Q1	Suggested Answers
	Make w the subject from $w+\mathrm{i}+3=v$: $w=v-\mathrm{i}-3$ Sub into $v^2-\mathrm{i}w+2=0$ $v^2-\mathrm{i}(v-\mathrm{i}-3)+2=0$ $v^2-\mathrm{i}v+1+3\mathrm{i}=0$ **Using quadratic formula,** $v=\dfrac{\mathrm{i}\pm\sqrt{-1-4(1+3\mathrm{i})}}{2}$ $=\dfrac{\mathrm{i}\pm\sqrt{-5-12\mathrm{i}}}{2}$ $=\dfrac{\mathrm{i}\pm(2-3\mathrm{i})}{2}$ use GC to find $\sqrt{-5-12\mathrm{i}}$ $=\dfrac{\mathrm{i}+(2-3\mathrm{i})}{2}$ or $\dfrac{\mathrm{i}-(2-3\mathrm{i})}{2}$ $=1-\mathrm{i}$ or $-1+2\mathrm{i}$ $w=-2-2\mathrm{i}$ or $-4+\mathrm{i}$
	Alternative (1) Sub $v=w+\mathrm{i}+3$ into $v^2-\mathrm{i}w+2=0$ $(w+\mathrm{i}+3)^2-\mathrm{i}w+2=0$ $w^2+2(\mathrm{i}+3)w+(\mathrm{i}+3)^2-\mathrm{i}w+2=0$ **Using quadratic formula,** $w=\dfrac{-(6+\mathrm{i})\pm\sqrt{(6+\mathrm{i})^2-4(10+6\mathrm{i})}}{2}$ $=\dfrac{-(6+\mathrm{i})\pm\sqrt{-5-12\mathrm{i}}}{2}=\dfrac{-(6+\mathrm{i})\pm(2-3\mathrm{i})}{2}$ use GC to find $\sqrt{-5-12\mathrm{i}}$ $w=-2-2\mathrm{i},-4+\mathrm{i}$ Using $v=w+\mathrm{i}+3$ We have $v=1-\mathrm{i},-1+2\mathrm{i}$
	Alternative (2) From the equation $w+\mathrm{i}+3=v$ Multiply by i, we have $\mathrm{i}w-1+3\mathrm{i}=\mathrm{i}v\cdots(1)$ $v^2-\mathrm{i}w+2=0\cdots(2)$ (1)+(2) : $v^2-\mathrm{i}v+1+3\mathrm{i}=0$ **Using quadratic formula,**

	$v=\dfrac{\mathrm{i}\pm\sqrt{-1-4(1+3\mathrm{i})}}{2}$ $=\dfrac{\mathrm{i}\pm\sqrt{-5-12\mathrm{i}}}{2}$ $=\dfrac{\mathrm{i}\pm(2-3\mathrm{i})}{2}$ use GC to find $\sqrt{-5-12\mathrm{i}}$ $=\dfrac{\mathrm{i}+(2-3\mathrm{i})}{2}$ or $\dfrac{\mathrm{i}-(2-3\mathrm{i})}{2}$ $=1-\mathrm{i}$ or $-1+2\mathrm{i}$ $w=-2-2\mathrm{i}$ or $-4+\mathrm{i}$
Q2	**Suggested Answers**
(a)	$x=a\theta-a\sin\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=a-a\cos\theta$ $y=a-a\cos\theta \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=a\sin\theta$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{a\sin\theta}{a(1-\cos\theta)}$ $=\dfrac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)}$ $=\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\dfrac{\theta}{2}$
(b)	When $\theta=\dfrac{\pi}{3}$, $x=\dfrac{\pi}{3}a-\dfrac{\sqrt{3}}{2}a$, $y=\dfrac{1}{2}a$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cot\dfrac{\pi}{6}=\dfrac{1}{\tan\frac{\pi}{6}}=\dfrac{1}{\left(\frac{1}{\sqrt{3}}\right)}=\sqrt{3}$ Equation of tangent is $y-\dfrac{1}{2}a=\sqrt{3}\left(x-\dfrac{\pi}{3}a+\dfrac{\sqrt{3}}{2}a\right)$ $\therefore y=\sqrt{3}x-\dfrac{\sqrt{3}}{3}\pi a+2a$ When $x=\dfrac{\pi}{3}a$, $y=\sqrt{3}\left(\dfrac{\pi}{3}a\right)x-\dfrac{\sqrt{3}}{3}\pi a+2a=2a$ Therefore, the tangent passes through $\left(\dfrac{1}{3}\pi a,\ 2a\right)$ Or When $y=2a$, $2a=\sqrt{3}x-\dfrac{\sqrt{3}}{3}\pi a+2a \quad \Rightarrow x=\dfrac{\frac{\sqrt{3}}{3}\pi a}{\sqrt{3}}=\dfrac{1}{3}\pi a$ Therefore, the tangent passes through $\left(\dfrac{1}{3}\pi a,\ 2a\right)$

