H2 2017 Preliminary Exam Paper 1 Question
Answer all questions [100 marks].

1 Without using a calculator, solve the inequality

$$\frac{3x^2+7x+1}{x+3} < 2x-1. \qquad [4]$$

2 The function p is defined by $\mathrm{p}: x \mapsto \frac{1-x^2}{1+x^2}, \quad x \in \mapsto.$

(i) Find algebraically the range of p, showing your working clearly. [3]

(ii) Show that $\mathrm{p}(x) = \mathrm{p}(-x)$ for all $x \in \mapsto$ [1]

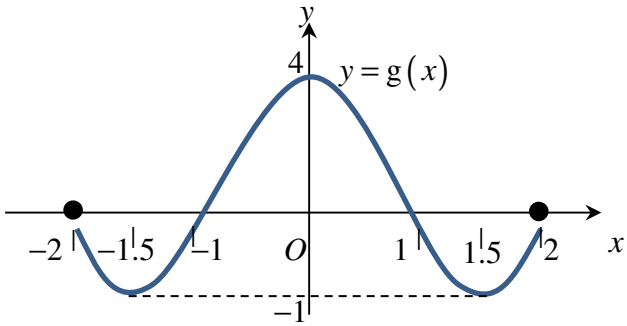
It is given that $\mathrm{q}(x) = \mathrm{p}\left(\frac{1}{2}x - 4\right), \ x \in \mapsto.$

(iii) State a sequence of transformations that will transform the graph of p on to the graph of q. Hence state the line of symmetry for the graph of q. [3]

3 The function f is defined by

$$\mathrm{f}: x \mapsto (x-k)^2, \quad x < k \text{ where } k > 5.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]



The diagram above shows the curve with equation $y = \mathrm{g}(x)$, where $-2 \le x \le 2$. The curve crosses the x–axis at $x = -2$, $x = -1$, $x = 1$ and $x = 2$, and has turning points at $(-1.5, -1)$, $(0, 4)$ and $(1.5, -1)$.

(ii) Explain why the composite function fg exists. [2]

(iii) Find in terms of k,

(a)the value of $\mathrm{fg}(-1)$ [1]

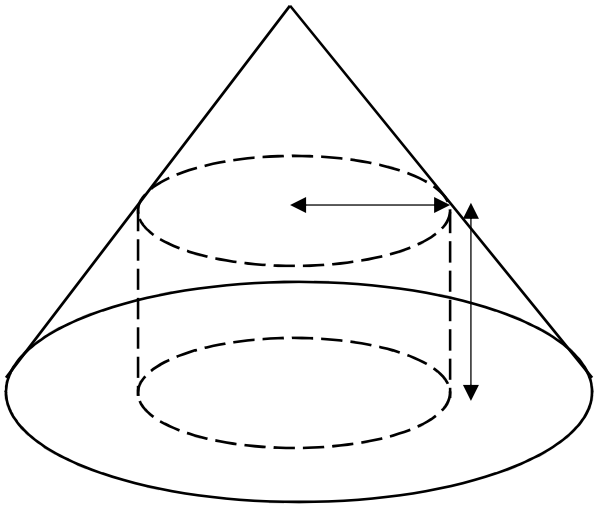
(b)the range of fg. [2]

4 It is given that $z = -1 - \mathrm{i}\sqrt{3}$.

(i) Given that $\frac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary, find the smallest positive integer n. [4]

The complex number w is such that $|wz| = 4$ and $\arg\left(\frac{w^*}{z^2}\right) = -\frac{5\pi}{6}$.

