# NANYANG JUNIOR COLLEGE
# JC2 PRELIMINARY EXAMINATION

Higher 2

**MATHEMATICS** **9740/01**

Paper 1 **11 Sep 2012**

**3 hours**

Additional Materials: Answer Papers
List of Formulae (MF15)

**READ THESE INSTRUCTIONS FIRST**

Write your name and class on every script you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

**1** It is given that $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ , where $a$, $b$, $c$ and $d$ are constants.

The curve $C$ with equation $y = \mathrm{f}(x)$ passes through (0, –1) and has a maximum point at (–1, 1). The area bounded by $C$, the $x$-axis and the lines $x = 2$ and $x = 3$ is $\frac{31}{4}$ units$^2$. Given that f($x$) > 0 for $2 < x < 3$, find the values of $a$, $b$, $c$ and $d$. **[5]**

**2** The vectors **a** and **b** are given by

$$\mathbf{a} = \mathbf{i} + (\sin\theta)\mathbf{j} + (\cos\theta)\mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + (\sin\phi)\mathbf{j} + (\cos\phi)\mathbf{k}, \text{ where } 0 \le \theta \le \phi \le \pi .$$

Find an expression for $|\mathbf{a} \times \mathbf{b}|$ in terms of $\delta$, where $\delta = \frac{1}{2}(\phi - \theta)$. **[5]**

Deduce that the angle $\alpha$ between **a** and **b** is given by $\sin\alpha = \sin\delta\sqrt{1+\cos^2\delta}$ . **[2]**

**3** The equation of a curve is given by $4x^3 + 3x^2y = y^3 - 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$, simplifying your answer. **[2]**

The curve meets the line $y = -x$ at point $P$. Find

**(i)** the coordinates of $P$ and **[2]**

**(ii)** the equation of the tangent at $P$. **[2]**

The tangent to the curve at $P$ cuts the $x$-axis at $R$ and the normal to the curve at $P$ cuts the $y$-axis at $Q$. If $O$ denotes the origin, use the results above to give a geometrical description of the quadrilateral $OQPR$. **[1]**

**4** **(a)** Solve the inequality $1-2\sin x>0$ for $0\le x\le\frac{\pi}{2}$. Hence, evaluate the exact value of

$$\int_0^{\frac{\pi}{2}}\left|1-2\sin x\right|\,\mathrm{d}x.$$ **[4]**

**(b)** Use the substitution $x=\sin\theta$ to show that $\displaystyle\int\frac{\cos\theta}{\sqrt{2\cos 2\theta-1}}\,\mathrm{d}\theta=\int\frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$.

Given that $\displaystyle\int_0^{\alpha}\frac{\cos\theta}{\sqrt{2\cos 2\theta-1}}\,\mathrm{d}\theta=\frac{\pi}{4}$, find the value of $\alpha$, given that $0<\alpha<\frac{\pi}{2}$. **[4]**

**5** A curve $y=\mathrm{f}(x)$ undergoes in succession, the following transformations:

**A**: A reflection in the $y$-axis

**B**: A translation of 2 units in the direction of the $x$-axis

**C**: A translation of $-a$ units in the direction of the $y$-axis, where $a>1$

**(i)** The equation of the resulting curve is $y=\mathrm{g}(x)$, where $\mathrm{g}(x)=\dfrac{1-3a}{-x+5}$. Determine the equation of the curve $y=\mathrm{f}(x)$, in terms of $a$ and $x$. **[4]**

**(ii)** Sketch, on the same diagram, the graphs of $y=\mathrm{g}(x)$ and $y=\mathrm{g}^{-1}(x)$, indicating clearly the equations of the asymptotes and the axial intercepts. **[4]**

**6** **(a)** In a triangle with vertices $A$, $B$ and $C$, angle $BAC$ is a right-angle and angle $ABC=\frac{\pi}{3}-x$.

**(i)** Show that $\dfrac{AB}{AC}=\dfrac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$. **[1]**

**(ii)** Hence, show that when $x$ is small enough for $x^2$ and higher powers of $x$ to be neglected, then $\dfrac{AB}{AC}\approx a+bx$, where $a$ and $b$ are exact constants to be determined. **[3]**

**(b)** A curve is defined by the equation

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=\sqrt{1+x^2}$$

and (0, 1) is a point on the curve.

**(i)** Find the Maclaurin's expansion of $y$ up to and including the term in $x^2$. **[3]**

**(ii)** Hence, find the series expansion of $\mathrm{e}^y$, up to and including the term in $x^2$. **[3]**

**7** **(a)** An arithmetic progression has first term $a$ and common difference $d$, where $a$ and $d$ are non-zero. The first, third and seventh terms of the arithmetic progression are three consecutive positive terms of a geometric progression with common ratio $r$.

