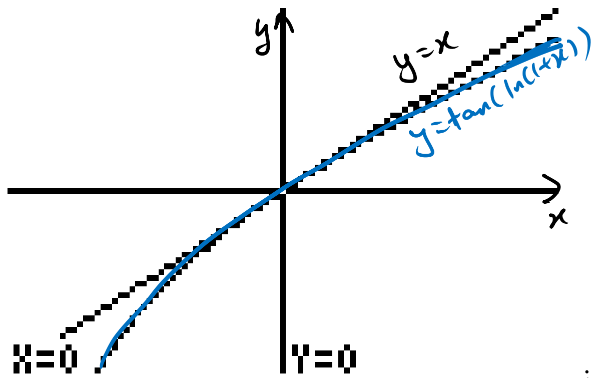
6. Maclaurin Series (solutions)

1	$(1-x)^{\frac{1}{3}} = 1+\frac{1}{3}(-x)+\frac{\frac{1}{3}(-\frac{2}{3})}{2!}(-x)^2+\cdots$ $= 1-\frac{1}{3}x-\frac{1}{9}x^2+\cdots$ ------(I) $(3+ax)(3+bx)^{-1}$ $= (3+ax)(3)^{-1}(1+\frac{1}{3}bx)^{-1}$ $= (1+\frac{1}{3}ax)\left(1+(-1)(\frac{1}{3}bx)+\frac{(-1)(-2)}{2!}(\frac{1}{3}bx)^2+\cdots\right)$ $= (1+\frac{1}{3}ax)\ (1-\frac{1}{3}bx+\frac{1}{9}b^2x^2+\cdots)$ $= 1+(\frac{1}{3}a-\frac{1}{3}b)x+(\frac{1}{9}b^2-\frac{1}{9}ab)x^2+\cdots$ $= 1+\frac{1}{3}(a-b)x+\frac{1}{9}(b^2-ab)x^2+\cdots$ ------(II) Comparing (I) & (II), $a-b=-1$ ----(1) $b^2-ab=-1$ -----(2) From (1), $a=b-1$ Subst a into (2), $b^2-(b-1)b=-1$ Thus $b=-1,\ a=-2$
2(i)	$y^2+e^x+3y=5$ ---------- (1) Differentiating (1) with respect to x, $2y\frac{dy}{dx}+e^x+3\frac{dy}{dx}=0$ $(2y+3)\frac{dy}{dx}+e^x=0$ -------- (2) Differentiating (2) with respect to x, $\left(2\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)+(2y+3)\frac{d^2y}{dx^2}+e^x=0$ $2\left(\frac{dy}{dx}\right)^2+(2y+3)\frac{d^2y}{dx^2}+e^x=0$ (Shown) ----- (3)
(ii)	When $x = 0$, $y^2+3y-4=0$ (from (1)) $(y-1)(y+4)=0$ $\Rightarrow y = 1$ or -4 (NA since $y > 0$)

	When $x=0$, $\dfrac{dy}{dx}=-\dfrac{1}{5}$ (from (2)) $\dfrac{d^2y}{dx^2}=-\dfrac{27}{125}$ (from (3)) $y=1-\dfrac{1}{5}x-\dfrac{27}{250}x^2+\ldots$
3(i)	$(1+x)y=\ln(1+2x)$ … (*) Differentiate (*) w.r.t. x : $(1+x)\dfrac{dy}{dx}+y=\dfrac{2}{1+2x}$ … (1) Differentiate (1) w.r.t. x : $(1+x)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\dfrac{dy}{dx}=2\left(\dfrac{-2}{(1+2x)^2}\right)$ $(1+x)\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+\dfrac{4}{(1+2x)^2}=0$ (shown)… (2)
(ii)	Differentiate (2) w.r.t. x : $(1+x)\dfrac{d^3y}{dx^3}+3\dfrac{d^2y}{dx^2}-\dfrac{16}{(1+2x)^3}=0$ … (3) When $x=0$ From (*) : $y=\ln 1=0$ From (1) : $\dfrac{dy}{dx}=2$ From (2) : $\dfrac{d^2y}{dx^2}=-8$ From (3) : $\dfrac{d^3y}{dx^3}=40$ Hence, $y=0+2x+\dfrac{-8}{2!}x^2+\dfrac{40}{3!}x^3+\ldots$ $=2x-4x^2+\dfrac{20}{3}x^3+\ldots$
(iii)	$y=(1+x)^{-1}\ln(1+2x)$ $=(1-x+x^2+\ldots)\left(2x-\dfrac{(2x)^2}{2}+\dfrac{(2x)^3}{3}+\ldots\right)$ $=2x+(-2-2)x^2+\left(\dfrac{8}{3}+2+2\right)x^3+\ldots$