(ii) Find the value of $|w|$ and the exact value of $\arg(w)$ in terms of π. [3]

	On an Argand diagram, points A and B represent the complex numbers w and z respectively. **(iii)** Referred to the origin O, find the exact value of the angle OAB in terms of π. Hence, or otherwise, find the exact value of $\arg(z-w)$ in terms of π. [2]
5	
	A metal cylinder of radius r cm and height h cm is inscribed in a circular cone paperweight of base radius 4 cm and height 6 cm (see diagram). It is determined that the volume of the cylinder, V cm^3, should be as large as possible to provide weight to the paperweight. Show that $V=\frac{4\pi}{9}\left(36h-12h^2+h^3\right)$. [2] Hence find the exact maximum value of V. [5] The metal cylinder is known to expand under heat. An experiment shows that the height of the cylinder is increasing at a rate of 0.04 cm s^{-1} at an instant when $h=1.5$. Find the rate of change of V at this instant. [2]
6	Timber cladding is the application of timber planks over timber planks to provide the layer intended to control the infiltration of weather elements. **(a)** Using method A, 20 rectangular planks are used and the lengths of the planks form an arithmetic progression with common difference d cm. The shortest plank has length 65 cm and the longest plank has length 350 cm. **(i)**Find the value of d. [2] **(ii)** Find the total length of all the planks. [2] **(b)** Using method B, a long plank of 2000 cm is sawn off by a machine into n smaller rectangular planks. The length of the first plank is a cm and each successive plank is $\frac{8}{9}$ as long as the preceding plank. **(i)**Show that the total length of the planks sawn off can never be greater than k times the length of the first plank, where k is an integer to be determined. [2]

(ii) Given that $a = 423$, find the greatest possible integral value of n and the corresponding length of the shortest plank. [4]

7 **(i)** Express $\frac{1}{r^2-1}$ in partial fractions, and deduce that

$$\frac{1}{r(r^2-1)} = \frac{1}{2}\left[\frac{1}{r(r-1)} - \frac{1}{r(r+1)}\right].$$ [2]

(ii) Hence, find the sum, S_n, of the first n terms of the series

$$\frac{1}{2\times 3} + \frac{1}{3\times 8} + \frac{1}{4\times 15} + \ldots .$$ [4]

(iii) Explain why the series converges, and write down the value of the sum to infinity. [2]

(iv) Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]

8 A drug is administered by an intravenous drip. The drug concentration, x, in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(1 + x - 2x^2\right),$$

where $0 \le x < 1$, and k is a positive constant. Initially, $x = 0$.

(i) Find an expression for x in terms of k and t. [5]

After one hour, the drug concentration reaches 75% of its maximum level.

(ii) Show that the exact value of k is $\frac{1}{3}\ln 10$, and find the time taken for the drug concentration to reach 90% of its maximum level. [3]

A second model is proposed with the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin^2\left(\frac{1}{2}t\right),$$

where x is the drug concentration, measured as a fraction of its maximum level, in the blood after t hours. Initially, $x = 0$.

(iii) Find an expression for x in terms of t. [3]

(iv) Explain, with the aid of a sketch, why this proposed second model is inappropriate. [2]

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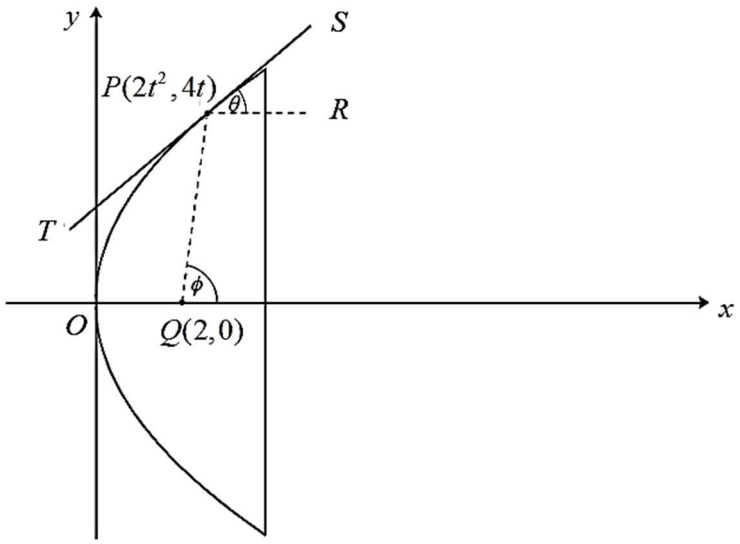


The figure above shows a cross-section of a searchlight whose inner reflective surface is modelled, in suitable units, by the curve

$$x = 2t^2, \quad y = 4t, \quad -\sqrt{2} \le t \le \sqrt{2}.$$

The inner reflective surface of the searchlight has the shape produced by rotating the curve about the *x*-axis.

