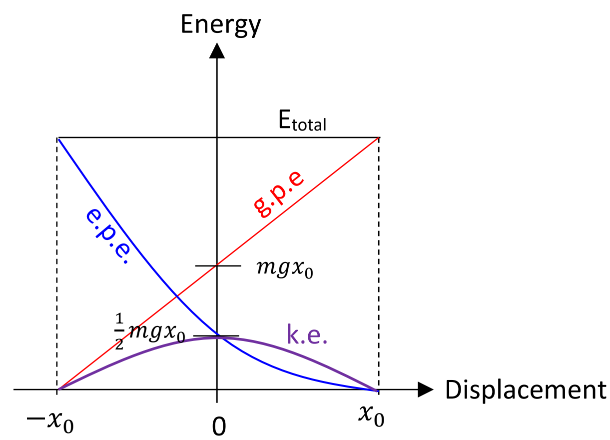
(d) Using your answers to (c), sketch, in the figure below, the variation with displacement x,

(i) the kinetic energy E_k,

(ii) the gravitational potential energy

(iii) the elastic potential energy

(iv) the total energy



(e) The total energy of oscillation is $E_T = \frac{1}{2}m\omega^2 x_0^2$ where the angular frequency of oscillation, $\omega = \sqrt{\frac{k}{m}}$. Using your answer in (a), express E_T in terms of m, g and x_0.

From $\omega = \sqrt{\frac{k}{m}}$, we have $m\omega^2 = k$ hence $E_T = \frac{1}{2}m\omega^2 x_0^2 = \frac{1}{2}kx_0^2 = \frac{1}{2}(kx_0)x_0 = \frac{1}{2}mgx_0$

(f) Comment on your expression for E_T obtained in (e) in relation to your answers in (c).

E_T is the total energy of oscillation and this is not the same as the total energy E_{total} in (c). The kinetic energy changes from 0 to $\frac{1}{2}mgx_0$. Likewise, the total potential energy (g.p.e + e.p.e) also changes from 0 to $\frac{1}{2}mgx_0$. Hence E_T is just the total energy associated with the oscillation = $\frac{1}{2}mgx_0$.

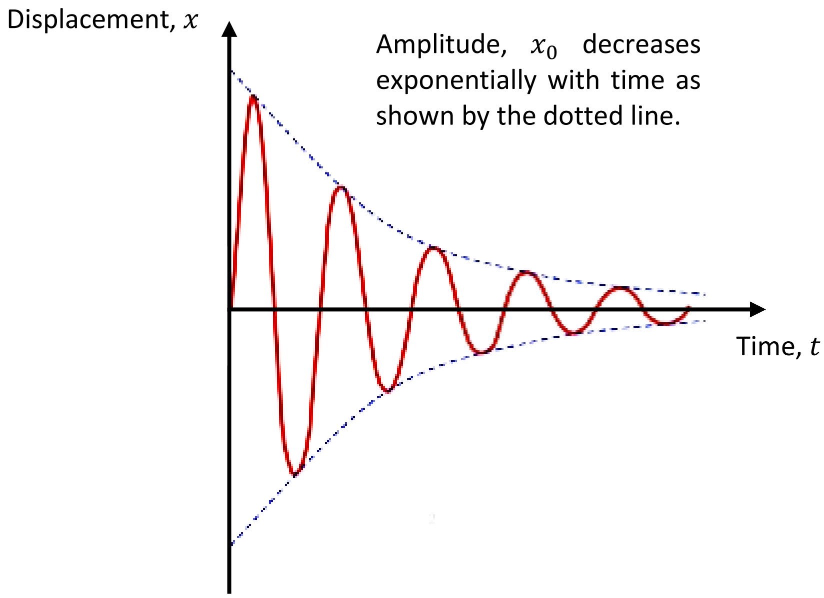
4. Damped Oscillations

For simple harmonic motion, the amplitude of oscillation remains constant. But in reality, most oscillations will show a decrease in amplitude over time due to energy loss. An oscillation that shows a decrease in amplitude over time due to resistive forces acting on the oscillator is called a damped oscillation.

The amount of damping depends on how much resistive force is acting on the oscillator and it determines how fast the amplitude decreases over time. There are 3 categories of damping as follows:

4.1 Light Damping

It shows definite oscillation, but the amplitude of oscillation decays exponentially with time. The period is slightly **greater than** it would be if there was no damping.



