2024 VJC Prelim Paper 1 Solutions

1 **(a)** Express $\frac{33-8x}{x^2+2x-15}+2$ as a single algebraic fraction. Hence, without using a calculator, solve exactly the inequality $\frac{33-8x}{x^2+2x-15} > -2$. [4]

1a	$\frac{33-8x}{x^2+2x-15}+2$ $=\frac{33-8x+2\left(x^2+2x-15\right)}{x^2+2x-15}$ $=\frac{2x^2-4x+3}{x^2+2x-15}$ $=\frac{2(x-1)^2+1}{(x+5)(x-3)}$ Solve: $\frac{2(x-1)^2+1}{(x+5)(x-3)}>0$ Since $(x-1)^2 \geq 0$ for all $x \in \mathbb{R}$, $2(x-1)^2+1>0$ for all $x \in \mathbb{R}$. Hence, we solve $(x+5)(x-3)>0$. $\therefore x>3$ or $x<-5$	There is a need to <u>explain</u> clearly why the numerator is always positive. The better method is to complete the square and state that $(x-1)^2 \geq 0$, so $2(x-1)^2+1>0$. Some only described the discriminant to be negative or said that there are no real roots – both statements do NOT imply that the numerator is always positive. Note: 1 is NOT a critical value.

(b) Using your answer to part **(a)**, find the set of values of x for which $\frac{33-8e^{2x}}{e^{4x}+2e^{2x}-15} > -2$. [2]

1b	Replace x with e^{2x}: $e^{2x}>3$ or $e^{2x}<-5$ (rej. $\because e^{2x}>0$ for all $x \in \mathbb{R}$) $\therefore \left\{x \in \mathbb{R}:\ x>\frac{\ln 3}{2}\right\}$	<u>Explain</u> clearly why $e^{2x}<-5$ is rejected. Many showed working up to $2x<\ln(-5)$ to reject the statement which is incorrect. Firstly, $\ln(-5)$ is undefined. Secondly, IF the inequality were $e^{2x}>-5$, would you say that $2x>\ln(-5)$ and reject to conclude that there are no solutions?!

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2 The sum of the first n terms of a sequence, u_r is given by $\sum_{r=1}^{n} u_r = 1 - \frac{n}{(n+1)!}$.

(a) Find u_n in terms of n, for $n \geq 2$, expressing your answer as a single algebraic fraction. [2]

2a	$u_n = \sum_{r=1}^{n} u_r - \sum_{r=1}^{n-1} u_r$ $= 1 - \frac{n}{(n+1)!} - \left(1 - \frac{n-1}{n!}\right)$ $= \frac{-n+(n-1)(n+1)}{(n+1)\times n!}$ $= \frac{n^2-n-1}{(n+1)!}$	Note that $\sum_{r=1}^{n} u_r$ is usually denoted by S_n. Here, $u_n = S_n - S_{n-1}$.

(b) Show that $\sum_{r=5}^{n} u_r < \frac{1}{30}$, for all $n \geq 5$. [2]

2b	$\sum_{r=5}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{4} u_r$ $= 1 - \frac{n}{(n+1)!} - \left(1 - \frac{4}{5!}\right)$ $= \frac{1}{30} - \frac{n}{(n+1)!}$ $< \frac{1}{30}$, since $\frac{n}{(n+1)!} > 0$ for all $n \geq 5$	The explanation to why $\frac{1}{30} - \frac{n}{(n+1)!}$ is less than $\frac{1}{30}$ is to correctly mention that $\frac{n}{(n+1)!} > 0$. Most commonly seen answer was that $\frac{n}{(n+1)!} \to 0$. While it is true, it does not explain why $\frac{1}{30} - \frac{n}{(n+1)!} < \frac{1}{30}$.

(c) Explain why $\sum_{r=1}^{\infty} u_r$ is a convergent series. [1]

2c	$1 - \frac{n}{(n+1)!} = 1 - \frac{n}{(n+1)\times n \times (n-1)!} = 1 - \frac{1}{(n+1)\times(n-1)!}$ As $n \to \infty$, $(n+1)\times(n-1)! \to \infty$, $\frac{1}{(n+1)\times(n-1)!} \to 0$, $\sum_{r=1}^{n} u_r \to 1 - 0 = 1$, which is a constant. Hence $\sum_{r=1}^{\infty} u_r$ is a convergent series.	Many gave incomplete explanations to say that as $n \to \infty$, $(n+1)! \to \infty$, so $\frac{n}{(n+1)!} \to 0$. This completely disregards the fact that the numerator, $n \to \infty$ too and did not show complete understanding of why $\frac{n}{(n+1)!} \to 0$.

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3 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{ax-6}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3,\ b \neq 9,$$

$$\mathrm{g}: x \mapsto \mathrm{e}^{-x} \quad \text{for } x \in \mathbb{R},\ x \geq \ln 3.$$

The function f is self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all values of x in the domain of f. It is given that f is self-inverse.

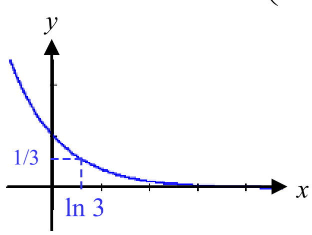
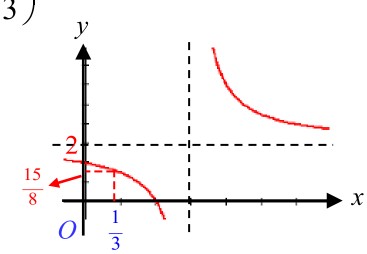
(a) Find the value of a. [3]

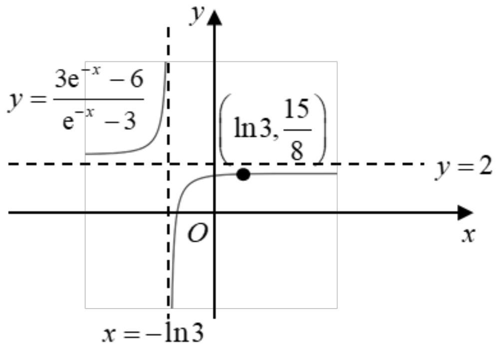
3a	Let $y = \frac{ax-6}{x-3}$ $xy - 3y = ax - 6$ $x = \frac{3y-6}{y-a}$ $\mathrm{f}^{-1}(x) = \frac{3x-6}{x-a}$ Given that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\frac{ax-6}{x-3} = \frac{3x-6}{x-a}$. $\therefore \quad a = 3$	Some wrote INCORRECTLY that $\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{f}(x) = x$. It is important to know that $\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{ff}(x) = x$.