	$= 2x-4x^2+\frac{20}{3}x^3+\ldots$ (verified to be same as part(ii))
4(i)	$\mathrm{f}(x)=\frac{1}{\sqrt{9-x^2}}$ $=\left(9-x^2\right)^{-\frac{1}{2}}$ $=\frac{1}{3}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$ $=\frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x^2}{9}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{x^2}{9}\right)^2+\ldots\right)$ $=\frac{1}{3}\left(1+\frac{x^2}{18}+\frac{x^4}{216}+\ldots\right)$ Range of validity: $\left\|-\frac{x^2}{9}\right\|<1\Rightarrow\left\|x^2\right\|<9$ $\therefore -3<x<3$
(ii)	When $x=\frac{1}{2}$, $\frac{1}{\sqrt{9-\left(\frac{1}{2}\right)^2}}=\frac{1}{3}\left(1+\frac{\left(\frac{1}{2}\right)^2}{18}+\frac{\left(\frac{1}{2}\right)^4}{216}+\ldots\right)$ $\frac{1}{\sqrt{\frac{35}{4}}}\approx\frac{1}{3}\left(\frac{3505}{3456}\right)$ $\sqrt{35}=\frac{3(3456)}{3505}\times 2=\frac{20736}{3505}$
5(i)	$\mathrm{f}(x)=\frac{5x}{(1+2x)\left(1+x^2\right)}=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}$ Solving $A=-2,\ \ B=1,\ \ C=2$, ie, $\mathrm{f}(x)=-\frac{2}{1+2x}+\frac{x+2}{1+x^2}$
(ii)	$-\frac{2}{1+2x}=-2(1+2x)^{-1}$ $=-2\left(1-2x+4x^2-8x^3+\ldots\right)$ $=-2+4x-8x^2+16x^3+\ldots$

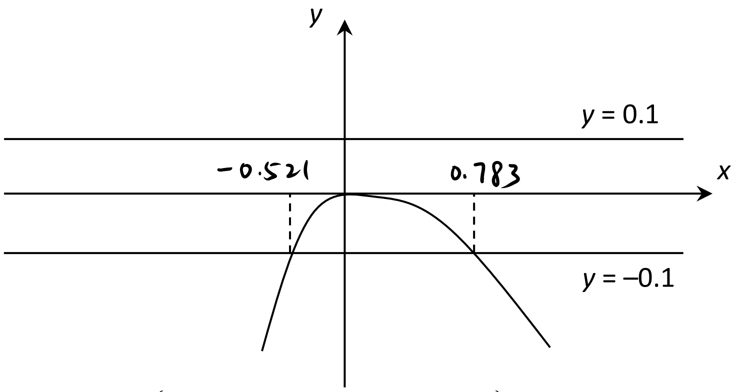
	$\frac{x+2}{1+x^2}=(x+2)(1+x^2)^{-1}$ $=(x+2)(1-x^2+...)$ $=2+x-2x^2-x^3+...$ $\therefore \mathrm{f}(x)=5x-10x^2+15x^3+...$
(iii)	$\int_0^1 \frac{x}{(1+2x)(1+x^2)}\,\mathrm{d}x \approx \int_0^1 (x-2x^2+3x^3)\,\mathrm{d}x$ $=\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{3}{4}x^4\right]_0^1$ $=\frac{1}{2}-\frac{2}{3}+\frac{3}{4}=\frac{7}{12}$
(iv)	$\int_0^1 \frac{x}{(1+2x)(1+x^2)}\,\mathrm{d}x = 0.164$ (3 s.f.) Range of validity given by $\lvert 2x\rvert<1$ and $\lvert x^2\rvert<1$ i.e, $\lvert x\rvert<\frac{1}{2} \Rightarrow -\frac{1}{2}<x<\frac{1}{2}$ The answer in part (iii) is not an appropriate approximation as the series expansion for $\frac{x}{(1+2x)(1+x^2)}$ is valid only for $\lvert x\rvert<\frac{1}{2}$. Hence it cannot be used for approximation for the interval up to $x=1$.
6(a)(i)	$\mathrm{f}(x)=\frac{2}{2-x}-\frac{1}{(1+x)^2}=\left(1-\frac{x}{2}\right)^{-1}-(1+x)^{-2}$ $=\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+...\right)-(1-2x+3x^2+...)$ $=(1-1)+\left(\frac{1}{2}+2\right)x+\left(\frac{1}{4}-3\right)x^2+...$ $=\frac{5x}{2}-\frac{11}{4}x^2+...$
(ii)	Equation of the tangent to curve at origin is $y=\frac{5x}{2}$

