H2 MATHEMATICS (9758)
TOPIC ***VECTORS (LINES)*** ***2022/JC1***

DISCUSSION

1 Find a vector equation and cartesian equation of the following lines:

(a) passing through the point with position vector $7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and parallel to to $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

(b) passing through the points (2, –2, 1) and (0, 4, 9)

(c) passing through the point with position vector $7\mathbf{i}$ and parallel to the line $\mathbf{r} = (2 - p)\,\mathbf{i} + p\mathbf{j} + 5\mathbf{k}$.

Solution:

(a) Vector equation of line: $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$

Cartesian equation of line:
$$\frac{x-7}{1} = \frac{y-2}{-3} = \frac{z-(-4)}{1}$$
$$x - 7 = \frac{y-2}{-3} = z + 4$$

(b) Direction of line $= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -8 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

Equation of line: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix},\ \lambda \in \mathbb{R}$

Cartesian equation of line:
$$\frac{x-2}{-1} = \frac{y-(-2)}{3} = \frac{z-1}{4}$$
$$2 - x = \frac{y+2}{3} = \frac{z-1}{4}$$

(c) Equation of parallel line: $\underset{\sim}{r} = \begin{pmatrix} 2-p \\ p \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} -p \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + p\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Direction of line $= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Equation of line: $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \lambda \in \mathbb{R}$

Cartesian equation of line:
$$\frac{x-7}{-1} = \frac{y-0}{1},\ z = 0$$
$$7 - x = y,\ z = 0$$

2 Convert the equations of following lines to their <u>vector equation</u> form:

(a) $3y = 2x - 6 = \dfrac{z-1}{2}$

(b) $x + 5 = 2 - 4y,\ \ z = 3$

<u>Solution:</u>

(a) $3y = 2x - 6 = \dfrac{z-1}{2}$

$$\frac{2x-6}{1} = \frac{3y}{1} = \frac{z-1}{2}$$

$$\frac{x-3}{\frac{1}{2}} = \frac{y-0}{\frac{1}{3}} = \frac{z-1}{2}$$

Vector equation of line: $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda' \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ 2 \end{pmatrix},\ \lambda' \in \mathbb{R}$

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 12 \end{pmatrix},\ \lambda \in \mathbb{R}$$

(b) $x + 5 = 2 - 4y,\ \ z = 3$

$$\frac{x-(-5)}{1} = \frac{4y-2}{-1},\ z = 3$$

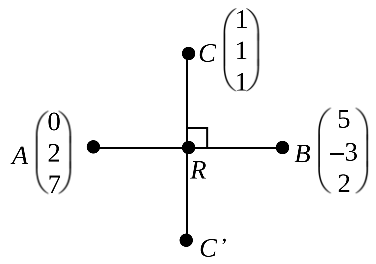
$$\frac{x-(-5)}{1} = \frac{y-\frac{1}{2}}{-\frac{1}{4}},\ z = 3$$

Vector equation of line: $\underset{\sim}{r} = \begin{pmatrix} -5 \\ \frac{1}{2} \\ 3 \end{pmatrix} + \lambda' \begin{pmatrix} 1 \\ -\frac{1}{4} \\ 0 \end{pmatrix},\ \lambda' \in \mathbb{R}$

$$\underset{\sim}{r} = \begin{pmatrix} -5 \\ \frac{1}{2} \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix},\ \lambda \in \mathbb{R}$$

3 Given three points $A(0,2,7)$, $B(5,-3,2)$ and $C(1,1,1)$, find the position vector of the point R on AB such that CR is perpendicular to AB. Hence find the perpendicular distance from C to AB and the position vector of the reflection of C in AB.