(b) Find the exact value of $\mathrm{f}^6(\pi)$. [1]

3b	$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ $\mathrm{ff}(x) = x$ $\mathrm{f}^2(x) = x$ $\mathrm{f}^6(\pi) = \mathrm{f}^2\mathrm{f}^2\mathrm{f}^2(\pi) = \pi$	

(c) Find the exact range of fg. [3]

3c	Given $\mathrm{f}(x) = \frac{3x-6}{x-3} = 3\left(\frac{x-2}{x-3}\right)$  $\mathrm{D_g} = [\ln 3, \infty) \xrightarrow{\mathrm{g}} \mathrm{R_g} = \left(0, \frac{1}{3}\right] \xrightarrow{\mathrm{f}} \mathrm{R_{fg}} = \left[\frac{15}{8}, 2\right)$	The first method is to find $\mathrm{R_g}$ and use it as the new domain of f. To do this, it is advisable to sketch the graphs of g and of f.

	Alternative $\text{fg}(x)=\dfrac{3e^{-x}-6}{e^{-x}-3}$, $x \geq \ln 3$ 	The alternative is to draw directly the graph of $y=\text{fg}(x)=\dfrac{3e^{-x}-6}{e^{-x}-3}$, and note that $D_{fg}=D_g=[\ln 3, \infty)$. Some who did this method, left out the horizontal asymptote of the graph or incorrectly stated that it to be $y=3$. BOTH methods should be learnt and internalised.

4 (a) Given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})=\mathbf{b}\times\mathbf{c}$, and $\mathbf{b}\neq\mathbf{c}$, find the relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [4]

4a	$(\underset{\sim}{a}+\underset{\sim}{b})\times(\underset{\sim}{a}+\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$ $\underset{\sim}{a}\times\underset{\sim}{a}+\underset{\sim}{a}\times\underset{\sim}{c}+\underset{\sim}{b}\times\underset{\sim}{a}+\underset{\sim}{b}\times\underset{\sim}{c}=\underset{\sim}{b}\times\underset{\sim}{c}$ $\underset{\sim}{0}+\underset{\sim}{a}\times\underset{\sim}{c}+\underset{\sim}{b}\times\underset{\sim}{a}=\underset{\sim}{0}\quad (\because \underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{0})$ $\underset{\sim}{a}\times(\underset{\sim}{c}-\underset{\sim}{b})=\underset{\sim}{0}\quad (\because \underset{\sim}{b}\times\underset{\sim}{a}=-\underset{\sim}{a}\times\underset{\sim}{b})$ Since $\underset{\sim}{a}\neq\underset{\sim}{0}$ and $\underset{\sim}{b}\neq\underset{\sim}{c}$, $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c}-\underset{\sim}{b}$.	The notation of $\underset{\sim}{0}$ is still NOT seen in most answers. Note that 0 and $\underset{\sim}{0}$ are NOT to be used interchangeably. Some answers still show misunderstanding that $\underset{\sim}{b}\times\underset{\sim}{a}$ and $\underset{\sim}{a}\times\underset{\sim}{b}$ are equal but they are NOT. Note that $\underset{\sim}{a}\times(\underset{\sim}{c}-\underset{\sim}{b})=\underset{\sim}{0}$ does not directly imply that $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c}-\underset{\sim}{b}$. One has to check and state/explain that neither $\underset{\sim}{a}$ nor $\underset{\sim}{c}-\underset{\sim}{b}$ is $\underset{\sim}{0}$.

(b) It is given instead that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ with $|\mathbf{a}|=2$, $|\mathbf{b}|=3$ and $|\mathbf{c}|=4$. Find the value of

$$\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}.$$ [3]

4b	$(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}).(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})=0$ $\underset{\sim}{a}.\underset{\sim}{a}+\underset{\sim}{b}.\underset{\sim}{b}+\underset{\sim}{c}.\underset{\sim}{c}+2\underset{\sim}{a}.\underset{\sim}{b}+2\underset{\sim}{b}.\underset{\sim}{c}+2\underset{\sim}{a}.\underset{\sim}{c}=0$ $\lvert\underset{\sim}{a}\rvert^2+\lvert\underset{\sim}{b}\rvert^2+\lvert\underset{\sim}{c}\rvert^2+2(\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{b}.\underset{\sim}{c}+\underset{\sim}{a}.\underset{\sim}{c})=0$ $2^2+3^2+4^2+2(\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{b}.\underset{\sim}{c}+\underset{\sim}{a}.\underset{\sim}{c})=0$ $\underset{\sim}{a}.\underset{\sim}{b}+\underset{\sim}{b}.\underset{\sim}{c}+\underset{\sim}{a}.\underset{\sim}{c}=-\dfrac{29}{2}$	

5 It is given that $\mathrm{f}(n)=\dfrac{n}{5^{n-1}}$ where n is a positive integer.

(a) By considering $\mathrm{f}(r)-\mathrm{f}(r+1)$, find an expression for $\displaystyle\sum_{r=2}^{n}\frac{4r-1}{5^r}$. [3]