There are different degrees of light damping affecting how quickly the amplitude decreases with time.

4.2 Critical Damping

It shows no real oscillation as such; the time taken for the displacement to become effectively zero is a minimum.

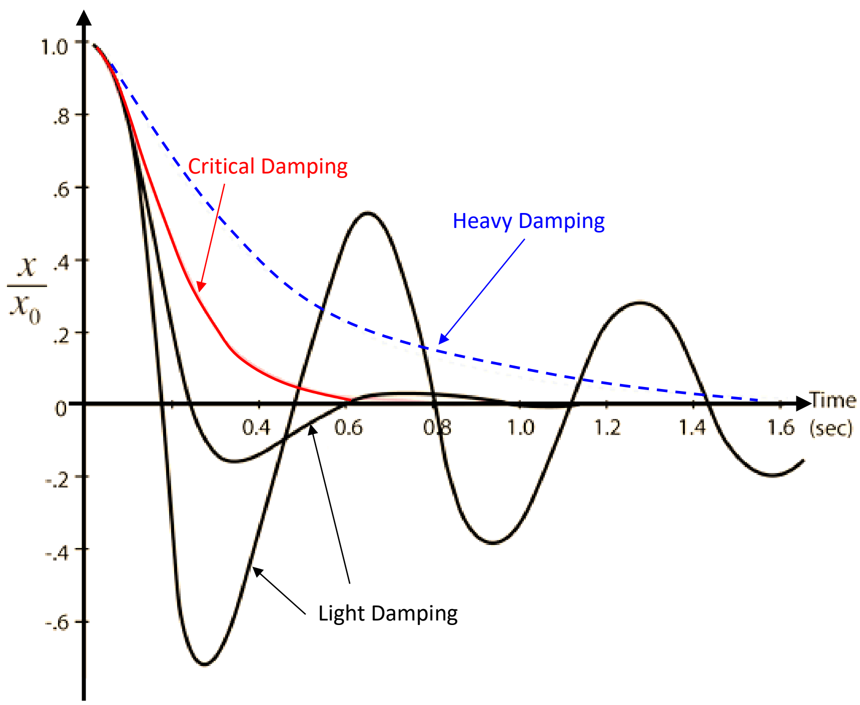
In critical damping, the resistance is just sufficient to prevent oscillation, yet not so great that the oscillator does not return to the zero position after being displaced.

The system returns to its equilibrium position in the shortest time possible without oscillating at all.

4.3 Heavy Damping

Just as in critical damping, there is no oscillation when the oscillator is heavily damped. The displacement of the oscillator decreases exponentially with time and the time it takes to return to the equilibrium position is longer than in critical damping.

The figure below compares the 3 different categories of damping.



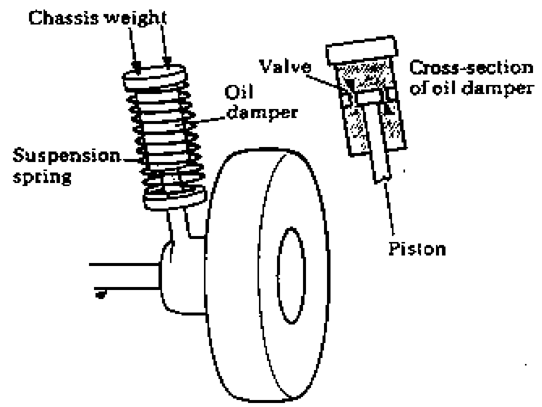
Examples of Damping

1. The balance wheel of a watch will cease to oscillate unless some energy can be supplied to it from the watch spring. The effect of frictional forces is to reduce the total mechanical energy of oscillating system.