Q3	**Suggested Answers**
(a)	$y=\left(\sin^{-1}x\right)^2$ Differentiate wrt x: $\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}$ $\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2\sin^{-1}x$ Differentiate wrt x: $\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{x}{\sqrt{1-x^2}}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{\sqrt{1-x^2}}$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=2$ **Alternatively,** $\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2\sin^{-1}x$ $\left(\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\left(2\sin^{-1}x\right)^2$ $\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y$ Differentiate wrt x : $\left(1-x^2\right)\cdot 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=2$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}-2=0$
(b)	Differentiate wrt x : $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\left(x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Differentiate wrt x : $\left(1-x^2\right)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-2x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3\left(x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\left(1-x^2\right)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x=0,\ y=0,\ \frac{\mathrm{d}y}{\mathrm{d}x}=0,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$ and $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=8$

	$\therefore y = 2\left(\frac{x^2}{2!}\right) + 8\left(\frac{x^4}{4!}\right) + \dots$ $y = x^2 + \frac{1}{3}x^4 + \dots$
(c)	$y \approx x^2 + \frac{1}{3}x^4$ $\left(\sin^{-1} x\right)^2 \approx x^2 + \frac{1}{3}x^4$ When $x = \frac{1}{2}$, $\left(\sin^{-1}\frac{1}{2}\right)^2 \approx \left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4$ $\frac{\pi^2}{36} \approx \frac{13}{48}$ $\pi \approx \frac{\sqrt{39}}{2}$ Percentage error of approximation $= \frac{\pi - \frac{\sqrt{39}}{2}}{\pi} \times 100\%$ $= 0.608\%$ (3 s.f.) Approximation is quite accurate as the value of x is **close to zero** **<u>and</u>** the series used terms up to x^4.

Q4	Suggested Answers
(a)	$\mathrm{f}(x)=\sqrt{3}\cos x+\sin x=R\cos(x-\alpha)$ $=R[\cos x\cos\alpha+\sin x\sin\alpha]$ $R\cos\alpha=\sqrt{3}$ --- (1) $R\sin\alpha=1$ --- (2) $(1)^2+(2)^2$: $R=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2$ $\dfrac{(2)}{(1)}$: $\alpha=\tan^{-1}\dfrac{1}{\sqrt{3}}=\dfrac{\pi}{6}$
(b)	$\displaystyle\int_0^{\frac{\pi}{6}}\left(\frac{1}{\mathrm{f}(x)}\right)^2\mathrm{d}x=\int_0^{\frac{\pi}{6}}\frac{1}{\left(2\cos\left(x-\frac{\pi}{6}\right)\right)^2}\mathrm{d}x$ $\displaystyle=\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(x-\frac{\pi}{6}\right)\mathrm{d}x$ $\displaystyle=\frac{1}{4}\left[\tan\left(x-\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}$ $\displaystyle=\frac{1}{4}\left[\tan 0-\tan\left(-\frac{\pi}{6}\right)\right]$ $\displaystyle=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12}$
(c)	$\displaystyle\int_0^{\frac{\pi}{12}}\frac{1}{\mathrm{f}(2x)}\mathrm{d}x=\int_0^{\frac{\pi}{12}}\frac{1}{2\cos\left(2x-\frac{\pi}{6}\right)}\mathrm{d}x$ $\displaystyle=\frac{1}{2}\int_0^{\frac{\pi}{12}}\sec\left(2x-\frac{\pi}{6}\right)\mathrm{d}x$ $\displaystyle=\frac{1}{2}\left(\frac{1}{2}\right)\ln\left[\left\|\sec\left(2x-\frac{\pi}{6}\right)+\tan\left(2x-\frac{\pi}{6}\right)\right\|\right]_0^{\frac{\pi}{12}}$ $\displaystyle=\frac{1}{4}\left[\ln\|\sec 0+\tan 0\|-\ln\left\|\sec\left(-\frac{\pi}{6}\right)+\tan\left(-\frac{\pi}{6}\right)\right\|\right]$ $\displaystyle=\frac{1}{4}\left[\ln 1-\ln\left\|\frac{2}{\sqrt{3}}+\left(-\frac{2}{\sqrt{3}}\right)\right\|\right]$ $\displaystyle=-\frac{1}{4}\ln\frac{1}{\sqrt{3}}$ $\displaystyle=\frac{1}{4}\ln\sqrt{3}=\frac{1}{8}\ln 3$