(b)	$\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}+\dots$ $=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots$ $e^{\sqrt{1+x}}=e^{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots}$ $=e^1e^{\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots}$ $=e\left(1+\left(\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\dots\right)+\frac{1}{2!}\left(\frac{1}{2}x-\frac{1}{8}x^2+\dots\right)^2+\frac{1}{3!}\left(\frac{1}{2}x+\dots\right)^3+\dots\right)$ $=e\left(1+\frac{1}{2}x+\left(-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{2}\right)^2\right)x^2+\left(\frac{1}{16}+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{1}{48}\right)x^3+\dots\right)$ $=e\left(1+\frac{1}{2}x+\frac{1}{48}x^3+\dots\right)$
7(i)	$\tan^{-1}y=\ln(1+x)$ $\frac{1}{1+y^2}\frac{dy}{dx}=\frac{1}{1+x}\Rightarrow(1+x)\frac{dy}{dx}=1+y^2$ (shown)
(ii)	$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=2y\frac{dy}{dx}$ $(1+x)\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}=2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2$ $(1+x)\frac{d^4y}{dx^4}+3\frac{d^3y}{dx^3}=2y\frac{d^3y}{dx^3}+6\frac{dy}{dx}\frac{d^2y}{dx^2}$ When $x=0$, $y=0$ $\frac{dy}{dx}=1$, $\frac{d^2y}{dx^2}=-1$, $\frac{d^3y}{dx^3}=4$, $\frac{d^4y}{dx^4}=-18$ $\therefore\tan(\ln(1+x))=x-\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{3}{4}x^4+\dots$
(iii)	 intersection at (0,0)

8(a)	$\frac{8}{(2-x)^2}$ $=8(2-x)^{-2}$ $=8\left[2^{-2}\left(1-\frac{x}{2}\right)^{-2}\right]$ $=2\left(1-\frac{x}{2}\right)^{-2}$ $=2\left(1+(-2)\left(-\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{2}\right)^2+...\right)$ $=2+2x+\frac{3}{2}x^2+...$ Expansion is valid for $\left\lvert\frac{x}{2}\right\rvert<1 \Rightarrow -2<x<2$
(b)	$y=\cot\left(2x+\frac{\pi}{4}\right)$ Differentiating w.r.t. x: $\frac{dy}{dx}=-2\text{cosec}^2\left(2x+\frac{\pi}{4}\right)$ $\frac{dy}{dx}=-2\left[1+\cot^2\left(2x+\frac{\pi}{4}\right)\right]$ $\frac{dy}{dx}=-2(1+y^2)$ (shown) Differentiating w.r.t. x: $\frac{d^2y}{dx^2}=-4y\frac{dy}{dx}$ Differentiating w.r.t. x: $\frac{d^3y}{dx^3}=-4y\frac{d^2y}{dx^2}-4\left(\frac{dy}{dx}\right)^2$ For $x=0$, $y=1$, $\frac{dy}{dx}=-4$, $\frac{d^2y}{dx^2}=16$, $\frac{d^3y}{dx^3}=-128$ $y=1-4x+8x^2-\frac{64}{3}x^3+...$

	Since $\cot\left(2x+\frac{\pi}{4}\right)=1-4x+8x^2-\frac{64}{3}x^3+...$ $\tan\left(2x+\frac{\pi}{4}\right)$ $\approx\left(1-4x+8x^2\right)^{-1}$ $=1-\left(-4x+8x^2\right)+\frac{(-1)(-2)}{2!}\left(-4x+8x^2\right)^2+...$ $=1+4x-8x^2+16x^2+...$ $=1+4x+8x^2+...$ Thus, $a=1,\ b=4,\ c=8$
9(i)	$\frac{dy}{dx}=\frac{x^2-3}{4-x^2}$ $y=\int\frac{x^2-3}{4-x^2}\,dx$ $=\int -1+\frac{1}{4-x^2}\,dx$ $=-x+\frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)+c$ (no need modulus $\because -2<x<2$) When $x=0,\ y=2:\ 2=\frac{1}{4}\ln(1)+c\Rightarrow c=2$ $\therefore y=-x+\frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)+2$
(ii)	When $x=0$, $y=2$ and $\frac{dy}{dx}=-\frac{3}{4}$. $\left(4-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=2x$ When $x=0$, $\frac{d^2y}{dx^2}=0$. $\left(4-x^2\right)\frac{d^3y}{dx^3}-2x\frac{d^2y}{dx^2}-2x\frac{d^2y}{dx^2}-2\frac{dy}{dx}=2$ When $x=0$, $\frac{d^3y}{dx^3}=\frac{2+2\left(-\frac{3}{4}\right)}{4}=\frac{1}{8}$. $\therefore y=2-\frac{3}{4}x+\frac{1}{8}\cdot\frac{x^3}{3!}+...$ $=2-\frac{3}{4}x+\frac{1}{48}x^3+...$