2. The internal forces of a spring can be dissipative, and prevent the spring from being perfectly elastic.

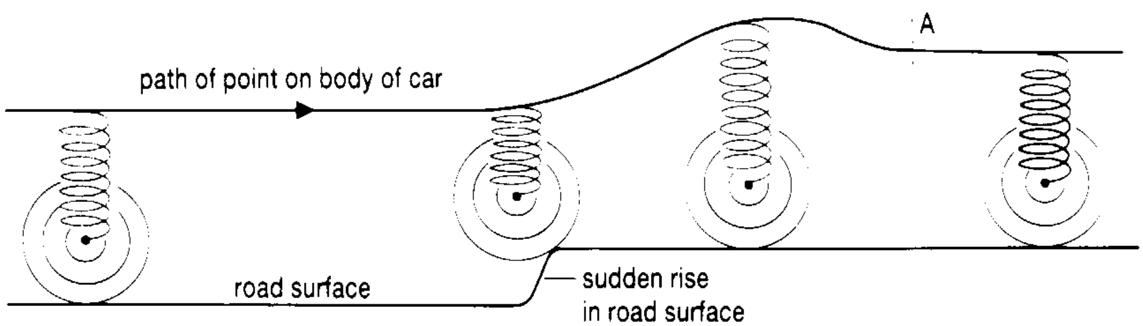
3. Critical damping is an important feature in moving coil meters which are used to measure current and voltage. When the reading changes, it is no use if the pointer oscillates for a while before settling down to the new reading. You need to make the new reading quickly in case it changes again. If the coil is critically damped, the pointer moves to its new reading in the shortest possible time without oscillating.

4. The critical damping in car suspension system:



The suspension system of a car is the link between the wheels and axles of a car and the body and the passengers, and consists of a spring which is damped by a shock absorber. If a car's shock absorbers are badly worn then the car becomes too bouncy and this is uncomfortable. A good suspension is one in which the damping is slightly under critical damping as this results in a comfortable ride and also leaves the car ready to respond to further bumps in the road quickly.

The following figure shows that by the time the car has reached A, the shock absorbing system is ready for another bump.



The spring of a car's suspension is ***critically damped*** so that when a car goes over a bump the passengers in the car quickly and smoothly regain equilibrium. This results in a more comfortable ride.

Video on different types of damping: https://www.youtube.com/watch?v=sP1DzhT8Vzo

You don't have to be concerned with the constants that characterise the different types of damping, only the types of damping. Different terms are used, so use the terms in these notes. For reference:

Light damping – underdamped

Critical damping – critical damping (returns to equilibrium the fastest)

Heavy damping - overdamped

5. Forced Oscillations and Resonance

5.1 Forced Oscillation

Imagine pushing a child on a swing. If each push is timed suitably, the child swings higher and higher. The pushes are a simple example of a periodic force, an external force applied at regular intervals. The periodic force provides a means of supplying energy to the system.

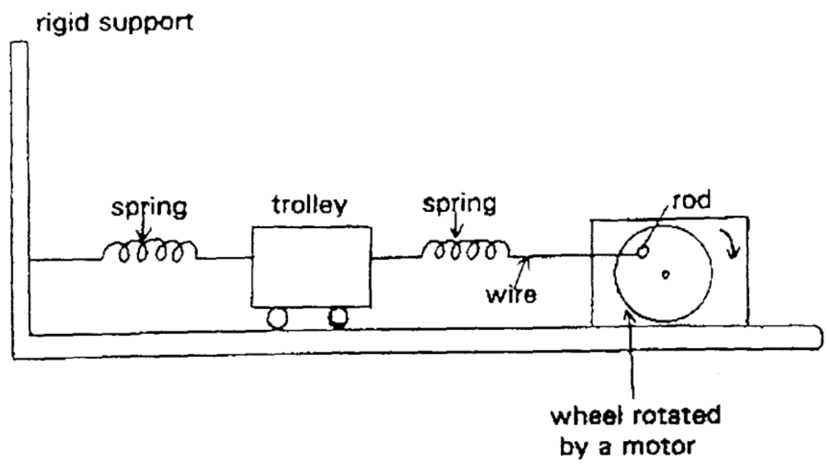
If the body is made to vibrate by a periodic force which has not the same frequency as the natural frequency of the body, it will keep in step with the force and its vibrations are called forced oscillations.

Examples of forced oscillations include:

- engine vibrations making bus windows oscillate,
- the spinning drum causing vibrations in a washing machine.

The figure below shows a way of forced oscillations in the laboratory using the trolley-and-spring oscillator driven by a motor.

Under steady conditions the amplitude of the forced oscillation is fixed for a fixed driving frequency.



