National Junior College
2016 – 2017 H2 Mathematics
Functions **Lecture Questions**

Part 1. Key Questions to answer:

- ❑ What are the set builder and interval notations, and how do you use them?

Prerequisite knowledge: Number Line, Set Notation and Language
Lecture Readings: Section 1

Question 1.1
What do the following symbols represent?

$$\mathbb{Z},\ \mathbb{Z}^+,\ \mathbb{Z}^-,\ \mathbb{R},\ \mathbb{R}^+,\ \mathbb{R}^-$$

Answer:
$\mathbb{Z}$: set of integers, $\mathbb{Z}^+$: set of positive integers, $\mathbb{Z}^-$: set of negative integers,
$\mathbb{R}$: set of real numbers, $\mathbb{R}^+$: set of positive real numbers, $\mathbb{R}^-$: set of negative real numbers

Question 1.2
What are set operators and how are they used as an extension to the above sets?

Question 1.3
How do we use the **set builder notation** or the **interval notation** to represent a set of values?

Range of Values	**Representation as a Set**	
	Set Builder Notation	**Interval Notation**
$-2<x<3$	$\{x\in\mathbb{R} \mid -2<x<3\}$	$(-2, 3)$
$-2\le x<3$	$\{x\in\mathbb{R} \mid -2\le x<3\}$	$[-2, 3)$
$-2\le x\le 3$		
$x\le -2$	$\{x\in\mathbb{R} \mid x\le -2\}$	$(-\infty, -2]$
$x>3$		

Question 1.4
What is the difference between the use of round and square brackets in the interval notation?

Part 2. Key Questions to answer:

- ❑ What are relations and functions?
- ❑ How do you determine graphically whether a relation is a function?
- ❑ How do you define and represent a relation/function?

Prerequisite knowledge: Curve Sketching
Lecture Readings: Section 2

Question 2.1 (**Numerical representation** of relations or functions)
Which of the following relations is a function?

x	$y = x + 1$
0	1
1	2
2	3
3	4

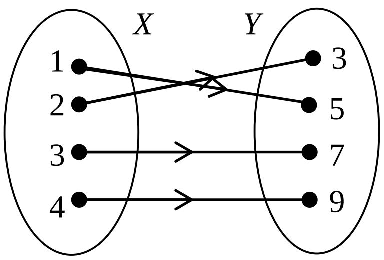
Function

$y^2 = 4 - x^2$	
x	y
0	± 2
1	$\pm\sqrt{3}$
2	0

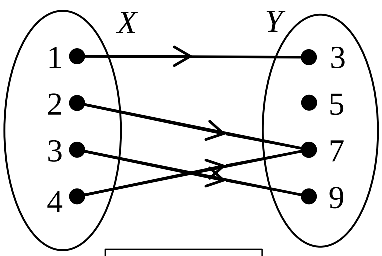
Not a Function

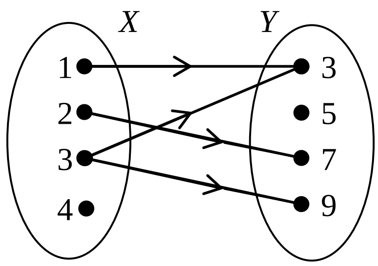
x	$y = \lvert x \rvert$
-2	2
2	2
-1	1
1	1

Question 2.2 (**Diagrammatical representation** of relations or functions)
Which of the following relations is a function?



Function





Not a Function

Learning Point(s):

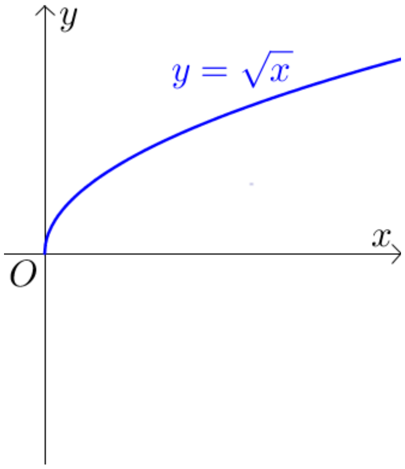
Question 2.3 (**Algebraic representation** of relations or functions)
Which of the following relations is a function?