Q5	Suggested Answers
(a)	**Method 1:**

$$\int \frac{4x-1}{x^2+4x+4}\,dx$$

$$= \int \frac{2(2x+4)-9}{x^2+4x+4}\,dx$$

$$= 2\int \frac{2x+4}{x^2+4x+4}\,dx - \int \frac{9}{(x+2)^2}\,dx$$

$$= 2\ln\left(x^2+4x+4\right) + \frac{9}{x+2} + C$$

$$= 4\ln(x+2) + \frac{9}{x+2} + C$$

Method 2: Use of partial fractions

$$\frac{4x-1}{x^2+4x+4} = \frac{4x-1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$4x-1 = A(x+2)+B$$

Solving, $A = 4$, $B = -9$

$$\int \frac{4x-1}{x^2+4x+4}\,dx = \int \frac{4}{x+2} - \frac{9}{(x+2)^2}\,dx$$

$$= 4\ln(x+2) + \frac{9}{x+2} + C$$

(b)

$$\int_0^1 \frac{|4x-1|}{x^2+4x+4}\,dx$$

$$= -\int_0^{\frac{1}{4}} \frac{4x-1}{x^2+4x+4}\,dx + \int_{\frac{1}{4}}^1 \frac{4x-1}{x^2+4x+4}\,dx$$

$$= -\left[4\ln(x+2) + \frac{9}{x+2}\right]_0^{\frac{1}{4}} + \left[4\ln(x+2) + \frac{9}{x+2}\right]_{\frac{1}{4}}^1$$

$$= -\left[\left(4\ln\frac{9}{4}+4\right) - \left(4\ln 2 + \frac{9}{2}\right)\right] + \left[(4\ln 3 + 3) - \left(4\ln\frac{9}{4}+4\right)\right]$$

$$= -4\ln\frac{9}{4} + 4\ln 2 + 4\ln 3 - 4\ln\frac{9}{4} - \frac{1}{2}$$

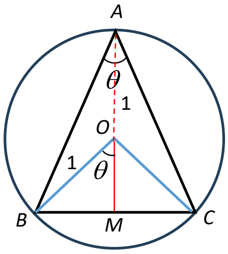
$$= 4\ln\frac{2\times 3}{\left(\frac{9}{4}\times\frac{9}{4}\right)} - \frac{1}{2}$$

$$= 4\ln\frac{32}{27} - \frac{1}{2}$$

Q6	Suggested Answers
(a)	$\dfrac{a+5d}{a}=\dfrac{a+13d}{a+5d}$ $(a+5d)^2=a(a+13d)$ $a^2+10ad+25d^2=a^2+13ad$ $25d^2=3ad$ Since $d\neq 0$, $3a=25d$ $d=\dfrac{3a}{25}$
(b)(i)	Sum to infinity = $\dfrac{b}{1-0.5}=2b$
(ii)	$S_n=\dfrac{b(1-0.5^n)}{1-0.5}=2b(1-0.5^n)$ $S_{2n}-S_n<0.004b$ since $S_{2n}>S_n$ as all the terms are positive $2b(1-0.5^{2n})-2b(1-0.5^n)<0.004b$ $0.5^n-0.5^{2n}<0.002$ since $b>0$ $0.5^n-0.5^{2n}-0.002<0$ Using GC, <table><tr><td>n</td><td>$0.5^n-0.5^{2n}-0.002$</td></tr><tr><td>8</td><td>$0.00189>0$</td></tr><tr><td>9</td><td>$-5.07\times10^{-5}<0$</td></tr></table> Smallest n = 9

Q7 — **Suggested Answers**

$BM = \sin\theta$ and $OM = \cos\theta$



Length of height, $AM = 1+\cos\theta$

Length of base, $BC = 2\sin\theta$

Area of triangle, $S = \frac{1}{2}\times(1+\cos\theta)\times 2\sin\theta$

$$= \sin\theta + \frac{1}{2}\sin 2\theta$$

Alternatively,

Area of triangle, $S = 2\left(\frac{1}{2}\times(1)^2\sin(\pi-\theta)\right)+\frac{1}{2}\times(1)^2\sin 2\theta$

$$= \sin\theta + \frac{1}{2}\sin 2\theta$$

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = \cos\theta + \cos 2\theta$$

For maximum area, $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 0$

$\cos\theta + \cos 2\theta = 0$

$\cos\theta + 2\cos^2\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$ (rejected since θ is acute)

Therefore, $\theta = \frac{\pi}{3}$

$$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -\sin\theta - 2\sin 2\theta$$

When $\theta = \frac{\pi}{3}$, $\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -\frac{3\sqrt{3}}{2} < 0$

Thus S is maximum when $\theta = \frac{\pi}{3}$

Since $\angle BOC = 2\theta$, $\angle BAC = \theta = \frac{\pi}{3}$ ($\angle$ at centre = 2 $\angle$ at circumference)