The amplitude of oscillation depends on

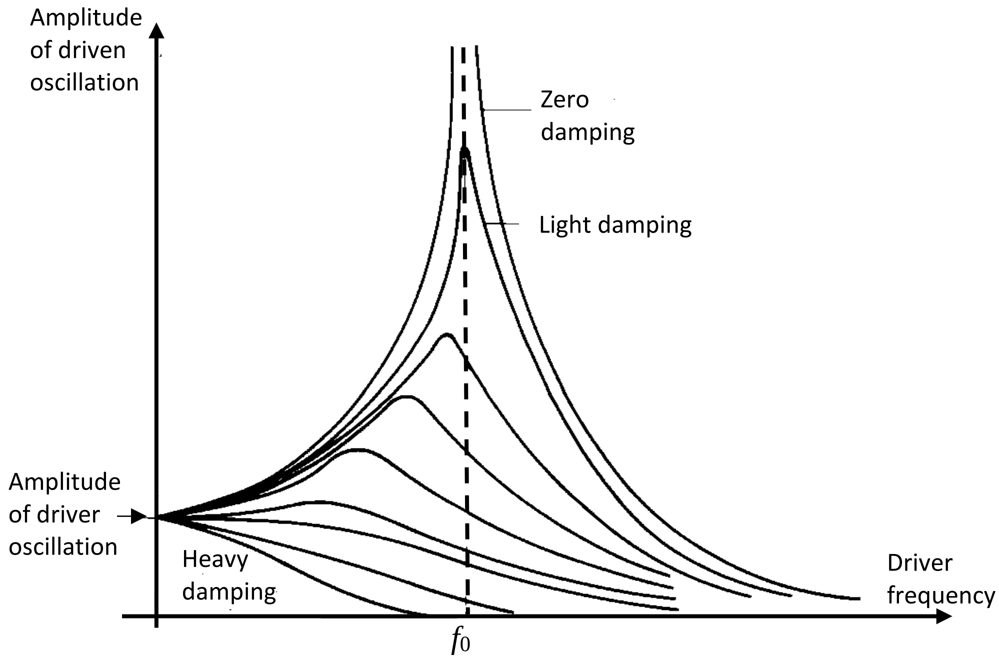
(1) how close the frequency of the driving force is to the natural frequency of the oscillator.

(2) the extent to which the oscillator is damped.

Note: Whatever its natural frequency, the forced oscillation takes on the frequency of the driving force.

5.2 Frequency response.

The frequency response graph in the figure below shows how the amplitude of the oscillator at steady state varies with the frequency of the driving force. The family of curves shown are for different degrees of damping. The natural frequency of the undamped oscillator is denoted by f_0.



The oscillator will vibrate with maximum amplitude when the frequency of the driving force matches the natural frequency of the oscillator (this effect is called resonance). Recall that damping causes the period of oscillation to increase (frequency to decrease), hence the frequencies at which the peaks occur decreases with increase damping.

The total energy of an oscillator $\left(E_T = \frac{1}{2} m\omega^2 x_0^2\right)$ depends on the amplitude of oscillation. The variation of the amplitude of oscillation with the frequency of the driving force also implies that for an oscillating system, the amount of energy transferred from the driving force to the oscillator depends on how close is the frequency of the driving force to the natural frequency of the oscillator. Maximum energy transfer occurs only when the frequency of driving force is equal to the natural frequency of the oscillator. This concept is important in the transmission and reception of electromagnetic waves.

From the figure above, we can see that damping has two effects on the forced oscillations:

(1) It decreases the ***maximum amplitude*** of the oscillator at resonance.

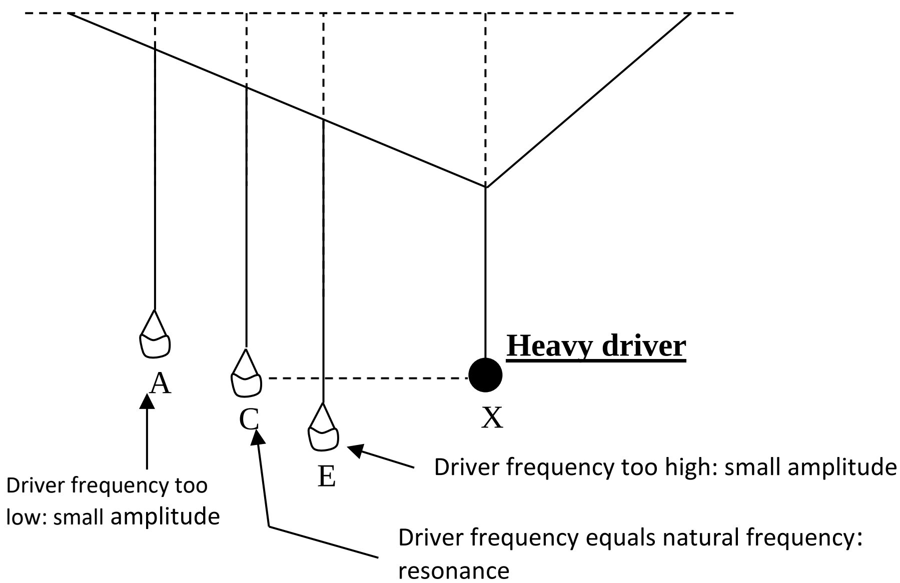
(2) It reduces the sharpness of resonance (ie the peaks becomes broader as damping increases)