<u>Solution:</u>



$$\overrightarrow{AB}=\begin{pmatrix}5\\-3\\2\end{pmatrix}-\begin{pmatrix}0\\2\\7\end{pmatrix}=\begin{pmatrix}5\\-5\\-5\end{pmatrix}=-5\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Equation of line AB: $\underset{\sim}{r}=\begin{pmatrix}0\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$

Since R lies on line AB, $\overrightarrow{OR}=\begin{pmatrix}0\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}-\lambda\\2+\lambda\\7+\lambda\end{pmatrix}$ for a λ value.

$$\overrightarrow{CR}=\overrightarrow{OR}-\overrightarrow{OC}=\begin{pmatrix}-\lambda\\2+\lambda\\7+\lambda\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-1-\lambda\\1+\lambda\\6+\lambda\end{pmatrix}$$

Since $CR\perp$ line AB, $\begin{pmatrix}-1-\lambda\\1+\lambda\\6+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=0$

$$1+\lambda+1+\lambda+6+\lambda=0$$

$$\lambda=-\frac{8}{3}$$

$$\therefore\ \overrightarrow{OR}=\begin{pmatrix}8/3\\2+(-8/3)\\7+(-8/3)\end{pmatrix}=\frac{1}{3}\begin{pmatrix}8\\-2\\13\end{pmatrix}$$

Perpendicular distance from C to $AB=\left|\overrightarrow{CR}\right|=\left|\begin{pmatrix}-1-(-8/3)\\1+(-8/3)\\6+(-8/3)\end{pmatrix}\right|=\left|\frac{5}{3}\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|=\frac{5}{3}\sqrt{6}$ units

Let C' be the reflection of C in line AB.

Using mid-point theorem, $\overrightarrow{OR}=\frac{1}{2}\left(\overrightarrow{OC}+\overrightarrow{OC'}\right)$

$$\overrightarrow{OC'}=2\overrightarrow{OR}-\overrightarrow{OC}$$

$$=2\left[\frac{1}{3}\begin{pmatrix}8\\-2\\13\end{pmatrix}\right]-\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}13\\-7\\23\end{pmatrix}$$

4 Given a line with vector equation $\mathbf{r}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}+5\mathbf{k}),\ \lambda\in\mathbb{R}$,

(i) show that point $A(3,6,13)$ lies on the line,

(ii) find the perpendicular distance from point $B(1,7,4)$ to the line.

Solution:

(i) $\begin{pmatrix}3\\6\\13\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\5\end{pmatrix},\ \lambda\in\mathbb{R}$

$3=1+\lambda \quad \Rightarrow \quad \lambda=2$

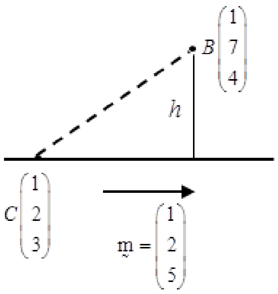
$6=2+2 \quad \Rightarrow \quad \lambda=2$

$13=3+5\lambda \quad \Rightarrow \quad \lambda=2$

$\therefore$ point A lies on the line.

(ii) Let point C be $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\underset{\sim}{m}=\begin{pmatrix}1\\2\\5\end{pmatrix}$.

$\overrightarrow{CB}=\begin{pmatrix}1\\7\\4\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\5\\1\end{pmatrix}$



$$h=\left|\overrightarrow{CB}\times\underset{\sim}{m}\right|$$

$$=\left|\begin{pmatrix}0\\5\\1\end{pmatrix}\times\frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{30}}\left|\begin{pmatrix}0\\5\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\5\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{30}}\left|\begin{pmatrix}23\\1\\-5\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{30}}\times\sqrt{555}$$

$$=\sqrt{\frac{37}{2}}\ \text{units}$$

5 Find whether the following pairs of lines are parallel, intersecting or skew. If they intersect, find the point of intersection and the acute angle between the lines.

(a) $\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ and $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}9\\12\\3\end{pmatrix},\ \mu\in\mathbb{R}$.

(b) $\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix},\ s\in\mathbb{R}$ and $\mathbf{r}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\0\end{pmatrix},\ t\in\mathbb{R}$.

(c) $\mathbf{r}=4\mathbf{i}+8\mathbf{j}+3\mathbf{k}+\alpha(\mathbf{i}+2\mathbf{j}+\mathbf{k}),\alpha\in\mathbb{R}$ and $\mathbf{r}=7\mathbf{i}+6\mathbf{j}+5\mathbf{k}+\beta(6\mathbf{i}+4\mathbf{j}+5\mathbf{k}),\beta\in\mathbb{R}$.