$\mathrm{h}: x \mapsto \pm\sqrt{x}, \quad x \in \{0, 4\},$ $\qquad$ $\mathrm{g}: x \mapsto \sqrt{x}, \quad x \in \{-4, 0\},$

$\mathrm{f}: x \mapsto \sqrt{x}, \quad x \in \{0, 4\},$ $\qquad$ $\mathrm{p}(x) = \sqrt{x}.$

- The relations g and h are not functions. Why?
- The relation f is a function.
  Note that $\mathrm{f}(x) = \sqrt{x}$ is known as the **rule** of function f.
  $\{0, 4\}$ is the **domain** ("inputs") of f, denoted by D_{f}.
  $\{0, 2\}$ is the **range** ("outputs") of f, denoted by R_{f}.
  A function is defined by both its **rule and domain**.
- If the domain of a function is not given explicitly, the convention is that the domain is the set of all numbers for which the rule makes sense and defines a real number.
  In the case of function p, the domain of p is $[0, \infty)$.
  The corresponding range of p is $R_p = [0, \infty)$.
- **Functions p and f are not the same** even though they have the same rule, as they have different domains.

Question 2.4 (**Graphical representation** of relations or functions)
Which of the following relations is a function? In the case where a relation is a function, state the range of the function.

<table>
<tr>
<td>$\mathrm{f}: x \mapsto \sqrt{x}, \quad x \in [0,\infty)$

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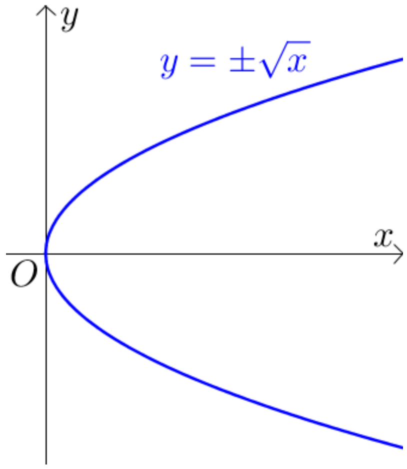
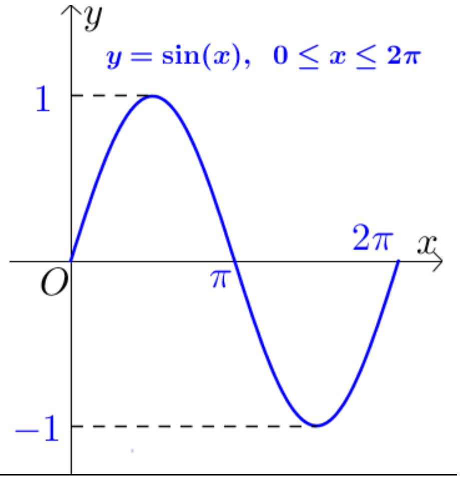
<td>$\mathrm{g}: x \mapsto \pm\sqrt{x}, \quad x \in [0,\infty)$

</td>
<td>$\mathrm{h}: x \mapsto \sin(x), \quad x \in [0,2\pi]$

</td>
</tr>
<tr>
<td>Relation f is a function.
$R_{\mathrm{f}} = [0,\infty)$.</td>
<td>Relation g is not a function.</td>
<td>________________</td>
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Learning Point(s):

Question 2.5

(i) Show that $x^2 + y^2 = 4$, where $x, y \in \mathbb{R}$, is not a function.

(ii) Define the function f such that all the following conditions are satisfied:

(1) any point with coordinates $\left(x,\ \mathrm{f}(x)\right)$ lies on the curve $x^2 + y^2 = 4$,

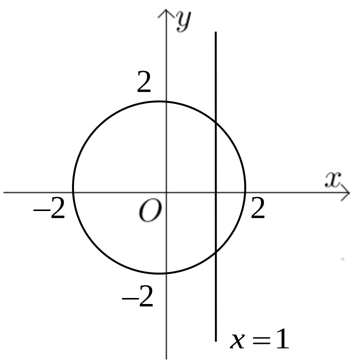
(2) the domain of f is maximal, and

(3) the range of f is a subset of $\mathbb{R}^-$, the set of all **negative** real numbers.