10	$y = e^{3\tan^{-1}x} \Rightarrow \frac{dy}{dx} = \frac{3}{1+x^2}e^{3\tan^{-1}x} = \frac{3y}{1+x^2}$ $\therefore \left(1+x^2\right)\frac{dy}{dx} = 3y$ (Shown) $\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 3\frac{dy}{dx} \Rightarrow \left(1+x^2\right)\frac{d^2y}{dx^2} + \left(2x-3\right)\frac{dy}{dx} = 0$ $\left(1+x^2\right)\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + \left(2x-3\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ $\Rightarrow \left(1+x^2\right)\frac{d^3y}{dx^3} + \left(4x-3\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ When $x=0$, $y=1$, $\frac{dy}{dx}=3$, $\frac{d^2y}{dx^2}=9$, $\frac{d^3y}{dx^3}=21$ Hence, $y = f\left(0\right) + xf'\left(0\right) + x^2\frac{f''\left(0\right)}{2!} + x^3\frac{f'''\left(0\right)}{3!} + \ldots$ $= 1 + 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \ldots$ $e^{2x+3\tan^{-1}x}$ $= e^{2x}e^{3\tan^{-1}x}$ $= \left(1 + 2x + \frac{(2x)^2}{2!} + \ldots\right)\left(1 + 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \ldots\right)$ $= 1 + 5x + \frac{25}{2}x^2 + \ldots$
11	$y = \sin^{-1}\left[\ln(x+1)\right]$ --- (1) $\sin y = \ln(x+1)$ Differentiate with respect to *x*: $\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x+1}$ (shown) --- (2) Differentiate with respect to *x*: $\cos y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+1)^2}$ $\cos y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \sin y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -\frac{1}{(x+1)^2}$ (shown) --- (3)
(i)	Differentiate with respect to *x*: $\cos y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}$

	$-\left[(\sin y)(2)\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right]=\frac{2}{(x+1)^3}$ $\cos y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3\sin y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3=\frac{2}{(x+1)^3}$ --- (4) When $x = 0$, (1): $y=\sin^{-1}0=0$ (2): $\cos(0)\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1$ (3): $\cos(0)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\sin(0)(1)^2=-1\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ (4): $\cos(0)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3\sin(0)(-1)(1)-\cos(0)(1)^3=2\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=3$ $y=0+1x+\frac{(-1)}{2!}x^2+\frac{3}{3!}x^3+\ldots$ $\therefore y=x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$
(ii)	$\left\|y-\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)\right\|<0.1$ $-0.1<\sin^{-1}[\ln(x+1)]-\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)<0.1$ Sketch $y=\sin^{-1}[\ln(x+1)]-\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)$, $y=0.1$ and $y=-0.1$.  Solution set: $\{x\in\mathbb{R}:-0.521<x<0.783\}$
(iii)	Using the Maclaurin series for $\sin^{-1}[\ln(x+1)]=x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$

	Differentiate both sides wrt x : $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\left[\ln(x+1)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3+...\right)$ $\frac{1}{(x+1)\sqrt{1-(\ln(x+1))^2}}=1-x+\frac{3}{2}x^2+...$
12(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{\tan^{-1}x}}{1+x^2}\Rightarrow y=\mathrm{e}^{\tan^{-1}x}+c$ When $x=0$, $y=1$ $\Rightarrow 1=\mathrm{e}^0+c\Rightarrow c=0$ Thus $y=\mathrm{e}^{\tan^{-1}x}$
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{\tan^{-1}x}}{1+x^2}=\frac{y}{1+x^2}$ $\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y$ $\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x-1)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ (shown)
(iii)	$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x-1)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(4x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ When $x=0$, $y=1$ (given) $\frac{\mathrm{d}y}{\mathrm{d}x}=1,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-1$ $\therefore y=1+x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots$
(iv)	$\frac{\mathrm{e}^{\tan^{-1}x}}{1+x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}=1+x-\frac{1}{2}x^2+\cdots$ **Alternative method:** $\frac{\mathrm{e}^{\tan^{-1}x}}{1+x^2}=\mathrm{e}^{\tan^{-1}x}(1+x^2)^{-1}$ $=\left(1+x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots\right)\left(1-x^2+\cdots\right)$ $=1+x-\frac{1}{2}x^2+\cdots$

13	$y = e^x \sin^2 x + 1$ Diff wrt x: $\frac{dy}{dx} = e^x(2\sin x\cos x) + e^x \sin^2 x$ $= e^x \sin 2x + e^x \sin^2 x$ $= e^x \sin 2x + y - 1$ $\therefore a = -1$ $\frac{d^2y}{dx^2} = e^x(2\cos 2x) + e^x \sin 2x + \frac{dy}{dx}$ $= 2e^x \cos 2x + \left(\frac{dy}{dx} - y + 1\right) + \frac{dy}{dx}$ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^x \cos 2x + 1$ $\therefore b = 1$
13(i)	When $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 2$ $y = 1 + \frac{2}{2!}x^2 + \ldots = 1 + x^2 + \ldots$ ---(*)
13(ii)	$(1-x)^{-1} = 1 + x + x^2 + \ldots$ To obtain the series expansion of $e^{-2x}\sin^2 2x + 1$ from the series expansion of $e^x \sin^2 x + 1$, replace x with $-2x$ in the series of $e^x \sin^2 x + 1$ in (*). Note that $e^{-2x}\sin^2(-2x) + 1 = e^{-2x}(-\sin 2x)^2 + 1$ $= e^{-2x}\sin^2 2x + 1$ $\frac{e^{-2x}\sin^2 2x + 1}{1-x} = \left(e^{-2x}\sin^2 2x + 1\right)(1-x)^{-1}$ $= \left(1 + (-2x)^2 + \ldots\right)\left(1 + x + x^2 + \ldots\right)$ $= \left(1 + 4x^2 + \ldots\right)\left(1 + x + x^2 + \ldots\right)$ $= 1 + x + x^2 + 4x^2 + \ldots$ $= 1 + x + 5x^2 + \ldots$