(d) $\mathbf{r}=\begin{pmatrix}4\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix},\ \lambda\in\mathbb{R}$ and z-axis.

[**Hint:** The z-axis has equation $\mathbf{r}=\gamma\begin{pmatrix}0\\0\\1\end{pmatrix},\ \gamma\in\mathbb{R}$.]

Solution:

(a) Since $\begin{pmatrix}9\\12\\3\end{pmatrix}=3\begin{pmatrix}3\\4\\1\end{pmatrix}$, the two lines are parallel lines.

(b) Since $\begin{pmatrix}2\\1\\1\end{pmatrix}\neq k\begin{pmatrix}1\\-2\\0\end{pmatrix}$ for any k, the 2 lines are not parallel.

Assuming that the two lines intersect, $\begin{pmatrix}1\\0\\3\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\0\end{pmatrix}$

$$1+2s=2+t \quad\Rightarrow\quad 2s-t=1 \quad \text{— (1)}$$
$$s=-1-2t \quad\Rightarrow\quad s+2t=-1 \quad \text{— (2)}$$
$$3+s=1 \quad\Rightarrow\quad s=-2 \quad \text{— (3)}$$

Solving (2) and (3): $s=-2$ and $t=\frac{1}{2}$.

Substitute $s=-2$ and $t=\frac{1}{2}$ into (1):

L.H.S. $=2(-2)-\frac{1}{2}=-\frac{9}{2}$.

R.H.S. $=1\neq$ L.H.S.

Since the two lines are not parallel and they do not intersect, they are skew lines.

(c) Since $\begin{pmatrix}1\\2\\1\end{pmatrix} \neq k\begin{pmatrix}6\\4\\5\end{pmatrix}$ for any k, the 2 lines are not parallel.

Assuming that the two lines intersect, $\begin{pmatrix}4\\8\\3\end{pmatrix}+\alpha\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}7\\6\\5\end{pmatrix}+\beta\begin{pmatrix}6\\4\\5\end{pmatrix}$

$$4+\alpha=7+6\beta \quad \Rightarrow \quad \alpha-6\beta=3 \quad \text{—} \quad (1)$$
$$8+2\alpha=6+4\beta \quad \Rightarrow \quad 2\alpha-4\beta=-2 \quad \text{—} \quad (2)$$
$$3+\alpha=5+5\beta \quad \Rightarrow \quad \alpha-5\beta=2 \quad \text{—} \quad (3)$$

Solving (1) and (2): $\alpha=-3$ and $\beta=-1$
Substitute $\alpha=-3$ and $\beta=-1$ into (3):

$$\text{L.H.S.}=-3-5(-1)=2$$
$$\text{R.H.S.}=2=\text{L.H.S.}$$

Hence the two lines intersect.

Position vector of intersection is $\begin{pmatrix}4\\8\\3\end{pmatrix}-3\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$.

Point of intersection is $(1,2,0)$.

Let the angle between two lines be θ.

$$\theta=\cos^{-1}\left(\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\frac{1}{\sqrt{77}}\begin{pmatrix}6\\4\\5\end{pmatrix}\right)$$
$$=\cos^{-1}\left(\frac{19}{\sqrt{6}\sqrt{77}}\right)$$
$$=0.487 \text{ rad (to 3 s.f.)} \quad \text{or} \quad 27.9^\circ \text{ (to 1 d.p.)}$$

The angle between the two lines is 27.9°.

(d) Since $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \neq k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for any k, the 2 lines are not parallel.

Let the vector equation of the line on the z-axis be $\underset{\sim}{r} = \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\gamma \in \mathbb{R}$

Assuming that the pair of lines intersects, $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \underbrace{\gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{z\text{-axis}}$

$$\begin{aligned} 4 + 2\lambda &= 0 \quad \text{—} \quad (1) \\ 2 + \lambda &= 0 \quad \text{—} \quad (2) \\ 1 - 2\lambda &= \gamma \quad \text{—} \quad (3) \end{aligned}$$

Solving (1): $\lambda = -2$
Solving (2): $\lambda = -2$

Substitute $\lambda = -2$ into (3): $\gamma = 5$.
$\therefore$ the pair of lines intersect at $(0, 0, 5)$.