(iii) Prove that your choice of f is indeed a function, and state the range of f.

Solution:

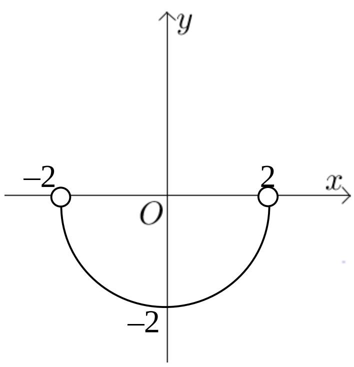
(i) From the graph of $x^2 + y^2 = 4$, the line $x = 1$ cuts the graph twice. Hence $x^2 + y^2 = 4$ is not a function.



(ii) $\mathrm{f} : x \mapsto -\sqrt{4 - x^2},\ x \in (-2,\ 2)$.

(iii) From the graph of $y = \mathrm{f}(x) = -\sqrt{4 - x^2}$, every vertical line $x = a$, where $a \in (-2,\ 2)$, cuts the graph once. Hence, f is a function.

$R_{\mathrm{f}} = [-2, 0)$.



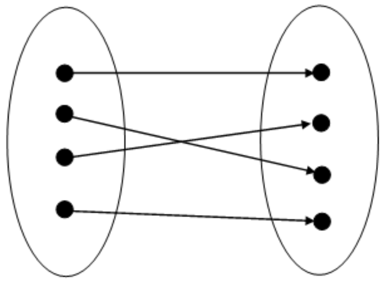
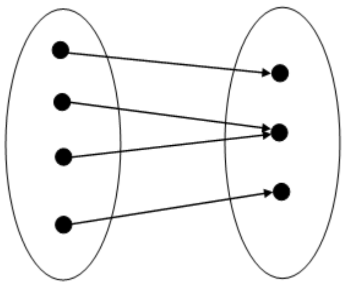
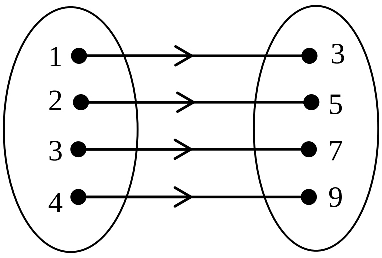
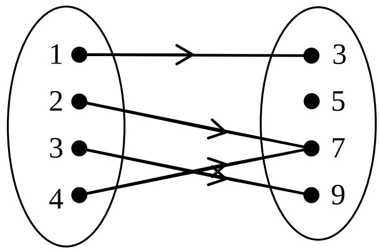
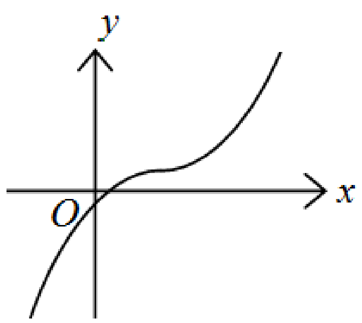
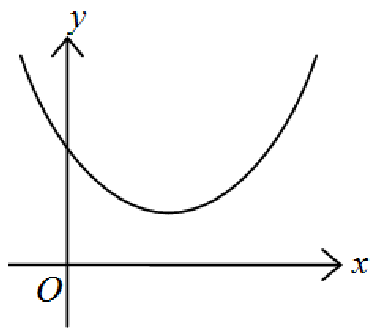
Learning Point(s):

Part 3. Key Questions to answer:

- ❑ What is a one-one function?
- ❑ How do you test whether a function is one-one?

Prerequisite knowledge: Curve Sketching, Piecewise and Quadratic expressions
Lecture Readings: Section 3

Question 3.1
Based on the table below, what makes a function a one-one function?

	One-one function	**Not a one-one function**
Set Diagram (Distinct points)		
Set Diagram (Distinct points)		
Graphical Representation (Continuous interval of points)		

Learning Point(s):

Question 3.2
Graphically, how do we test if a function is one-one?