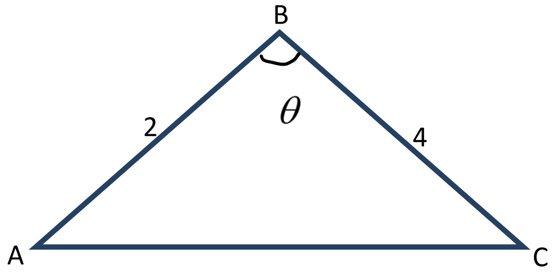
14	Let T be the point of the base of the tower. Then, $QT=\dfrac{h}{\tan\left(\frac{\pi}{4}+x\right)}=\dfrac{h}{\left(\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right)}\approx\dfrac{h}{\left(\frac{1+x}{1-x}\right)}$ $(\because x \text{ small } \Rightarrow \tan x\approx x)$ $\approx h(1-x)(1+x)^{-1}$ (shown) Also $PT=\dfrac{h}{\tan\frac{\pi}{6}}=\sqrt{3}h$ By Pythagoras' Theorem, $PQ^2=PT^2+QT^2$ $=\left(\sqrt{3}h\right)^2+\left(h(1-x)(1+x)^{-1}\right)^2$ $=3h^2+h^2(1-x)^2(1+x)^{-2}$ $=3h^2+h^2(1-2x+x^2)(1-2x+3x^2+...)$ $=3h^2+h^2(1-4x+8x^2+...)$ $\approx 4h^2(1-x+2x^2)$ (shown)
15(i)	$\dfrac{BC}{\sin x}=\dfrac{AB}{\sin y}$ $\sin y=\dfrac{AB}{BC}\sin x$ $\sin y=k\sin x$ (shown)
(ii)	$\cos y\dfrac{d^3y}{dx^3}-(\sin y)\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)-\left[(\sin y)(2)\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)+\left(\dfrac{dy}{dx}\right)^2\left(\cos y\dfrac{dy}{dx}\right)\right]=-k\cos x$ $\cos y\dfrac{d^3y}{dx^3}-3\sin y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)-\cos y\left(\dfrac{dy}{dx}\right)\left(\dfrac{dy}{dx}\right)^2=-k\cos x$ when $x=0,\ y=0,\ \dfrac{dy}{dx}=k,\ \dfrac{d^2y}{dx^2}=0,\ \dfrac{d^3y}{dx^3}=k\left(k^2-1\right)$ $\therefore y=0+kx+0+\dfrac{k\left(k^2-1\right)}{3!}x^3+...$ $=kx+\dfrac{k\left(k^2-1\right)}{6}x^3+...$

16(i)	$\measuredangle ABC = \pi - \frac{2}{3}\pi - \theta = \frac{\pi}{3} - \theta$ Using Sine rule, $\frac{1}{\sin\left(\frac{\pi}{3}-\theta\right)} = \frac{BC}{\sin\frac{2}{3}\pi}$ $BC = \frac{\sin\frac{2}{3}\pi}{\sin\left(\frac{\pi}{3}-\theta\right)}$ $= \frac{\sin\frac{2}{3}\pi}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$ $= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta}$ $= \frac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$ [Shown]
(ii)	Since θ is a sufficiently small angle, $BC \approx \frac{\sqrt{3}}{\sqrt{3}\left(1-\frac{\theta^2}{2}\right)-\theta}$ $= \frac{\sqrt{3}}{\sqrt{3}-\frac{\sqrt{3}}{2}\theta^2-\theta}$ $= \sqrt{3}\left[\sqrt{3}-\left(\theta+\frac{\sqrt{3}}{2}\theta^2\right)\right]^{-1}$ $= \sqrt{3}\left(\sqrt{3}\right)^{-1}\left[1-\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)\right]^{-1}$ $= \left[1-\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)\right]^{-1}$ $= 1+(-1)\left[-\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)\right]+\frac{(-1)(-2)}{2!}\left[-\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)\right]^2+\ldots\ldots$