(i) Show that the curve has cartesian equation $y^2 = 8x$, and find the volume of revolution of the curve, giving your answer as a multiple of π. [3]

$P\left(2t^2, 4t\right)$ is a point on the curve with parameter *t*. *TS* is the tangent to the curve at *P*, and *PR* is the line through *P* parallel to the *x*-axis. *Q* is the point $(2, 0)$. The angles that *PS* and *QP* make with the positive *x*-direction are θ and ϕ respectively.

(ii) By considering the gradient of the tangent *TS*, show that $\cot\theta = t$. [2]

(iii) Find the gradient of the line *QP* in terms of *t*. Hence show that $\phi = 2\theta$, and show that angle *TPQ* is equal to θ. [5]

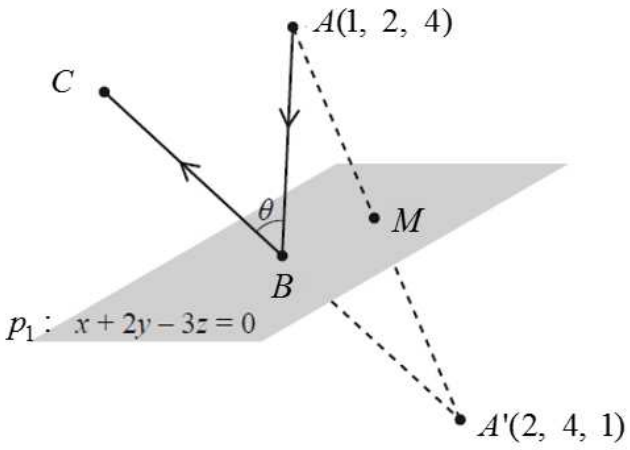
A lamp bulb is placed at *Q*.

(iv) Use your answer to part **(iii)** to describe the direction of the reflected light from the bulb. [1]

(v) Find a cartesian equation of the locus of the mid-point *M* on *PQ* as *t* varies. [2]

10 Federal Aviation Administration data shows that there were an increase in aviation incidents caused by laser illuminations reported by pilots in 2015 and 2016. A simplified laboratory model is set up to investigate the effects of a laser beam on plexiglass, a common material used to make cockpit windscreen.

The piece of plexiglass is represented by a plane p_1 with equation $x+2y-3z=0$.



Referred to the origin, a laser beam *ABC* is fired from the point A with coordinates $(1,\ 2,\ 4)$, and is reflected at the point *B* on p_1 to form a reflected ray *BC* as shown in the diagram above. It is given that *M* is the midpoint of AA', where the point A' has coordinates $(2,\ 4,\ 1)$.

(i) Show that AA' is perpendicular to p_1. [2]

(ii) By finding the coordinates of *M*, show that M lies in p_1. [2]

The vector equation of the line *AB* is $\mathbf{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda\in\mapsto.$

(iii) Find the coordinates of *B*. [2]

The acute angle between the incident ray *AB* and the reflected ray *BC* is θ (see diagram).

(iv) Given that $A'BC$ is a straight line, find the value of θ. Hence, or otherwise, write down the acute angle between the line *AB* and p_1. [3]

To reduce the effect of laser illumination on the pilot sitting in the cockpit at point A', a scientist proposes to include a protective film, represented by a plane p_2, such that the perpendicular distance from p_1 to p_2 is 0.5.

(v) State the possible cartesian equations of p_2. [2]

To further investigate the effects of a laser beam on plexiglass, separate laser beams are fired such that the incident ray *AD* is now a variable line which is also fired from the same point *A* and is reflected at the variable point *D* on p_1 to form a reflected ray *DE*.

	(vi) Given that AD is perpendicular to the previous ray AB, find the minimum possible distance between B and D. [2] **(vii)** Find the acute inclination of the reflected ray DE to the z-axis when DE is inclined at $60°$ to the x-axis and $45°$ to the y-axis. [3]

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