5 a	$\mathrm{f}(r)-\mathrm{f}(r+1)=\dfrac{r}{5^{r-1}}-\dfrac{r+1}{5^r}$ $=\dfrac{5r-r-1}{5^r}=\dfrac{4r-1}{5^r}$ $\displaystyle\sum_{r=2}^{n}\frac{4r-1}{5^r}=\sum_{r=2}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r+1)\right]$ $=\mathrm{f}(2)-\mathrm{f}(3)$ $+\mathrm{f}(3)-\mathrm{f}(4)$ $+\dots$ $+\mathrm{f}(n-1)-\mathrm{f}(n)$ $+\mathrm{f}(n)-\mathrm{f}(n+1)$ $=\mathrm{f}(2)-\mathrm{f}(n+1)$ $=\dfrac{2}{5}-\dfrac{n+1}{5^n}$	

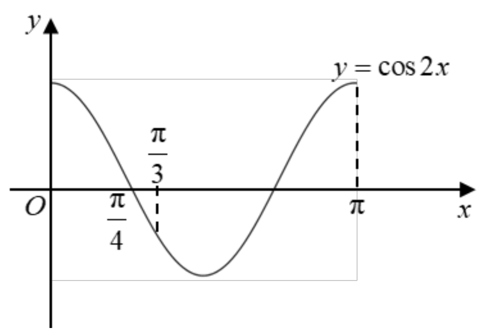
(b) Hence find an expression for $\displaystyle\sum_{r=1}^{n}\frac{4r+6}{5^{r+1}}$. [3]

5 b	$\displaystyle\sum_{r=1}^{n}\frac{4r+6}{5^{r+1}}$ Replace r by $r-1$ $\displaystyle\sum_{r-1=1}^{r-1=n}\frac{4(r-1)+6}{5^{r-1+1}}=\sum_{r=2}^{n+1}\frac{4r+2}{5^r}$ $\displaystyle=\sum_{r=2}^{n+1}\frac{4r-1}{5^r}+\sum_{r=2}^{n+1}\frac{3}{5^r}$ $=\dfrac{2}{5}-\dfrac{n+2}{5^{n+1}}+3\left[\dfrac{\frac{1}{25}\left(1-\left(\frac{1}{5}\right)^n\right)}{1-\frac{1}{5}}\right]$ $=\dfrac{11}{20}-\dfrac{n+2}{5^{n+1}}-\dfrac{3}{20}\left(\dfrac{1}{5}\right)^n$	When we use the replacement of variable method, we need to change the upper limit as well. Learn to recognise a sum of a GP, like in this case: $\displaystyle\sum_{r=2}^{n+1}\frac{3}{5^r}$ is a sum of a GP with first term $\dfrac{3}{5^2}$ and common ratio $\dfrac{1}{5}$.

6 **(a)** Find the exact value of $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$. [3]

6 a	$\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ $\displaystyle=2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\cdot\sin^{-1}x\,dx$ $\displaystyle=\left[\left(\sin^{-1}x\right)^2\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$ $\displaystyle=\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)^2-\left(\sin^{-1}\frac{1}{2}\right)^2$ $\displaystyle=\left(\frac{\pi}{3}\right)^2-\left(\frac{\pi}{6}\right)^2$ $\displaystyle=\frac{\pi^2}{12}$	Be familiar with all the standard forms. Know how to check if the integrand is truly of the form $\frac{f'(x)}{f(x)}$, or of the form $f'(x)\left[f(x)\right]^n$. Check through all standard forms before even thinking about integration by parts. Those who did it by parts spent much more time on this question.

(b) Find the exact value of $\int_0^{\frac{\pi}{3}}\left|\cos 2x\right|dx$. [3]

6 b	 $\displaystyle\int_0^{\frac{\pi}{3}}\left|\cos 2x\right|dx=\int_0^{\frac{\pi}{4}}\cos 2x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\cos 2x\,dx$ $\displaystyle=\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}-\left[\frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $\displaystyle=\frac{1}{2}-\left(\frac{\sqrt{3}}{4}-\frac{1}{2}\right)$ $\displaystyle=1-\frac{\sqrt{3}}{4}$	Sketching a graph is good way to tell when the expression is positive or negative. If you cannot visualise clearly, sketch it! There is a need to split the integral into parts where the expression $\cos 2x$ is positive and negative within the interval $\left(0,\frac{\pi}{3}\right)$.

(c) Find $\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$, where k is a positive constant. [4]

6 c	$\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left(x^2-2kx\right)+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left[\left(x-k\right)^2-k^2\right]+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left(x-k\right)^2+4k^2}\,\mathrm{d}x$ $=\int \frac{1}{\left(2k\right)^2-\left(x-k\right)^2}\,\mathrm{d}x$ $=\frac{1}{2\left(2k\right)}\ln\left\|\frac{2k+\left(x-k\right)}{2k-\left(x-k\right)}\right\|+C$ $=\frac{1}{4k}\ln\left\|\frac{k+x}{3k-x}\right\|+C$ **<u>Alternative</u>** $\int \frac{1}{\left(k+x\right)\left(3k-x\right)}\,\mathrm{d}x$ $=\int\left[\frac{1}{4k\left(k+x\right)}+\frac{1}{4k\left(3k-x\right)}\right]\mathrm{d}x$ $=\frac{1}{4k}\ln\left\|k+x\right\|-\frac{1}{4k}\ln\left\|3k-x\right\|+C$ $=\frac{1}{4k}\ln\left\|\frac{k+x}{3k-x}\right\|+C$	

7 **(a)** The curve C has equation $y=\mathrm{f}(x)$ where

$$\mathrm{f}(x)=\frac{ax^2+bx+c}{x+d},$$

and a, b, c and d are constants, and $a\neq 0$.