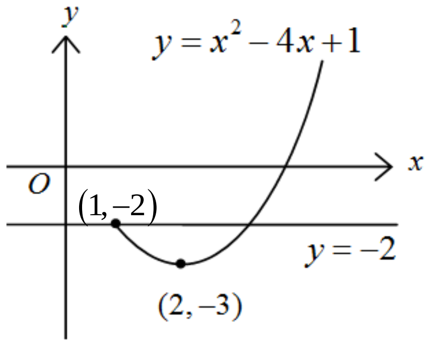
Solution:

Learning Point(s):

Question 3.3

Justify if $\mathrm{f}: x \mapsto x^2 - 4x + 1,\ x \in [1, \infty)$ is a one-one function.

If it is not one-one, find a restriction of f such that it is one-one.



Solution:
f is not one-one, because the line $y = -2$ cuts the curve $y = \mathrm{f}(x)$ at two points.

Alternatively to prove or disprove that a function is one-one, you may use the **algebraic definition** for one-one functions (self-reading).

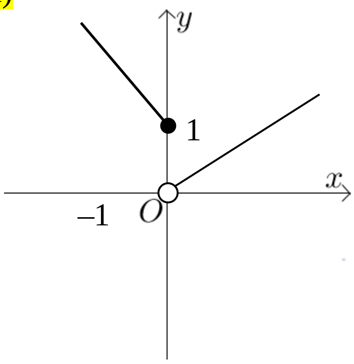
Observe that if we **restrict** the domain of f to $[1, 2]$ or $[2, \infty)$, then the function with the restricted domain is one-one.

Question 3.4
Determine if the following are one-one functions, giving your reasons. If it is not one-one, find a restriction of the function with a maximal domain such that it is one-one and has the same range as the original function.

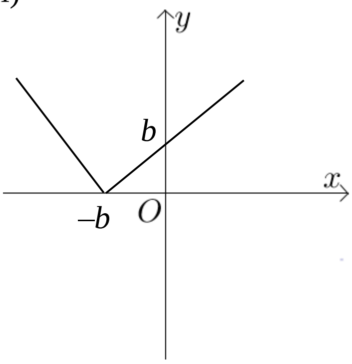
(i) $\mathrm{f}(x) = \begin{cases} 1 - x & x \le 0, \\ ax & x > 0, \end{cases}$ (ii) $\mathrm{g}: x \mapsto |x + b|,\ x \in \mathbb{R},$ (iii) $\mathrm{h}: x \mapsto x^2 - 2cx + 1,\ x \in [c, \infty),$

where a, b and c are **positive** constants.
Solution:

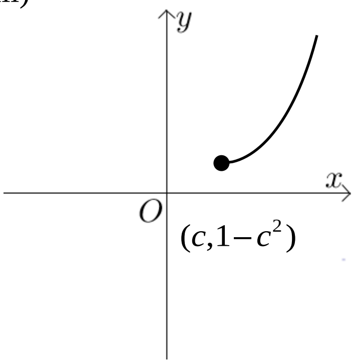
(i)



(ii)



(iii)



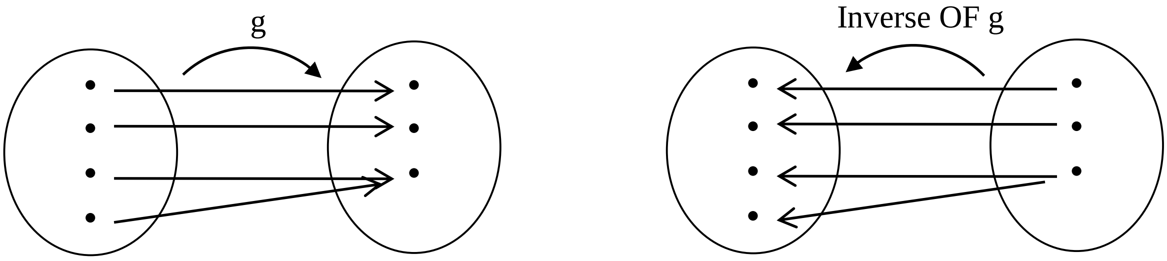
Learning Point(s):

Part 4. Key Questions to answer:

- ❑ What is meant by the inverse of a function?
  - o Is the inverse of a function necessarily a function?
- ❑ For a function f, how do you determine whether the function f^{-1} exists?
  - o If f^{-1} exists, what is its corresponding rule, domain and range?
- ❑ What is the relationship between the graphs of f and f^{-1}?