**(i)** Show that $r = 2$. **[3]**

**(ii)** The first term of the geometric progression is one-tenth that of the first term of the arithmetic progression. Find the smallest value of $n$ such that the sum of the first $n$ terms of the geometric progression exceeds the sum of the first $2n$ terms of the arithmetic progression. **[3]**

**(b)** Each time that a ball falls vertically on to a horizontal floor, it rebounds to three-fifth of the height from which it fell. It is initially dropped from a point $h$ m above the floor.

**(i)** Find the distance travelled by the ball just before it strikes the floor for the third time in terms of $h$. **[1]**

**(ii)** Show that the total distance travelled by the ball cannot exceed $4h$ m. **[3]**

**8** **(a)** Solve the equation $iz^5 = -32$, giving your roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. Sketch on an Argand diagram the points $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$ representing these roots, where $P_1$ represents the root with the smallest argument and $P_1\,P_2\,P_3\,P_4\,P_5$ is a polygon described in an anticlockwise sense. Find the area of $P_1\,P_2\,P_3\,P_4\,P_5$. **[6]**

**(b)** On an Argand diagram sketch clearly the locus of $P$ where $P$ represents the complex number $z$ such that $z$ satisfies both $|z - 2 - 2i| \le 1$ and $\arg(z - 1) = \arg(1 + 3i)$. Find the range of values of $\arg(z - 3 - 2i)$, given that $|z - 2 - 2i| \le 1$ and $\arg(z - 1) = \arg(1 + 3i)$. **[4]**

**9** **(a)** A sequence of negative real numbers $x_1, x_2, x_3, \ldots$ satisfies the relation

$$x_{n+1} = -\sqrt{1-2x_n}\text{, for } n \geq 1.$$

Given that the sequence converges to $l$, find the exact value of $l$. **[2]**

**(b)** **(i)** By expressing $\dfrac{2}{r(r+2)}$ in the form $\dfrac{A}{r}+\dfrac{B}{r+2}$, where $A$ and $B$ are real constants to be determined, show that $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$. **[3]**

**(ii)** Prove the result in **(i)** using mathematical induction. **[4]**

**(iii)** Deduce the value of $\displaystyle\sum_{r=2}^{\infty}\frac{1}{r(r+2)}$. **[2]**

**10** The points $P$ and $Q$ have position vectors $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $5\mathbf{i}+7\mathbf{j}+6\mathbf{k}$ respectively.

**(i)** The plane $\prod$ passes through $Q$ and is perpendicular to $PQ$. The equation of $\prod$ is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where $\lambda, \mu \in \mathbb{R}$ and vectors $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to each other. Write down a suitable vector $\mathbf{a}$ and explain why $3\mathbf{i}-2\mathbf{j}$ can be taken as $\mathbf{b}$. Find a suitable vector $\mathbf{c}$. **[5]**

The point $R$ has position vector $6\mathbf{i}+43\mathbf{j}+8\mathbf{k}$. The line $l$ passing through the points $P$ and $R$ intersect $\prod$ at point $S$ with position vector $\mathbf{s}$.

**(ii)** Explain why the lines with equations $\mathbf{r} = \mathbf{a} + \alpha\mathbf{b}$ and $\mathbf{r} = \mathbf{s} + \beta\mathbf{c}$, where $\alpha, \beta \in \mathbb{R}$, will intersect. **[2]**

**(iii)** Find the position vector of point $S$ and determine whether point $S$ lies on $PR$ produced. **[5]**

**11** In a city of 5 million people, the number of customers of a new company increases at a rate proportional to the number of people who are not its customers. The company determines that in order for its business to be profitable, it must have at least 1 million customers at any point in time.

The company has 1 million customers at the end of the third year of operation, starting from a customer base of 0. Assume that the population remains unchanged at any point in time.

**(i)** State a differential equation involving $y$ and $t$, where $y$ is the number of customers (in millions) and $t$ is the number of years from the time the company starts its operation. Hence show that $y = 5\left(1 - e^{kt}\right)$, where $k$ is a constant to be determined. **[5]**

**(ii)** Sketch the graph of $y$ against $t$. **[2]**

At the end of the 6th year of operation, the company has a customer base of 1.8 million. However, due to new competitors, the growth rate of its number of customers is now modelled by the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -0.1 .$$

It is known that under this model, its number of customers would drop to 1.65 million at the end of the following year of operation.

**(iii)** Determine in which year the company's business will become unprofitable. **[5]**

**End of Paper**