Given that C has asymptote $y=x+1$, find the value of a and show that $b=d+1$. [2]

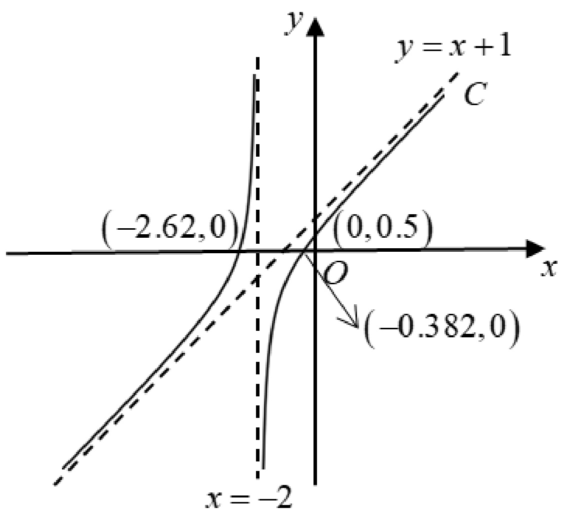
7a	$y=\mathrm{f}(x)=\dfrac{ax^2+bx+c}{x+d}=x+1+\dfrac{R}{x+d}$ From the numerator: $ax^2+bx+c=(x+1)(x+d)+R$ Comparing the coefficients, we get: $a=1,\ \ b=d+1$ **<u>Alternative</u>** $\mathrm{f}(x)=\dfrac{ax^2+bx+c}{x+d}=ax+(b-ad)+\dfrac{c-bd+ad^2}{x+d}$ $ax+(b-ad)\equiv x+1$ $\begin{cases}a=1\\b-ad=1\end{cases}\Rightarrow a=1$ and $b=d+1$	Many candidates were able to express it in the correct form of $x+1+\dfrac{R}{x+d}$ and compared coefficients after expansion. A handful did not regard the remainder and made common mistakes like saying $ax^2+bx+c=(x+1)+(x-d)$.

If f is an increasing function for all $x\in\mathbb{R}$, $x>-d$, show that $c<d$. [3]

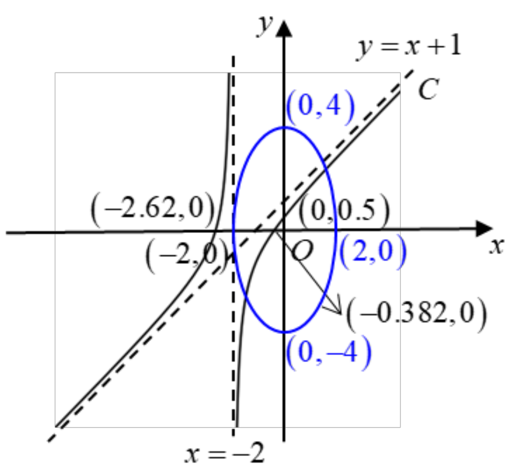
7a	Comparing coefficients from previous part, we get: $a=1,\ \ b=d+1$ and $c=d+R\Rightarrow R=c-d$ [OR alternative: $c-bd+ad^2=c-(d+1)d+d^2=c-d$] $\therefore y=\mathrm{f}(x)=x+1+\dfrac{c-d}{x+d}$ If f is increasing for $x>-d$, it must be the case that $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$ for all $x\in\mathbb{R},\ x\neq -d$. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{c-d}{(x+d)^2}>0$ Since $(x+d)^2>0$ for all $x\in\mathbb{R}$, $x\neq -d$, $(x+d)^2-(c-d)>0$. i.e. $x^2+2dx+d^2-c+d>0$ for all $x\in\mathbb{R},\ x\neq -d$ This means that discriminant < 0 $\Rightarrow(2d)^2-4(1)(d^2-c+d)<0$ $4c-4d<0$ $\Rightarrow c<d$ (shown)	Differentiating the expression $x+1+\dfrac{c-d}{x+d}$ is much easier than differentiating $\dfrac{x^2+bx+c}{x+d}$. We need to explain why $(x+d)^2-(c-d)>0$ for all $x\in\mathbb{R},\ x\neq -d$ will lead to $c<d$.

(b) It is further given that $c=1$ and $d=2$.

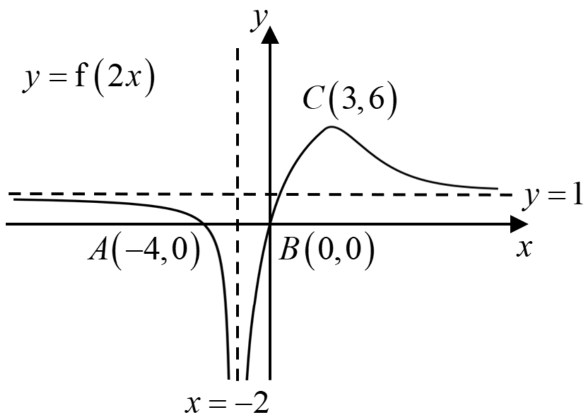
(i) Sketch *C*. [3]

7b (i)	Sketch $y=\mathrm{f}(x)=x+1-\dfrac{1}{x+2}$ 	When asked to sketch a graph, always label the coordinates of any stationary points, axial intercepts, and the equations of any asymptotes. If question did not ask for exact values, leave all answers rounded off to 3 s.f..

(ii) By sketching a suitable graph in the same diagram in part **(b)(i)**, find the number of real roots to the equation $\left(\dfrac{x^2+3x+1}{x+2}\right)^2+4x^2=16$. [2]

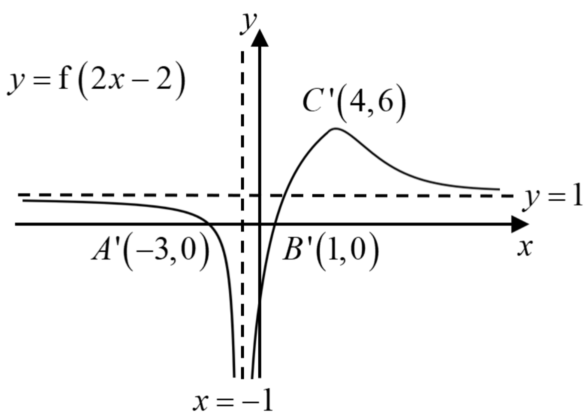
7b (ii)	Sketch $\dfrac{y^2}{4^2}+\dfrac{x^2}{2^2}=1$ on the same diagram, i.e. ellipse with centre at $(0,0)$ and passing through (–2, 0), (2, 0), (0, –4) and (0, 4).  The 2 points of intersection implies that there are 2 real roots to the equation $\left(\dfrac{x^2+3x+1}{x+2}\right)^2+4x^2=16$.	When sketching any conic section, like a circle or ellipse, the graph should show the correct 'shape'. In this case, the ellipse should look 'tall', passing through (–2, 0). Many did not recognise that the graph to be drawn is $\dfrac{y^2}{4^2}+\dfrac{x^2}{2^2}=1$, and went on to make y the subject unnecessarily, often resulting in only the positive square root taken ($y=\sqrt{16-4x^2}$) which is incorrect.