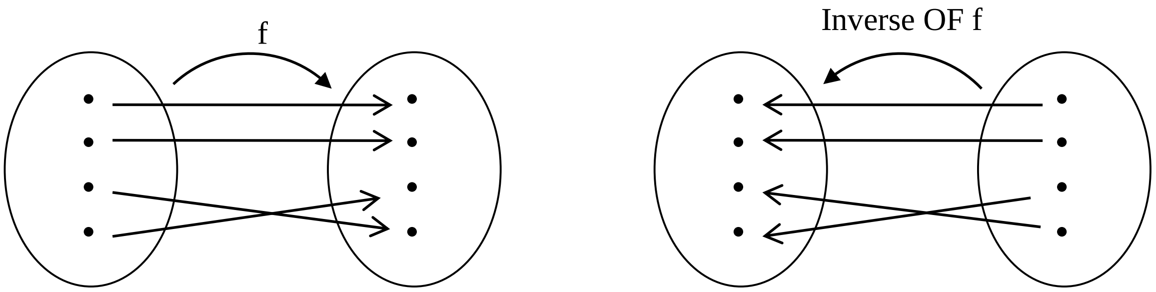
Prerequisite knowledge: Curve Sketching, Line of Symmetry, Changing subject in a formula
Lecture Readings: Section 4

In layman terms, if a function describes the action of "DO", then the inverse of a function describes the action of "UNDO". See the illustration below.



Note that while g is a function, the inverse of g is **NOT** a function. Why?

Question 4.1
Is the inverse of f a function?



Answer:
Yes. Since the inverse of f is a function, we say that its **inverse function exists** and denote it as f^{-1}.
Note that the inverse function f^{-1} is **NOT** $\frac{1}{f}$ (i.e. inversing a function does not mean we are taking reciprocal of the function).

Learning Point(s):

Question 4.2

For any function f, deduce the condition for f^{-1} to exist. Given that f^{-1} exists, what is its domain and range?

Solution:

Question 4.3

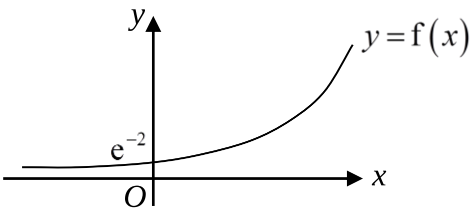
It is given that $\mathrm{f}: x \mapsto \mathrm{e}^{x-2}, x \in \mathbb{R}$. Determine if f^{-1} exists. If so, find the rule of f^{-1} and state its domain.

Solution:

From the graph, any horizontal line $y = k$, where $k \in (0, \infty)$,

intersects the graph of f exactly once.

This means that f is a one-one function. Hence f^{-1} exists.