Let the angle between two lines be θ.

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{1}{\sqrt{9}} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{1}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \cos^{-1}\left(-\frac{2}{3} \right) \\ &= 2.3005\ldots \text{ rad} \quad \text{or} \quad 131.810\ldots^\circ \end{aligned}$$

The angle between the two lines would therefore be $180^\circ - 131.810\ldots^\circ = 48.2^\circ$ (to 1 d.p.), or $(\pi - 2.3005\ldots)\text{rad} = 0.841$ rad (to 3 s.f.).

6 Given two lines $l_1: 4(1-x) = y = 2z+4$ and $l_2: 8-4x = y+3 = 2z$, explain why l_1 and l_2 are parallel. Find the distance between them.

Solution:

$l_1:\ -4(x-1) = y = 2[z-(-2)]$

$$\frac{x-1}{-\frac{1}{4}} = \frac{y-0}{1} = \frac{z-(-2)}{\frac{1}{2}}$$

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda' \begin{pmatrix} -\frac{1}{4} \\ 1 \\ \frac{1}{2} \end{pmatrix},\ \lambda' \in \mathbb{R}$$

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$$

$l_2:\ -4(x-2) = y-(-3) = 2(z-0)$

$$\frac{x-2}{-\frac{1}{4}} = \frac{y-(-3)}{1} = \frac{z-0}{\frac{1}{2}}$$

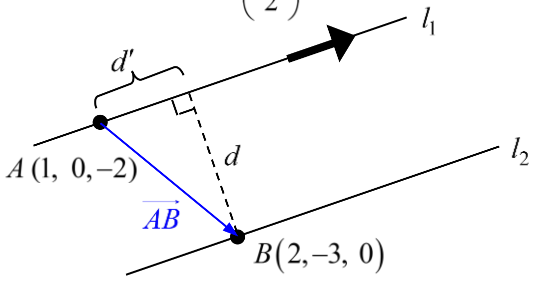
$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu' \begin{pmatrix} -\frac{1}{4} \\ 1 \\ \frac{1}{2} \end{pmatrix},\ \mu' \in \mathbb{R}$$

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix},\ \mu \in \mathbb{R}$$

Since the direction vector used for l_1, i.e. $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$, is parallel to that used for l_2, i.e. $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$

$\therefore$ l_1 and l_2 are parallel.

Let $\underset{\sim}{m} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.



Let $A = (1,\ 0, -2)$ on l_1,
$B = (2, -3,\ 0)$ on l_2.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{m} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

Method ①:

Length of projection of $\overrightarrow{AB}$ on $\underset{\sim}{\mathrm{m}}$, $d' = \left| \overrightarrow{AB} \cdot \dfrac{\underset{\sim}{\mathrm{m}}}{\left|\underset{\sim}{\mathrm{m}}\right|} \right|$

$$= \frac{\left| \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} \right|}{\sqrt{21}}$$

$$= \frac{\left|-1-12+4\right|}{\sqrt{21}}$$

$$= \frac{9}{\sqrt{21}}$$

$\left|\overrightarrow{AB}\right| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$.

Perpendicular distance between the two lines, $d = \sqrt{\left|\overrightarrow{AB}\right|^2 - (d')^2} = \sqrt{14 - \dfrac{9^2}{21}} = \sqrt{\dfrac{213}{21}} = \sqrt{\dfrac{71}{7}}$.