8 **(a)** The diagram shows the graph with equation $y = \mathrm{f}(2x)$. The graph passes through the points $A(-4,0)$, $B(0,0)$ and $C(3,6)$, and has asymptotes $x = -2$ and $y = 1$.

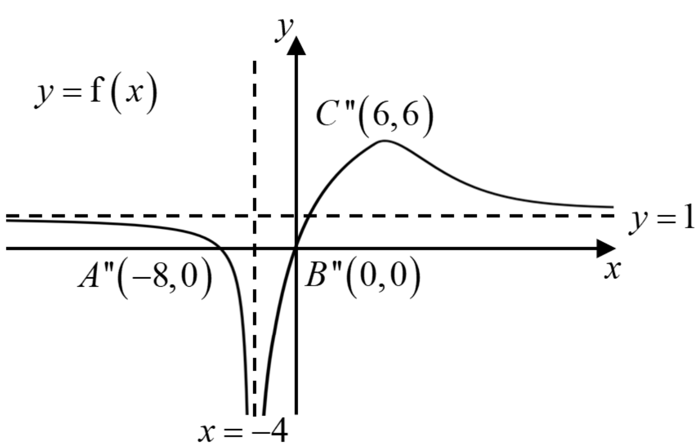


On separate clearly labelled diagrams, deduce the graphs of

(i) $y = \mathrm{f}(2x-2)$, [2]

8a (i)	$y = \mathrm{f}(2x-2) = \mathrm{f}\left(2\left(x-1\right)\right)$  	Note that it is a transformation <u>from</u> $y = \mathrm{f}(2x)$ <u>to</u> $y = \mathrm{f}(2x-2) = \mathrm{f}\left(2\left(x-1\right)\right)$ i.e. **replace** x with $x-1$ i.e. translate 1 unit (not 2 units) in the positive x-direction

(ii) $y = \mathrm{f}(x)$. [2]

8a (ii)	 	Note that it is a transformation <u>from</u> $y = \mathrm{f}(2x)$ <u>to</u> $y = \mathrm{f}(x)$ i.e. **replace** x with $\frac{x}{2}$ i.e. stretch factor is 2, not ½.

(b) The curve C_1 undergoes the transformations in the order given below:

1. A translation of 2 units in the negative x direction.
2. A stretch parallel to the x axis, factor 2, y axis invariant.
3. A translation of 1 unit in the positive y direction.

The resulting curve C_2 has equation

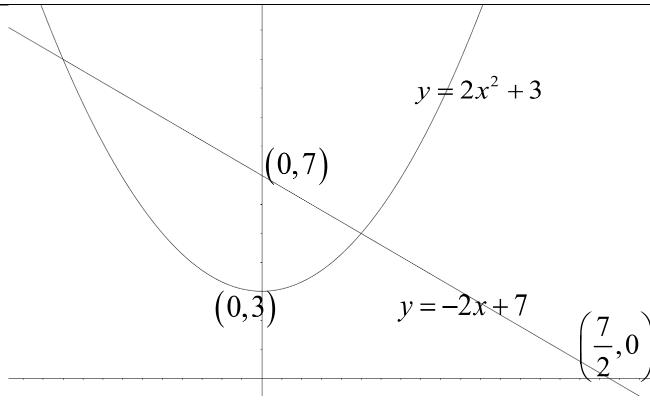
$$y=\frac{x^2+9x+22}{x+4}\ ,\quad x\in\mathbb{R},\quad x\neq -4\,.$$

Find, in the simplest form, the equation for C_1. [4]

8b	Let C_1 have equation $y=\mathrm{f}(x)$. 1. $y=\mathrm{f}(x)\rightarrow y=\mathrm{f}(x+2)$; 2. $y=\mathrm{f}(x+2)\rightarrow y=\mathrm{f}\left(\frac{1}{2}x+2\right)$; 3. $y=\mathrm{f}\left(\frac{1}{2}x+2\right)\rightarrow y=\mathrm{f}\left(\frac{1}{2}x+2\right)+1$. Now $y=\mathrm{f}\left(\frac{1}{2}x+2\right)+1=\frac{x^2+9x+22}{x+4}$. $\mathrm{f}\left(\frac{1}{2}x+2\right)=\frac{x^2+9x+22}{x+4}-1=\frac{x^2+8x+18}{x+4}$ $\mathrm{f}\left(\frac{1}{2}(x+4)\right)=\frac{(x+4)^2+2}{x+4}=x+4+\frac{2}{x+4}$ Let $w=\frac{1}{2}(x+4)\Rightarrow x+4=2w$ $\therefore \mathrm{f}(w)=2w+\frac{2}{2w}=2w+\frac{1}{w}$ $\therefore y=\mathrm{f}(x)=2x+\frac{1}{x},\quad x\neq 0.$ **<u>Alternative</u>**: May be easier to use the "reverse method": 3. $\frac{x^2+9x+22}{x+4}-1=\frac{x^2+8x+18}{x+4}$. 2. Replace x with $2x$: $\frac{(2x)^2+8(2x)+18}{(2x)+4}=\ldots=\frac{2x^2+8x+9}{x+2}$. 1. Replace x with $x-2$: $\frac{2(x-2)^2+8(x-2)+9}{(x-2)+2}=\ldots=2x+\frac{1}{x},\quad x\neq 0$.	Need to reverse the order and do the correct replacement. Note that $y=\frac{4x^2+2}{2x}$ is not in simplified form The alternative method is more challenging.