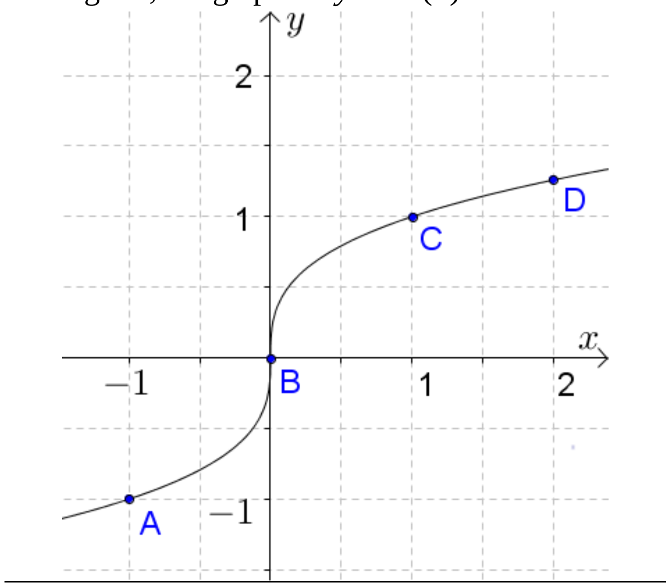


$$y = \mathrm{f}(x) \quad \Rightarrow \quad x = \mathrm{f}^{-1}(y)$$
$$y = \mathrm{e}^{x-2} \Rightarrow \ln y = x - 2 \Rightarrow x = \ln y + 2 \Rightarrow \mathrm{f}^{-1}(x) = \ln x + 2, \quad x \in (0, \infty)$$

Learning Point(s):

Question 4.4

The graph of $y = \mathrm{f}(x)$ is shown below. The points *A*, *B*, *C* and *D* lie on the graph. Sketch, on the same diagram, the graph of $y = \mathrm{f}^{-1}(x)$. Deduce the relationship between the graphs of f and f^{-1}.



Points on $y = \mathrm{f}(x)$	Points on $y = \mathrm{f}^{-1}(x)$
$A(-1, -1)$	
$B(0, 0)$	
$C(1, 1)$	
$D(2, 1.26)$	

Learning Point(s):

Question 4.5

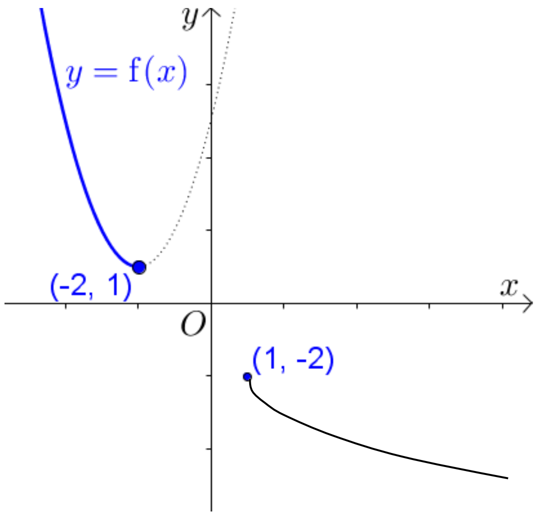
(i) Show that $f : x \mapsto (x+2)^2 + 1,\ x \in (-\infty, -2]$ has an inverse function f^{-1}.

(ii) Define f^{-1} in a similar form.

(iii) Hence sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram, demonstrating clearly the relationship between the two graphs.

Solution:

(i) From the graph of f, every horizontal line $y = a$, where $a \in R_f$ cuts the graph once. Therefore f is one-one, and it follows that f^{-1} exists.



(ii) Let $y = f(x) = (x+2)^2 + 1,\quad$ where $x \le -2.$

$\Rightarrow\ (x+2)^2 = y - 1$

$\Rightarrow\ x + 2 = \pm\sqrt{y-1}$

$\Rightarrow\ x = -2 \pm \sqrt{y-1}$

$\Rightarrow\ x = -2 - \sqrt{y-1}.$

$D_{f^{-1}} = R_f = [1, \infty).$

$f^{-1} : x \mapsto -2 - \sqrt{x-1}, \quad x \in [1, \infty)$

Do you know why we rejected $x = -2 + \sqrt{x-1}$?

Learning Point(s):

Note that sometimes algebraic manipulations may be required before changing the subject of a formula. Example: $y = f(x) = (x-2)(x-3)$. In other cases, it may not be possible to find an analytical expression for $f^{-1}(x)$. Example: $y = f(x) = x \ln x$

Part 5. Key Questions to answer:

- ❑ What is meant by a composite function?
- ❑ Given functions f and g, how do you determine whether the composite function fg exists?
  - o If fg exists, what is its corresponding rule, domain and range?
  - o Is fg the same as gf?