Method ②:

Perpendicular distance between the two lines, $d = \left| \overrightarrow{AB} \times \dfrac{\underset{\sim}{\mathrm{m}}}{\left|\underset{\sim}{\mathrm{m}}\right|} \right|$

$$= \frac{\left|\overrightarrow{AB} \times \underset{\sim}{\mathrm{m}}\right|}{\left|\underset{\sim}{\mathrm{m}}\right|}$$

$$\overrightarrow{AB} \times \underset{\sim}{\mathrm{m}} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -6-8 \\ -2-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -14 \\ -4 \\ 1 \end{pmatrix}$$

$$\therefore \quad d = \frac{\left|\overrightarrow{AB} \times \underset{\sim}{\mathrm{m}}\right|}{\left|\underset{\sim}{\mathrm{m}}\right|} = \frac{\sqrt{213}}{\sqrt{21}} = \sqrt{\frac{213}{21}} = \sqrt{\frac{71}{7}}.$$

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7 **[2012/I/9]**

(i) Find a vector equation of the line through the points A and B with position vectors $7\mathbf{i}+8\mathbf{j}+9\mathbf{k}$ and $-\mathbf{i}-8\mathbf{j}+\mathbf{k}$ respectively.

(ii) The perpendicular to this line from the point C with position vector $\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ meets the line at the point N. Find the position vector N and the ratio of $AN:NB$.

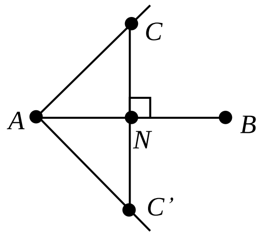
(iii) Find a Cartesian equation of the line which is a reflection of the line AC in the line AB.

Solution:

(i) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\-8\\1\end{pmatrix}-\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}-8\\-16\\-8\end{pmatrix}=-8\begin{pmatrix}1\\2\\1\end{pmatrix}$

Equation of line AB: $\underset{\sim}{r}=\begin{pmatrix}7\\8\\9\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$

(ii)



Since N lies on the line AB, $\overrightarrow{ON}=\begin{pmatrix}7\\8\\9\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}7+\lambda\\8+2\lambda\\9+\lambda\end{pmatrix}$ for a λ value.

Then $\overrightarrow{CN}=\overrightarrow{ON}-\overrightarrow{OC}=\begin{pmatrix}7+\lambda\\8+2\lambda\\9+\lambda\end{pmatrix}-\begin{pmatrix}1\\8\\3\end{pmatrix}=\begin{pmatrix}6+\lambda\\2\lambda\\6+\lambda\end{pmatrix}$.

Since $\overrightarrow{CN}\perp$ line AB, $\begin{pmatrix}6+\lambda\\2\lambda\\6+\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=0$

$$6+\lambda+4\lambda+6+\lambda=0$$
$$6\lambda=-12$$
$$\lambda=-2$$

Hence $\overrightarrow{ON}=\begin{pmatrix}7-2\\8-2(2)\\9-2\end{pmatrix}=\begin{pmatrix}5\\4\\7\end{pmatrix}$.

$\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}5\\4\\7\end{pmatrix}-\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}-2\\-4\\-2\end{pmatrix}$

$$\overrightarrow{NB} = \overrightarrow{OB} - \overrightarrow{ON}$$
$$= \begin{pmatrix} -1 \\ -8 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} -6 \\ -12 \\ -6 \end{pmatrix}$$
$$= 3\begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix} = 3\overrightarrow{AN}$$

Hence $AN : NB = 1:3$.

(iii) Let C' be the point of reflection of C in the line AB.

$$\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OC'}\right)$$
$$\overrightarrow{OC'} = 2\overrightarrow{ON} - \overrightarrow{OC}$$
$$= 2\begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 9 \\ 0 \\ 11 \end{pmatrix}$$

$$\overrightarrow{AC'} = \overrightarrow{OC'} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$$

$\therefore$ required equation is $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$

Cartesian equation is $x - 7 = \dfrac{y-8}{-4} = z - 9$.

8 **(a)** Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

(b) The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. The points P and Q are fixed and R varies. Given that $\mathbf{p}$ is non-zero and $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$, describe geometrically the set of all possible positions of the point R.

Solution:

(a) $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b}$ represents a line passing through the point with position vector $\underset{\sim}{a}$ and is parallel to the direction vector $\underset{\sim}{b}$.