Video on resonance: https://www.youtube.com/watch?v=ATRzwhk7UvY

5.3 Barton's Pendulums

Barton's pendulums, shown in the figure below can be used to illustrate forced/coupled oscillations and resonance.

Suspend a number of pendulums A, C, E of different lengths having light bobs from a thread. Suspend another simple pendulum X with a heavy bob having a length equal to that of a bob C from the same thread.



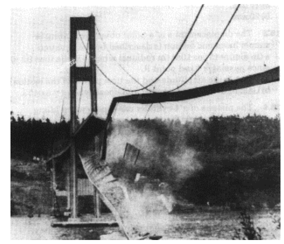
When X oscillates the impulse will disturb the pendulums A, C and E. They will oscillate. C will show the largest amplitude because C ***resonates*** with X.

5.4 Other Examples of Resonance

1. A microwave oven makes use of resonance. The microwaves are produced with a frequency which is the same as a natural frequency of vibration of water molecules. When some food, which always contains some water molecules, is placed in the oven, the water molecules resonate, absorbing energy from the microwaves and heats up. The microwaves do not heat up the containers in which the food is placed because the natural frequencies of vibration of the molecules in the plastic and glass containers are nowhere close to the frequency of the microwave used in the microwave oven.

2. An opera singer hitting a top note may shatter a wineglass if the frequency emitted by the singer matches one of the frequencies of the stationary wave set up along the rim of the glass.

3. The picture on the right shows the collapse, in 1940, of the **Tacoma Narrows Bridge** in Washington State. One of the reasons it collapsed is due to resonant vibrations. The driving force in this case was the wind, and the collapse occurred because the frequency at which vortices were generated as the wind blew through the bridge matched a major natural frequency of the bridge structure. There are more physics and engineering issue regarding the collapse, you can read and watch the video here:



https://practical.engineering/blog/2019/3/9/why-the-tacoma-narrows-bridge-collapsed

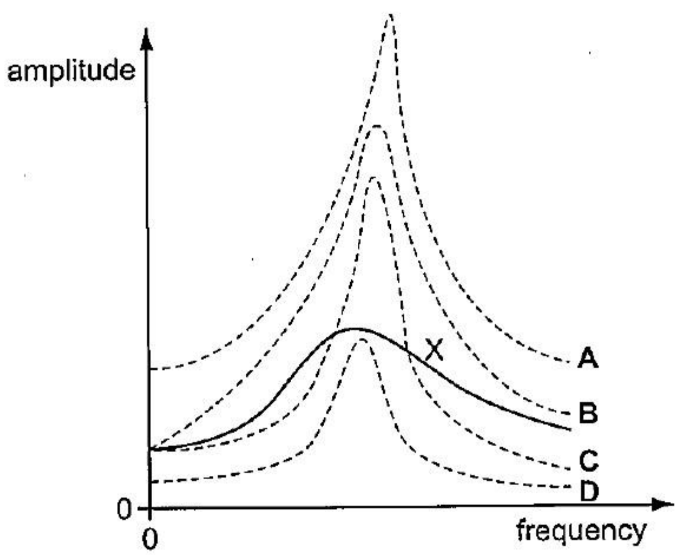
4. The transmission and reception of TV and radio programmes is due to electrical resonance.

5. All musical instruments produce distinctive sound due to mechanical resonance.

Example 5.1

The solid line X on the graph shows how the amplitude of an oscillating system varies with the frequency at which it is driven. The driving oscillator has a constant amplitude at all frequencies.

Which dashed line shows the response of the system under conditions where there is less damping?



Answer: B

With less damping, the amplitude must be higher for all frequencies except when frequency is zero, the amplitude is unchanged. The resonant or natural frequency will also be higher.