RAFFLES INSTITUTION

2023 YEAR 5 TERM 3 COMMON TEST

FURTHER MATHEMATICS 9649

2 hours

Additional materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, remove the cover page and fasten it to all your work securely together. The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 65.

This document consists of 4 printed pages.

1 **(a)** Find, in terms of n, $\sum_{r=2}^{n}\left(\frac{1}{2^r}+\frac{1}{r^2-1}\right)$. **[4]**

(b) Give a reason why the series $\sum_{r=2}^{\infty}\left(\frac{1}{2^r}+\frac{1}{r^2-1}\right)$ converges, and write down its value. **[2]**

2 The curve C has equation $y=\frac{x^2+ax}{x-a}$, where a is a positive constant.

Sketch C, indicating clearly the equations of any asymptotes, the coordinates of turning points and the coordinates of the points where the curve crosses the axes. **[6]**

State the set of values that y can take. **[1]**

3 Sketch the curve C with equation $4x^2+y^2=1$ where $y\geq 0$. The region R is described as the region bounded by the x-axis and C.

Use differentiation to find the area of the largest rectangle that can be inscribed in the region R. **[7]**

4 **(a)** Obtain the expansion of $(9-x)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}}$ up to and including the term in x^3. Give the coefficients as exact fractions in their simplest form. [5]

(b) Find the set of values of x for which the expansion in part (**a**) is valid. [2]

5 ~~Use de Moivre's theorem to show that~~

$$\cancel{\sin 5\theta = \cos^5\theta(t^5 - 10t^3 + 5t)} \text{ ~~where~~ } \cancel{t = \tan\theta}.$$

~~Deduce that~~ $\cancel{\tan^2\frac{\pi}{5}}$ ~~is a root of the equation~~ $\cancel{x^2 - 10x + 5 = 0}$.

~~Hence find the exact value of~~ $\cancel{\tan^2\frac{\pi}{5}}$. ~~[7]~~

6 ~~The complex number z satisfies~~ $\cancel{|z+2-3\mathrm{i}| = |z-p|}$, ~~where~~ $\cancel{p}$ ~~is a complex number with~~ $\cancel{\mathrm{Re}(p) > 0}$.

~~Describe, with the aid of a sketch, the locus of the point which represents~~ $\cancel{z}$ ~~in an Argand diagram.~~ ~~[3]~~

~~Given that the points representing~~ $\cancel{1+8\mathrm{i}}$ ~~and~~ $\cancel{-2-\mathrm{i}}$ ~~lie on the locus of~~ $\cancel{z}$, ~~find in any order,~~

~~**(i)**~~ $\cancel{p}$,

~~**(ii)** the least possible value of~~ $\cancel{|z-p|}$. ~~[5]~~

7(a) Given that $u_1 = \frac{7}{2}$ and $u_n = \frac{1}{2}u_{n-1} + n^2$ for $n \geq 2$, prove by mathematical induction that $u_n = 2n^2 - 4n + 6 - \left(\frac{1}{2}\right)^n$ for all positive integers n. **[5]**

(b) ~~Without using a calculator, show that~~ $\cancel{1 > 2\sqrt{2} - 2}$. ~~[1]~~

~~Prove by mathematical induction that~~

$$\cancel{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \ldots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2},$$

~~for all positive integers~~ $\cancel{n}$. ~~[4]~~

[Turn over

8(a) On the 1st of January 2023, Mr Yu visited his newly acquired fishing farm for the first time and there were 2000 fish. His planned business model is such that on the last day of each month, 30% of the fish in the farm are sold off and another 150 fish are added in.

Based on the above information, Mr Yu wants to come up with a recurrence relation to predict the number of fish in his farm on the first day of subsequent months.

(i) Mr Yu visits his farm once a month on the first day of every month. Let u_n be the number of fish in his farm on his nth visit. Given that $u_1 = 2000$, write down a recurrence relation for u_n. **[1]**

(ii) Solve the recurrence relation in part **(i)**, leaving your answer in terms of n. **[3]**

Mr Yu's friend Mr Sia, who owns a prawn farm, found out about his model and informed him that he did not take into account the fish breeding and dying off. Upon checking past records, Mr Yu found that the number of fish reduced by 25% each month, due to the nett effect of breeding and dying off. With this additional information, Mr Yu adjusts his model to take this into account before the selling off and adding of fish is done. From the end of February 2023 onwards, the monthly sales percentage is adjusted to $p\%$.

(iii) Given that the number of added fish remains at 150 and the number of fish in the long term steadies at 375, find the value of p. **[3]**

(b) Mr Sia shares that he only sells his prawns when the number of prawns in his farm reach 60 000, due to his superior breeding method. The initial population of prawns in his farm, on 1st January 2023 was 3000 and the population on 1st February 2023 was 3500.

It is known from previous years that the increase in prawn population from month n to month $(n+1)$ is twice the increase from month $(n-1)$ to month n.

Let P_n denote the population of prawns n months after 1st January 2023.

Given that $P_0 = 3000$, find an expression for P_n in terms of n. Hence state the month and year when Mr Sia sells his prawns with at least 3000 prawns in his farm for the breeding cycle to start again. **[6]**

********* End of Paper *********