(b) $\left(\underset{\sim}{r} - \underset{\sim}{p}\right) \times \underset{\sim}{q} = \underset{\sim}{0}$

$\left(\underset{\sim}{r} - \underset{\sim}{p}\right) // \underset{\sim}{q}$

$\Rightarrow \underset{\sim}{r} - \underset{\sim}{p} = \lambda \underset{\sim}{q} \quad , \lambda \in \mathbb{R}$

$\Rightarrow \quad \underset{\sim}{r} = \underset{\sim}{p} + \lambda \underset{\sim}{q}, \lambda \in \mathbb{R}$

Hence, R lies on a line that passes through point P and is parallel to the vector $\overrightarrow{OQ}$.

9 **[2017/I/10]**

Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0, 0, 0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

A new cable is installed which passes through points $P(1, 2, -1)$ and $Q(5, 7, a)$.

(i) Find the value of a for which C and the new cable will meet.

To ensure that the cables do not meet, the engineers use $a = -3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be 90°. Show that this is not possible.

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length.

Solution:

(i) $l_C : \underset{\sim}{r} = \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$

$$\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 7 \\ a \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}$$

$$l_{PQ} : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}, \mu \in \Re$$

If C and new cable meet, there exists a solution for the following equations.

$$\begin{pmatrix} 3\lambda \\ \lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 1+4\mu \\ 2+5\mu \\ -1+(a+1)\mu \end{pmatrix}$$

$3\lambda = 1 + 4\mu \quad \Rightarrow \quad 3\lambda - 4\mu = 1 \quad$ — (1)

$\lambda = 2 + 5\mu \quad \Rightarrow \quad \lambda - 5\mu = 2 \quad$ — (2)

$-2\lambda = -1 + (a+1)\mu$

Solving (1) and (2): $\mu = -\frac{5}{11}, \lambda = -\frac{3}{11}$.

$$-2\left(-\frac{3}{11}\right) - (a+1)\left(-\frac{5}{11}\right) = -1$$

$$a = -\frac{22}{5}$$

(ii) $\overrightarrow{RP}=\begin{pmatrix}1\\2\\-1\end{pmatrix}-\begin{pmatrix}3\lambda\\\lambda\\-2\lambda\end{pmatrix}=\begin{pmatrix}1-3\lambda\\2-\lambda\\2\lambda-1\end{pmatrix}$ $\qquad$ $\overrightarrow{RQ}=\begin{pmatrix}5\\7\\-3\end{pmatrix}-\begin{pmatrix}3\lambda\\\lambda\\-2\lambda\end{pmatrix}=\begin{pmatrix}5-3\lambda\\7-\lambda\\2\lambda-3\end{pmatrix}$

$$\begin{aligned}\begin{pmatrix}1-3\lambda\\2-\lambda\\2\lambda-1\end{pmatrix}\cdot\begin{pmatrix}5-3\lambda\\7-\lambda\\2\lambda-3\end{pmatrix}&=5-3\lambda-15\lambda+9\lambda^2+14-2\lambda-7\lambda+\lambda^2+4\lambda^2-6\lambda-2\lambda+3\\&=22-35\lambda+14\lambda^2\\&=14\left[\left(\lambda-\frac{5}{4}\right)^2+\frac{1}{112}\right]\\&>0\end{aligned}$$

Hence, angle PRQ can never be 90 degree (shown).

(iii) $\overrightarrow{PR}=-\begin{pmatrix}1-3\lambda\\2-\lambda\\2\lambda-1\end{pmatrix}=\begin{pmatrix}3\lambda-1\\\lambda-2\\1-2\lambda\end{pmatrix}$

$$PR=\sqrt{(3\lambda-1)^2+(\lambda-2)^2+(1-2\lambda)^2}=\sqrt{14\lambda^2-14\lambda+6}$$

<u>Method ①:</u> Complete the Square

There are 3 methods to find value of λ.