9 Find the area of the region bounded by the graphs of $y = 2x^2 + 3$ and $y = -2x + 7$. [3]

9



Solving $y = 2x^2 + 3$ and $y = -2x + 7$

$$-2x + 7 = 2x^2 + 3$$

$$2x^2 + 2x - 4 = 0$$

$$x^2 + x - 2 = 0$$

$x = -2$ or $x = 1$

Area

$$= \int_{-2}^{1} (-2x + 7)\,dx - \int_{-2}^{1} (2x^2 + 3)\,dx$$

$$= \left[-x^2 + 4x - \frac{2x^3}{3} \right]_{-2}^{1}$$

$$= \left(-1 + 4 - \frac{2}{3} \right) - \left(-4 - 8 + \frac{2(8)}{3} \right)$$

$$= 9$$

For questions that do not prohibit the use of GC nor require exact solutions, one can use GC to evaluate numerical answers.

There is NO need to split the integral to

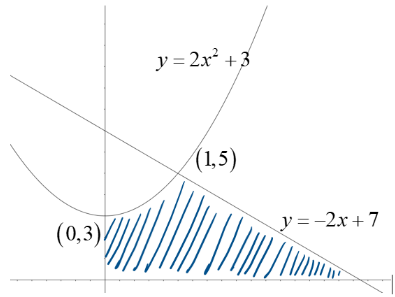
$$\int_{-2}^{0} \left(-2x + 7 - \left(2x^2 + 3\right) \right) dx + \int_{0}^{1} \left(-2x + 7 - \left(2x^2 + 3\right) \right) dx$$

as the line is above the curve for $-2 \le x \le 1$

State the area of the region bounded by the 2 graphs if both graphs are translated 7 units in the negative y-direction. [1]

9 Area = 9

The region R is bounded by $y=2x^2+3$, $y=-2x+7$, the x-axis and the y-axis. Find the exact volume of the solid generated when R is rotated 2π about the y-axis. [5]

9	Substitute $x=1$ into $y=-2x+7$ get $y=5$. Expressing x^2 as the subject $y=2x^2+3$ $\quad$ $y=-2x+7$ $x^2=\dfrac{y-3}{2}$ and $x^2=\left(\dfrac{y-7}{-2}\right)^2=\dfrac{1}{4}(y-7)^2$ Volume $=\pi\int_0^5 \frac{1}{4}(y-7)^2\,dy-\pi\int_3^5\left(\frac{y-3}{2}\right)dy$ $=\pi\left[\frac{(y-7)^3}{12}\right]_0^5-\pi\left[\frac{(y-3)^2}{4}\right]_3^5$ $=\dfrac{323\pi}{12}$	It is important to identify the correct region, R.  For this question, you are required to find the volume generated when R is rotated about ***y*-axis**, not x-axis. For hollow volumes, always use $\int_a^b \pi x_1^2\,dy-\int_c^d \pi x_2^2\,dy$ (for rotation about y-axis)

10 Given that $z=2-\mathrm{i}$ is a root of the equation $4z^4-12z^3+17z^2+pz+q=0$, where p and q are real, find p and q. [4]

10	$4(2-\mathrm{i})^4-12(2-\mathrm{i})^3+17(2-\mathrm{i})^2+p(2-\mathrm{i})+q=0$ $4(-7-24\mathrm{i})-12(2-11\mathrm{i})+17(3-4\mathrm{i})+p(2-\mathrm{i})+q=0$ $(-28-24+51+2p+q)+(-96+132-68-p)\mathrm{i}=0$ Comparing real and imaginary parts, $-1+2p+q=0$ ...(1) $-32-p=0$ ...(2) Solving, $p=-32,\quad q=65$ **<u>Alternative</u>** Since all the coefficients of $4z^4-12z^3+17z^2+pz+q=0$ are real, by Conjugate Root Theorem, $z^*=2+i$ is also a root. A quadratic factor is $[z-(2-\mathrm{i})][z-(2+\mathrm{i})]=z^2-4z+5$ $4z^4-12z^3+17z^2+pz+q=(z^2-4z+5)(az^2+bz+c)$ Comparing coefficients, $a=4$ $b-4a=-12\Rightarrow b=4$ $c-4b+5a=17\Rightarrow c=13$ $-4c+5b=p\Rightarrow p=-32$ $5c=q\Rightarrow q=65$	Tip: Question did not prohibit the use of GC, hence can use GC to quickly evaluate $(2-\mathrm{i})^4$, $(2-\mathrm{i})^3$ and $(2-\mathrm{i})^2$

Using the values of p and q found, find the other roots of the equation $4z^4-12z^3+17z^2+pz+q=0$ in exact form. [4]

10	Since all the coefficients of $4z^4-12z^3+17z^2-32z+65=0$ are real, by Conjugate Root Theorem, $z^*=2+\mathrm{i}$ is also a root. A quadratic factor is $\left[z-(2-i)\right]\left[z-(2+i)\right]=z^2-4z+5$ By long division (or by observation), $4z^4-12z^3+17z^2-32z+65=\left(z^2-4z+5\right)\left(4z^2+4z+13\right)$ $4z^4-12z^3+17z^2-32z+65=0$ $\left(z^2-4z+5\right)\left(4z^2+4z+13\right)=0$ $z=2-\mathrm{i},\ 2+\mathrm{i}$ or $z=\dfrac{-4\pm\sqrt{16-4(4)(13)}}{8}$ $=\dfrac{-4\pm\sqrt{-192}}{8}$ $=\dfrac{-4\pm4\sqrt{-12}}{8}$ $=\dfrac{-1\pm2\sqrt{3}\mathrm{i}}{2}$ $=-\dfrac{1}{2}\pm\sqrt{3}\mathrm{i}$ The other roots are $z=2+\mathrm{i},\ -\dfrac{1}{2}+\sqrt{3}\mathrm{i}$ or $-\dfrac{1}{2}-\sqrt{3}\mathrm{i}$	Need to mention the use of Conjugate Root Theorem as all coefficients of the polynomial are real. Tip: use $(a+b)(a-b)=a^2-b^2$ to quickly expand $\left[z-(2-\mathrm{i})\right]\left[z-(2+\mathrm{i})\right]$ $=\left[(z-2)+i\right]\left[(z-2)-i\right]$ $=(z-2)^2-i^2$ Remember to answer the question – there are in total **3 other roots** of the equation.