	$=1+\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)+\left(\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}\right)^2+......$ $=1+\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}+\frac{\theta^2}{3}+......$ $\approx 1+\frac{\theta}{\sqrt{3}}+\frac{5}{6}\theta^2,\qquad \text{where } p=\frac{5}{6}$
17(i)	<u>**Method 1**</u> $y=e^{\sin^{-1}3x}$ $\ln y=\sin^{-1}3x$ Diff. w.r.t x, $\frac{1}{y}\frac{dy}{dx}=\frac{3}{\sqrt{1-9x^2}}$ $\sqrt{1-9x^2}\frac{dy}{dx}=3y \text{ ....}(1)$ Diff. (1) w.r.t x, $\sqrt{1-9x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{1}{2}\left(1-9x^2\right)^{-\frac{1}{2}}\left(-18x\right)\right)=3\frac{dy}{dx}$ $\sqrt{1-9x^2}\frac{d^2y}{dx^2}-\frac{9x}{\sqrt{1-9x^2}}\frac{dy}{dx}=3\frac{dy}{dx}$ $\left(1-9x^2\right)\frac{d^2y}{dx^2}-9x\frac{dy}{dx}=3\sqrt{1-9x^2}\frac{dy}{dx}$ $\left(1-9x^2\right)\frac{d^2y}{dx^2}-9x\frac{dy}{dx}=9y \text{ ....}(2)$ [Shown]

	Method 2 $y = e^{\sin^{-1}3x}$ Differentiating w.r.t. x, $\frac{dy}{dx} = e^{\sin^{-1}3x}\left(\frac{3}{\sqrt{1-9x^2}}\right)$ $\frac{d^2y}{dx^2} = e^{\sin^{-1}3x}\left(\frac{3}{\sqrt{1-9x^2}}\right)\left(\frac{3}{\sqrt{1-9x^2}}\right) + e^{\sin^{-1}3x}\left(-\frac{3}{2}(1-9x^2)^{-\frac{3}{2}}(-18x)\right)$ $= \frac{9e^{\sin^{-1}3x}}{1-9x^2} + \frac{9e^{\sin^{-1}3x}(3x)}{(1-9x^2)\sqrt{1-9x^2}}$ $(1-9x^2)\frac{d^2y}{dx^2} = 9y + 9x\frac{dy}{dx}$ $(1-9x^2)\frac{d^2y}{dx^2} - 9x\frac{dy}{dx} = 9y$ $\qquad$ $(shown)$
(ii)	Diff. (2) w.r.t x, $\left(1-9x^2\right)\frac{d^3y}{dx^3} + \left(-18x\right)\frac{d^2y}{dx^2} - 9x\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 9\frac{dy}{dx}$ $\left(1-9x^2\right)\frac{d^3y}{dx^3} - 27x\frac{d^2y}{dx^2} = 18\frac{dy}{dx}$ ....(3) When $x = 0$, $y = e^{\sin^{-1}3(0)} = 1$ From (1), $\sqrt{1-0}\frac{dy}{dx} = 3(1) \Rightarrow \frac{dy}{dx} = 3$ From (2), $\sqrt{1-0}\frac{d^2y}{dx^2} - 9(0)(3) = 9(1) \Rightarrow \frac{d^2y}{dx^2} = 9$ From (3), $\sqrt{1-0}\frac{d^3y}{dx^3} - 27(0)(9) = 18(3) \Rightarrow \frac{d^3y}{dx^3} = 54$ $\therefore y = 1 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(54) + ...$ $= 1 + 3x + \frac{9}{2}x^2 + 9x^3 + ...$

(iii)	$e^{-\frac{\pi}{2}} = e^{\sin^{-1} 3x}$ $-\frac{\pi}{2} = \sin^{-1} 3x$ $\sin\left(-\frac{\pi}{2}\right) = 3x$ $x = \frac{1}{3}\sin\left(-\frac{\pi}{2}\right)$ $= -\frac{1}{3}$ When $x = -\frac{1}{3}$, $e^{-\frac{\pi}{2}} \approx 1 + 3\left(-\frac{1}{3}\right) + \frac{9}{2}\left(-\frac{1}{3}\right)^2 + 9\left(-\frac{1}{3}\right)^3$ $= \frac{1}{6}$
18 (a)	$\mathrm{f}(x) = \sin\left(2x + \frac{\pi}{4}\right)$ $\mathrm{f}'(x) = 2\cos\left(2x + \frac{\pi}{4}\right)$ $\mathrm{f}''(x) = -2^2 \sin\left(2x + \frac{\pi}{4}\right)$ $\mathrm{f}(0) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\mathrm{f}'(0) = 2\cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ $\mathrm{f}''(0) = -2^2 \sin\left(\frac{\pi}{4}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$ $\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \dots$ $\therefore \mathrm{f}(x) = \frac{\sqrt{2}}{2} + \sqrt{2}x - \sqrt{2}x^2 + \dots$