$$\begin{aligned}PR&=\sqrt{14\lambda^2-14\lambda+6}\\&=\sqrt{14\left(\lambda^2-\lambda+\frac{6}{14}\right)}\\&=\sqrt{14\left[\lambda^2-\lambda+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{3}{7}\right]}\\&=\sqrt{14\left(\lambda-\frac{1}{2}\right)^2+\frac{5}{2}}\end{aligned}$$

Minimum PR occurs at $\lambda=\frac{1}{2}$

<u>Method ②:</u> Using Differentiation

Method 2A (minimise PR)

$$PR=\left(14\lambda^2-14\lambda+6\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}PR}{\mathrm{d}\lambda}=\frac{1}{2}\left(14\lambda^2-14\lambda+6\right)^{-\frac{1}{2}}\left(28\lambda-14\right)$$

$$=\frac{28\lambda-14}{2\sqrt{14\lambda^2-14\lambda+6}}$$

At stationary point, $\frac{\mathrm{d}PR}{\mathrm{d}\lambda}=0$

$$\frac{28\lambda-14}{2\sqrt{14\lambda^2-14\lambda+6}}=0$$

$$\lambda=\frac{1}{2}$$

(You must prove that PR is a minimum using 1st derivative test or 2nd derivative test)

Using 1st derivative test:

λ	$\left(\frac{1}{2}\right)^-$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{2}\right)^+$
$\frac{\mathrm{d}PR}{\mathrm{d}\lambda}$	<0	0	>0
Slope	\	–	/

Will not show 2nd derivative test because it will be too tedious here

Minimum PR occurs at $\lambda=\frac{1}{2}$

Method 2B (minimise PR^2)

Minimising PR is also same as minimising PR^2

$$PR^2=14\lambda^2-14\lambda+6$$

$$\frac{\mathrm{d}\left(PR^2\right)}{\mathrm{d}\lambda}=28\lambda-14$$

At stationary point, $\frac{\mathrm{d}\left(PR^2\right)}{\mathrm{d}\lambda}=0$

$$28\lambda-14=0$$

$$\lambda=\frac{1}{2}$$

(You must prove that PR^2 is a minimum using 1st derivative test or 2nd derivative test)

Using 1st derivative test:

λ	$\left(\frac{1}{2}\right)^-$	$\left(\frac{1}{2}\right)$	$\left(\frac{1}{2}\right)^+$
$\frac{\mathrm{d}\left(PR^2\right)}{\mathrm{d}\lambda}$	<0	0	>0
Slope	\	–	/

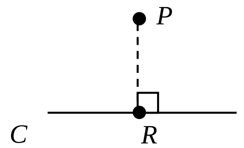
Using 2nd derivative test:

$$\frac{\mathrm{d}^2\left(PR^2\right)}{\mathrm{d}\lambda^2}=28>0 \text{ (minimum)}$$

Minimum PR occurs at

<u>Method ③:</u> Foot of Perpendicular

Since length of PR have to be smallest,
R will be the foot of perpendicular from P to line C



$$\overrightarrow{OR}=\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix} \text{ for a value of } \lambda$$

$$l_C:\ \underset{\sim}{r}=\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix},\lambda\in\mathbb{R}$$

$$\overrightarrow{PR}=\begin{pmatrix}3\lambda\\\lambda\\-2\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}3\lambda-1\\\lambda-2\\1-2\lambda\end{pmatrix}$$

Since $\overrightarrow{PR} \perp l_C$

$$\overrightarrow{PR} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 3\lambda - 1 \\ \lambda - 2 \\ 1 - 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$3(3\lambda - 1) + \lambda - 2 - 2(1 - 2\lambda) = 0$$

$$14\lambda - 7 = 0$$

$$\lambda = \frac{1}{2}$$

$$\overrightarrow{OR} = \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Coordinates of R $\left(\frac{3}{2}, \frac{1}{2}, -1\right)$.

$$\overrightarrow{PR} = \begin{pmatrix} 3\left(\frac{1}{2}\right) - 1 \\ \frac{1}{2} - 2 \\ 1 - 2\left(\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix}$$

$$PR = \left|\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix}\right|$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \sqrt{\frac{10}{4}} = \frac{1}{2}\sqrt{10}$$