11 Dendrologists are specialised scientists who study trees and woody plants. Their work is diverse and can encompass various activities related to the identification, classification, biology, and ecology of trees. A group of dendrologists are studying the growth of 2 species of trees, codenamed Tree *Vee* and Tree *Jay*.

In the 1st year, the height of Tree *Vee* and Tree *Jay* are both H cm.

In the 2nd year, Tree *Vee*'s height increases by s cm and subsequently, the increase in height every year is 10% less than the previous year's increase. Show that the height of Tree *Vee* in the 4th year is given by $\left(H+2.71s\right)$ cm. [1]

11

Tree *Vee*

n	Increase in height (cm)
1	-
2	s
3	$0.9\ s$
4	$0.9^2\ s$
…	

Height of Tree *Vee* in the 4th year in cm

$= H+s+0.9s+0.9^2 s$

$=H+2.71s$

Show that the height of Tree *Vee* in the nth year is given by $\left[H+10s\left(1-0.9^{n-1}\right)\right]$ cm. [3]

11	Height of Tree *Vee* in the nth year in cm $=H+s+0.9s+0.9^2 s+\ldots+0.9^{n-2} s$ $=H+\dfrac{s\left(1-0.9^{n-1}\right)}{1-0.9}$ $=H+10s\left(1-0.9^{n-1}\right)$	It is a "show" question. Need to <u>write down the terms of the series</u> before using sum of GP formula to get the final result.

Hence, write down in terms of H and s, the theoretical maximum height (in cm) of Tree *Vee*. [1]

11	Theoretical maximum height (cm) of Tree *Vee* $=H+10s$	

In the 2nd year, Tree *Jay*'s height increases by t cm and subsequently, the increase in height every year is 0.5 cm less than the previous year's increase. Show that the height of Tree *Jay* in the 10th year is given by $\left(H-18+9t\right)$ cm. [2]

<table>
<tr>
<td>11</td>
<td>
<table>
<tr><th colspan="2">Tree Jay</th></tr>
<tr><th>n</th><th>Increase in height (cm)</th></tr>
<tr><td>1</td><td>-</td></tr>
<tr><td>2</td><td>t</td></tr>
<tr><td>3</td><td>$t-0.5$</td></tr>
<tr><td>4</td><td>$t-2(0.5)$</td></tr>
<tr><td>…</td><td></td></tr>
</table>
Height of Tree Jay (cm) in the 10th year
$=H+t+\left(t-0.5\right)+\left(t-2\left(0.5\right)\right)+...+\left(t-8\left(0.5\right)\right)$
$=H+\frac{9}{2}\left[2t+\left(9-1\right)\left(-0.5\right)\right]$
$=H+\frac{9}{2}\left(2t-4\right)$
$=H-18+9t$
</td>
<td>Again, it is a "show" question.

Best to <u>write down the terms of the series</u> before using sum of AP formula to get the final result</td>
</tr>
</table>

It is now given that $t=20$.

After the 10th year, Tree *Jay*'s height increases at a constant rate of 7 cm per year. Express Tree Jay's height (in cm) in the nth year (where $n\geq 11$) in terms of H and n. [2]

<table>
<tr>
<td>11</td>
<td>
Height of Tree Jay (cm) in the 10th year
$=H-18+9(20)$
$=H+162$

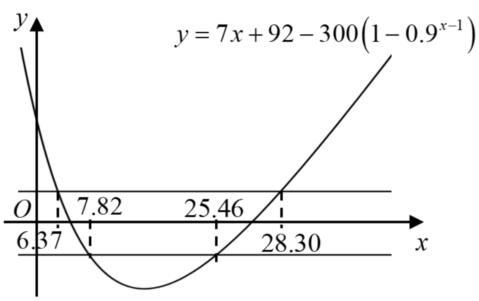
<table>
<tr><th colspan="2">Tree Jay</th></tr>
<tr><th>n</th><th>Height (cm)</th></tr>
<tr><td>10</td><td>H+162</td></tr>
<tr><td>11</td><td>H+169</td></tr>
<tr><td>12</td><td>H+176</td></tr>
<tr><td>…</td><td></td></tr>
</table>

Height of Tree Jay (cm) in the nth year
$=H+162+(n-10)(7)$
$=H+7n+92$
</td>
<td></td>
</tr>
</table>

It is further given that $s = 30$, and the 1st year is the year 2024. Find the years in which the heights of Tree *Vee* and Tree *Jay* are within 7 cm of each other, **after 2034**. [3]

<table>
<tr><td>11</td><td>
Difference in height (cm) of Tree Vee and Tree Jay in nth year
$= [H + 7n + 92] - [H + 10s(1 - 0.9^{n-1})]$
$= 7n + 92 - 300(1 - 0.9^{n-1})$
Using GC,
<table>
<tr><th>n</th><th>Difference in height (cm) of Tree Vee and Tree Jay= $7n + 92 - 300(1 - 0.9^{n-1})$</th></tr>
<tr><td>25</td><td>– 9.07</td></tr>
<tr><td>26</td><td>– 4.46</td></tr>
<tr><td>27</td><td>0.383</td></tr>
<tr><td>28</td><td>5.44</td></tr>
<tr><td>29</td><td>10.7</td></tr>
</table>
2024 + (26 – 1) = 2049

The years are 2049, 2050 and 2051.