(b)	 By cosine rule, $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta$ $\approx 2^2 + 4^2 - 2(2)(4)\left(1 - \frac{\theta^2}{2}\right)$, since θ is sufficiently small $= 4 + 8\theta^2$ $AC \approx (4 + 8\theta^2)^{\frac{1}{2}}$ (shown) $= 4^{\frac{1}{2}}(1 + 2\theta^2)^{\frac{1}{2}}$ $= 2\left(1 + \frac{1}{2}(2\theta^2) + \ldots\right)$ $\approx 2 + 2\theta^2$ where $a = 2,\ b = 2$
19(i)	$y = \ln\left(\frac{\cos x}{\mathrm{e} - x}\right) = \ln(\cos x) - \ln(\mathrm{e} - x)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x}{\cos x} + \frac{1}{\mathrm{e} - x} = -\tan x + \frac{1}{\mathrm{e} - x}$ (shown) Method 1 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sec^2 x + \frac{1}{(\mathrm{e} - x)^2}$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -2\sec^2 x \tan x + \frac{2}{(\mathrm{e} - x)^3}$ When $x = 0$, $y = \ln\left(\frac{1}{\mathrm{e}}\right) = -\ln \mathrm{e} = -1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{\mathrm{e}^2} - 1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{2}{\mathrm{e}^3}$ $\therefore y = -1 + \frac{1}{\mathrm{e}}x + \frac{\left(\frac{1}{\mathrm{e}^2} - 1\right)}{2!}x^2 + \frac{\left(\frac{2}{\mathrm{e}^3}\right)}{3!}x^3 + \ldots$ $= -1 + \frac{1}{\mathrm{e}}x + \frac{1}{2}\left(\frac{1}{\mathrm{e}^2} - 1\right)x^2 + \frac{1}{3\mathrm{e}^3}x^3 + \ldots$

(ii)	Replace x by $-ex$ on both LHS and RHS Then $y=\ln\left(\frac{\cos(-ex)}{e-(-ex)}\right)=\ln\left(\frac{\cos(ex)}{e(1+x)}\right)=\ln\left(\frac{\cos(ex)}{(1+x)}\right)+\ln\frac{1}{e}$ $\ln\left(\frac{\cos(ex)}{(1+x)}\right)+\ln\frac{1}{e}=-1+\frac{1}{e}(-ex)+\frac{1}{2}\left(\frac{1}{e^2}-1\right)(-ex)^2+\frac{1}{3e^3}(-ex)^3+...$ $\Rightarrow \ln\left(\frac{\cos(ex)}{(1+x)}\right)=-x+\frac{1}{2}\left(\frac{1}{e^2}-1\right)(ex)^2-\frac{1}{3e^3}(ex)^3+...$ $\ln\left(\frac{\cos(ex)}{(1+x)}\right)=-x+\frac{1}{2}\left(1-e^2\right)x^2-\frac{1}{3}x^3+...$
20	$y=(1-\sin x)^{\frac{1}{2}}$ $y^2=1-\sin x$ $2y\frac{dy}{dx}=-\cos x$ ---- (1) $2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=\sin x=1-y^2$ $\therefore 2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+y^2-1=0$ ---- (2) $\therefore 2y\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)\frac{dy}{dx}+4\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2y\frac{dy}{dx}=0$ ---- (3) Let $x=0$, $y=1$, $\frac{dy}{dx}=-\frac{1}{2}$, $\frac{d^2y}{dx^2}=-\frac{1}{4}$, $\frac{d^3y}{dx^3}=\frac{1}{8}$ Hence $y=1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\frac{\left(\frac{1}{8}\right)}{3!}x^3+...$ $=1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+...$ $(1-\sin x)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+...$ Differentiating with respect to x $\frac{-\cos x}{2(1-\sin x)^{\frac{1}{2}}}=-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+..$ or from (1) $\cos x=-2y\frac{dy}{dx}$ $\cos x=-2(1-\sin x)^{\frac{1}{2}}\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+...\right)$ $=-2\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+...\right)\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+...\right)$

	$= -2\left(-\frac{1}{2}-\frac{1}{4}x+\frac{1}{16}x^2+\frac{1}{4}x+\frac{1}{8}x^2+\frac{1}{16}x^2+...\right)$ $= -2\left(-\frac{1}{2}+\frac{1}{4}x^2+...\right)=1-\frac{1}{2}x^2+...$ **<u>Alternative solution</u>** $\cos x=\sqrt{1-\sin^2 x}$ $\cos x=\sqrt{(1-\sin x)(1+\sin x)}$ $\cos x=(1-\sin x)^{\frac{1}{2}}(1+\sin x)^{\frac{1}{2}}=(1-\sin x)^{\frac{1}{2}}(1-\sin(-x))^{\frac{1}{2}}$ $\cos x=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+...\right)\left(1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{48}x^3+...\right)$ $=1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{2}x-\frac{1}{4}x^2-\frac{1}{8}x^2+\ldots$ $=1-\frac{1}{2}x^2+...$
21(i)	$\tan^{-1}y = 3e^x-3$ $\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=3e^x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=3e^x(1+y^2)$ Alternatively: $y=\tan(3e^x-3)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2(3e^x-3)\times 3e^x$ $=3e^x(1+\tan^2(3e^x-3))$ $=3e^x(1+y^2)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3e^x(1+y^2)+2y\frac{\mathrm{d}y}{\mathrm{d}x}(3e^x)$ $=\frac{\mathrm{d}y}{\mathrm{d}x}+6ye^x\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\frac{\mathrm{d}y}{\mathrm{d}x}(1+6ye^x)$