Prerequisite knowledge: Curve Sketching
Lecture Readings: Section 5

Question 5.1
Let f and g be two given functions and suppose that the function gf exists. What do you understand by the function gf?

Solution:
Suppose that we apply f to the element 2, and obtain the following result:

$$\mathrm{f}(2) = -3.$$

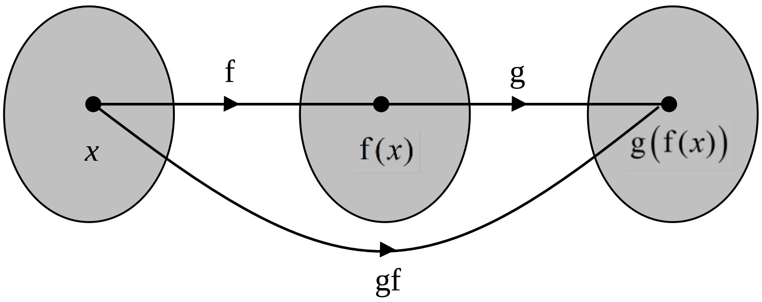
We would like to apply another function g to the above result, that is,

$$\mathrm{g}\left(\mathrm{f}(2)\right) = \mathrm{g}\left(-3\right).$$

Assume that $\mathrm{g}(-3)$ yields a real number, say 5. That means 2 is mapped to –3 by f and –3 is mapped to 5 by g. In short, 2 is mapped to 5 by the **composite function** gf (note the order of g and f).

Diagrammatically, we have $2 \xrightarrow{\ \mathrm{f}\ } -3 \xrightarrow{\ \mathrm{g}\ } 5$

In general, we have



Observe that the composite function gf takes in the same inputs as that of f. Therefore $D_{\mathrm{gf}} = D_{\mathrm{f}}$.

However, this composite function gf does not exist if g is defined by

$$\mathrm{g} : x \mapsto \sqrt{x}, \qquad x \in [0, \infty).$$

This is because "the output -3 is not accepted by g as an input" because $\sqrt{-3}$ is undefined.

Question 5.2
Deduce the condition for existence of the composite function gf.

Solution:

Question 5.3
The functions f and g are defined by

$$\mathrm{f}: x \mapsto x-5, \quad x\in[2,\infty) \qquad \text{and} \qquad \mathrm{g}: x \mapsto \sqrt{x+3}, \quad x\in[-3,\infty).$$

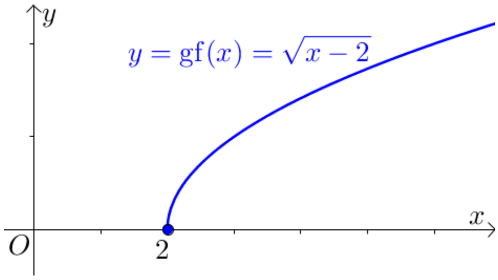
Show that gf exists. Hence define the function gf and find its range.

Solution:

Since $[-3,\infty) = R_\mathrm{f} \subseteq D_\mathrm{g}$, the composite function gf exists.
Therefore

$$\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x)) = \mathrm{g}(x-5) = \sqrt{(x-5)+3} = \sqrt{x-2},$$
$$D_{\mathrm{gf}} = D_\mathrm{f} = [2,\infty).$$

From the graph of $y = \mathrm{gf}(x)$, $R_{\mathrm{gf}} = [0,\infty)$.



Observe that in this example, $R_{\mathrm{gf}} = R_\mathrm{g}$. We have this result because $R_\mathrm{f} = D_\mathrm{g}$. The following question demonstrates that the result of $R_{\mathrm{gf}} = R_\mathrm{g}$ does **NOT** hold if $R_\mathrm{f} \neq D_\mathrm{g}$.