<u>Alternative (graphical solution)</u>


Required values are $n = 26, 27, 28$.
The years are 2049, 2050 and 2051.
</td><td>
As n must be an integer, use a table of values to solve the <u>inequality</u>
$|7n + 92 - 300(1 - 0.9^{n-1})| < 7$

Tip: to obtain the years, you can consider using this table
<table>
<tr><th>n</th><th>Year</th></tr>
<tr><td>n =1</td><td>2024</td></tr>
<tr><td>n = 2</td><td>2025</td></tr>
<tr><td>…</td><td></td></tr>
<tr><td>n = 26</td><td>2024+26-1
=2049</td></tr>
</table>
</td></tr>
</table>

12 Game developers closely monitor the number of people playing their game. Understanding player numbers and behaviour can not only help in optimising in-game purchases, advertisement placements, and other revenue-generating aspects, it can also help the company manage server loads and ensure the game runs smoothly without performance issues.

Two game developers are interested in the number of players playing the mobile game "Mobile Saga". They attempt to model the number of players x, in **hundred thousands**, at time t months after the launch of the game using a differential equation. On the day of the launch, there were 55 000 players.

(a) One game developer suggests that x and t are related by the differential equation $\frac{dx}{dt}=\frac{3}{5}x-kt^2$, where k is a positive constant.

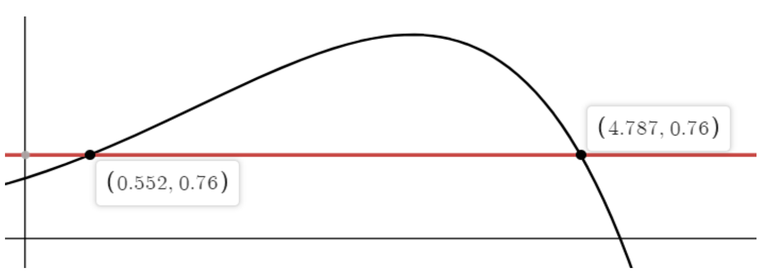
(i) By substituting $x=ue^{\frac{3}{5}t}$, show that the differential equation can be written as $\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$. [2]

12 a i	$x=ue^{\frac{3}{5}t} \quad ---(1)$ $\frac{dx}{dt}=0.6x-kt^2 \quad ---(2)$ Differentiating (1) w.r.t t $\frac{dx}{dt}=\frac{3}{5}ue^{\frac{3}{5}t}+e^{\frac{3}{5}t}\frac{du}{dt} \quad ---(3)$ Substitute (1) & (3) into (2) $\frac{3}{5}ue^{\frac{3}{5}t}+e^{\frac{3}{5}t}\frac{du}{dt}=0.6\left(ue^{\frac{3}{5}t}\right)-kt^2$ $e^{\frac{3}{5}t}\frac{du}{dt}=-kt^2$ $\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$	For such DE questions involving substitution, **identify the variables** to be substituted before performing any differentiation **Given DE**: $\frac{dx}{dt}=\frac{3}{5}x-kt^2$ in terms of $\frac{dx}{dt},t,x$ ↓ **Result**: $\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$ - in terms of $\frac{du}{dt},t$ Looking at the above, one has to use $x=ue^{\frac{3}{5}t}$ substitute $\frac{dx}{dt}$ with $\frac{du}{dt}$. Hence, differentiate $x=ue^{\frac{3}{5}t}$ wrt t, bearing in mind both u and t are variables, not constants.

(ii) Hence show that $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)e^{\frac{3}{5}t}$. [4]

12 a ii	$\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$ Integrating w.r.t. t: $u=-k\int t^2e^{-\frac{3}{5}t}\,dt$ $=-k\left[-\frac{5}{3}t^2e^{-\frac{3}{5}t}-\int(2t)\left(-\frac{5}{3}e^{-\frac{3}{5}t}\right)dt\right]$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}-\frac{10k}{3}\int te^{-\frac{3}{5}t}\,dt$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}-\frac{10k}{3}\left[\left(-\frac{5}{3}te^{-\frac{3}{5}t}\right)-\int-\frac{5}{3}e^{-\frac{3}{5}t}\,dt\right]$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}+\frac{50k}{9}te^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $xe^{-\frac{3}{5}t}=\frac{5k}{3}t^2e^{-\frac{3}{5}t}+\frac{50k}{9}te^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+Ce^{\frac{3}{5}t}$ When $t=0$, $x=0.55$ $0.55=\frac{250k}{27}+C \qquad \Rightarrow C=\frac{11}{20}-\frac{250k}{27}$ Hence $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)e^{\frac{3}{5}t}$	

(iii) Company A intends to place an advertisement in the game only if there are more than 76 000 players playing the game. Given that $k=\frac{1}{10}$, find the length of time for which Company A will place an advertisement in "Mobile Saga", giving your answer correct to the nearest month. [2]

12 a iii	$x=\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}$  $\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}=0.76$ From the GC, longest duration = $4.787-0.552=4.235 \approx 4$ months	Note that t is not discrete, hence **cannot use a table** to solve the inequality. The question asked for the length of time, i.e. **duration**, when there are more than 76 000 players playing the game. It is not sufficient to solve for 1 value of t.

(b) The other game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$. Given further that there were 180 000 players playing "Mobile Saga" after 1 month, find x in terms of t. [4]

12 b	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$ Integrating w.r.t t: $\frac{\mathrm{d}x}{\mathrm{d}t}=-10\int\frac{1}{(1+t)^3}\,\mathrm{d}t=\frac{5}{(1+t)^2}+C$ Integrating w.r.t. t: $x=5\int\frac{1}{(1+t)^2}\,\mathrm{d}t=-\frac{5}{(1+t)}+Ct+D$ When $t=0,\ x=0.55$ $0.55=-5+D \quad \Rightarrow D=\frac{111}{20}$ When $t=1,\ x=1.8$ $1.8=-\frac{5}{(1+1)}+C+\frac{111}{20} \quad \Rightarrow C=-\frac{5}{4}$ $\therefore x=-\frac{5}{1+t}-\frac{5}{4}t+\frac{111}{20}$	After obtaining $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{5}{(1+t)^2}+C$, one has to integrate **wrt t** once again to obtain $x=-\frac{5}{(1+t)}+Ct+D$, and NOT $x=-\frac{5}{(1+t)}+Cx+D$.