	when $x=0$, $y=0$ $\frac{dy}{dx}=3$ $\frac{d^2y}{dx^2}=3$ $y=0+3x+\frac{3}{2!}x^2+...$ $=3x+\frac{3}{2}x^2+...$
21(ii)	$(1+ax)^{10}\sin bx$ $=(1+10ax+\frac{10(9)}{2!}(ax)^2+\frac{(10)(9)(8)}{3!}(ax)^3+...)(bx-\frac{(bx)^3}{3!})$ $=(1+10ax+45a^2x^2+120a^3x^3+...)(bx-\frac{1}{6}b^3x^3+...)$ $=bx+10abx^2+...$ Comparing coefficients of x: $b=3$ Comparing coefficients of x^2: $10ab=\frac{3}{2}$ $a=\frac{1}{20}$
22a(i)	$\frac{d}{dx}\sqrt{1-x^2}=-\frac{2x}{2\sqrt{1-x^2}}$ $=-\frac{x}{\sqrt{1-x^2}}$
a(ii)	$\int\sin^{-1}x\,dx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx$ $=x\sin^{-1}x+\sqrt{1-x^2}+C$
b(i)	$y^3+y=2x^2-x$ $3y^2\frac{dy}{dx}+\frac{dy}{dx}=4x-1$ $6y\left(\frac{dy}{dx}\right)^2+3y^2\frac{d^2y}{dx^2}+\frac{d^2y}{dx^2}=4$ $6y\left(\frac{dy}{dx}\right)^2+\frac{d^2y}{dx^2}\left(3y^2+1\right)=4$ (shown)

b(ii)	When $x=0$, $y^3+y=0$ $y\left(y^2+1\right)=0$ $y=0$ or $y^2+1=0\left(NA \because y^2+1>0 \text{ for all real } y\right)$ $3(0)^2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}=4(0)-1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-1$ $6(0)(-1)^2+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(3(0)^2+1\right)=4 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4$ $y=0+\frac{(-1)}{1!}x+\frac{4}{2!}x^2+...$ $=-x+2x^2+\ldots$
b(iii)	Area $=\int_0^{0.5}-y\,\mathrm{d}x+\int_{0.5}^{0.6}y\,\mathrm{d}x$ $=\int_0^{0.5}\left(x-2x^2\right)\mathrm{d}x+\int_{0.5}^{0.6}\left(-x+2x^2\right)\mathrm{d}x$ By GC, area of shaded region $=0.0473$ unit^2. The approximation will be better if more terms in the Maclaurin's series are included in the integral.
23a	$\mathrm{e}^{3x}\ln(1+ax)$ $=\left(1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}+...\right)\left(ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}-...\right)$ $=ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}+3ax^2-\frac{3a^2x^3}{2}+\frac{9ax^3}{2}+...$ $=ax+\left(3a-\frac{a^2}{2}\right)x^2+\left(\frac{a^3}{3}-\frac{3a^2}{2}+\frac{9a}{2}\right)x^3+...$ Since there is no term in x^2, $3a-\frac{a^2}{2}=0$ $a^2-6a=0$ $a=0$ (rejected $\because$ a is non-zero) or 6. Therefore, the coefficient of $x^3=\left(\frac{6^3}{3}-\frac{3\times6^2}{2}+\frac{9\times6}{2}\right)=45$

23b	$\frac{1+3x}{\sqrt{9-x^2}}=(1+3x)(9-x^2)^{-\frac{1}{2}}$ $=(1+3x)\left[9\left(1-\frac{x^2}{9}\right)\right]^{-\frac{1}{2}}$ $=(1+3x)\left[9^{-\frac{1}{2}}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}\right]$ $=\frac{1}{3}(1+3x)\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$ $=\frac{1}{3}(1+3x)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{1}{9}x^2\right)+...\right]$ $=\frac{1}{3}(1+3x)\left(1+\frac{1}{18}x^2+...\right)$ $=\frac{1}{3}\left(1+3x+\frac{1}{18}x^2+\frac{1}{6}x^3+...\right)$ $=\frac{1}{3}+x+\frac{1}{54}x^2+\frac{1}{18}x^3+...$ For expansion to be valid, $\left\|-\frac{x^2}{9}\right\|<1$ $\left\|x^2\right\|<9$ $\therefore -3<x<3$