Question 5.4
The functions f and g are defined by

$$\mathrm{f}: x \mapsto x-5, \quad x\in[6,\infty) \qquad \text{and} \qquad \mathrm{g}: x \mapsto \sqrt{x+3}, \quad x\in[-3,\infty).$$

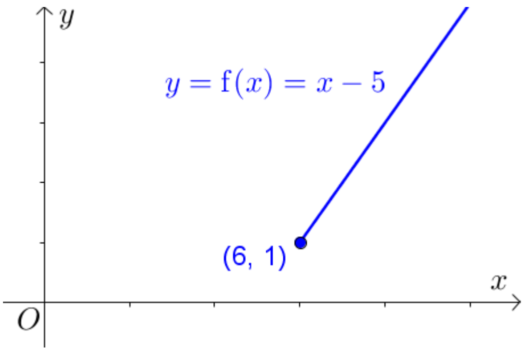
(i) Sketch, on separate diagrams, the graphs of functions f and g.

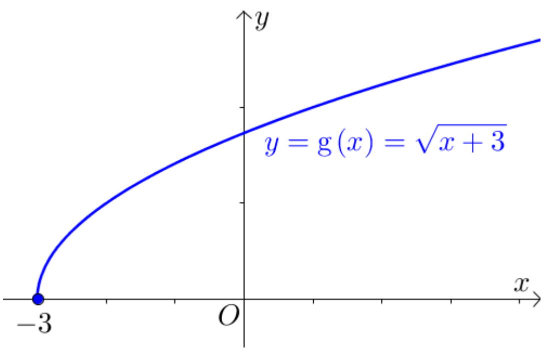
(ii) Show that gf exists, and find its rule, domain and range.

(iii) State whether fg exists, and explain your answer.

Solution:

(i)





(ii) $R_\mathrm{f} = [1,\infty)$. $D_\mathrm{g} = [-3,\infty)$. Since $R_\mathrm{f} \subseteq D_\mathrm{g}$, gf exists.
$\mathrm{gf}(x) = \mathrm{g}(x-5) = \sqrt{x-2}$.
$D_{\mathrm{gf}} = D_\mathrm{f} = [6,\infty)$.
$D_\mathrm{f} \xrightarrow{\mathrm{f}} R_\mathrm{f} = [1,\infty) \xrightarrow{\mathrm{g}} R_{\mathrm{gf}} = [2,\infty)$.

(iii) $R_\mathrm{g} = [0,\infty)$. $D_\mathrm{f} = [6,\infty)$. Since $R_\mathrm{g} \not\subseteq D_\mathrm{f}$, fg does not exist.

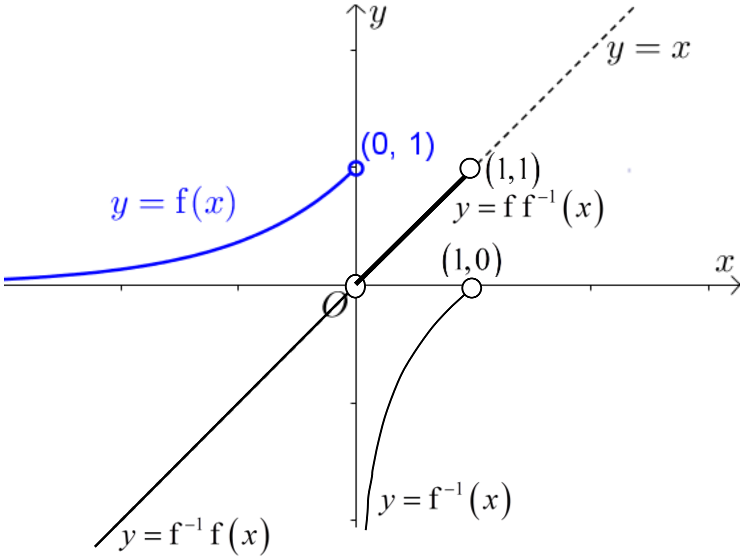
Learning Point(s):

Question 5.5
The function f is defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{x}, \quad x \in (-\infty, 0).$$

Sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$, $y = \mathrm{f}^{-1}\mathrm{f}(x)$ and $y = \mathrm{f}\mathrm{f}^{-1}(x)$. Indicate clearly any relationship(s) between the graphs. Hence solve $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{f}\mathrm{f}^{-1}(x)$.

Solution:



From the graph, there is no solution to $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{f}\mathrm{f}^{-1}(x)$.

Learning Point(s):