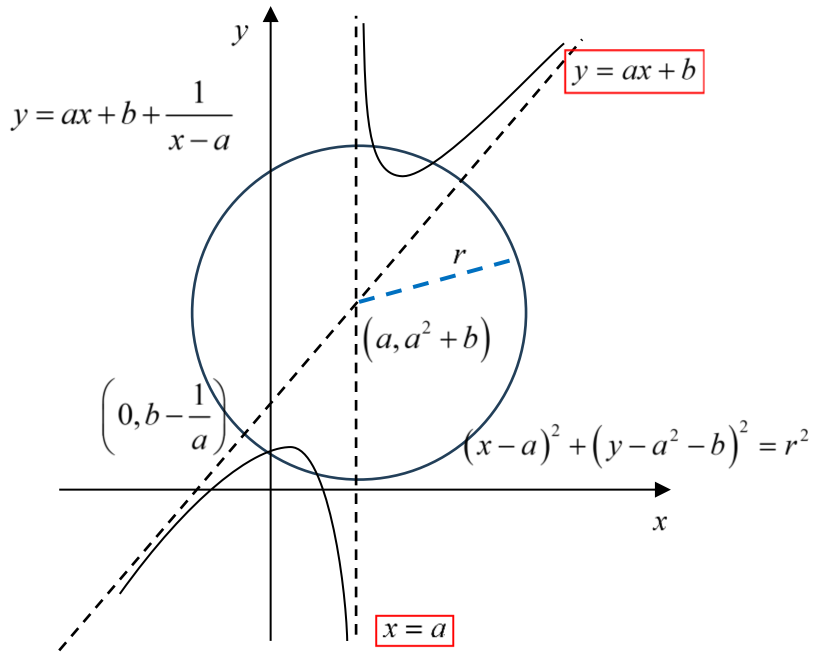
As the triangle is isosceles, all the angles in the triangle are $\frac{\pi}{3}$.

Therefore, maximum area occurs when triangle ABC is equilateral, ie, when $\theta = \frac{\pi}{3}$

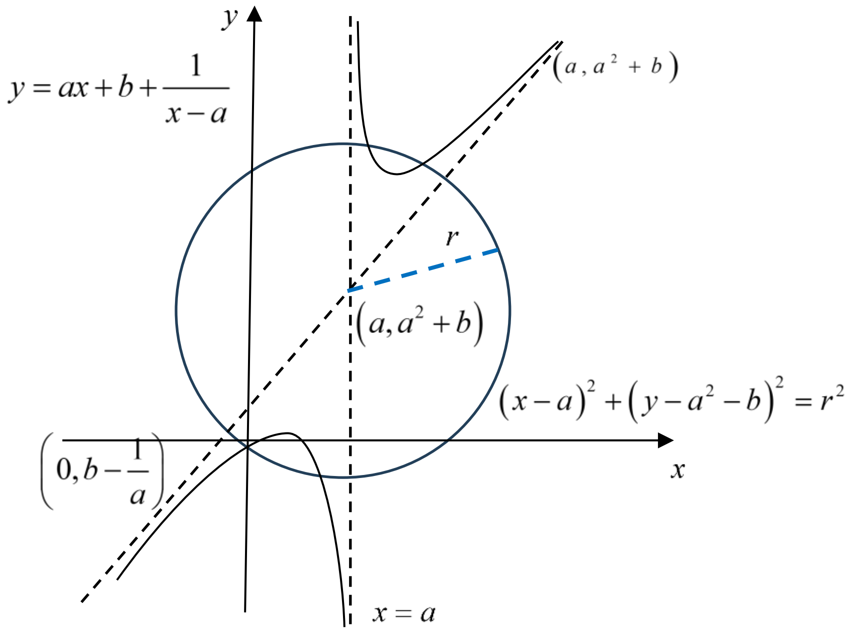
Maximum area $= \sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} = \frac{3}{4}\sqrt{3}$

<table>
<tr><th>Q8</th><th>Suggested Answers</th></tr>
<tr><td>(a)
(b)</td><td>

y-intercept $b-\frac{1}{a}>0$



OR y-intercept $b-\frac{1}{a}<0$ (give one of the diagrams will do)



</td></tr>
<tr><td>(c)</td><td>

$y=ax+b+\frac{1}{x-a}=2x+1+\frac{1}{x-2}$

$y=+\sqrt{r^2-(x-a)^2}+a^2+b=\sqrt{16-(x-2)^2}+5$ **(upper semicircle only)**

$a=2$ and $b=1$ (and $r=4$)

Using GC, the x-coordinates of the points of intersection are 2.29 and 3.52

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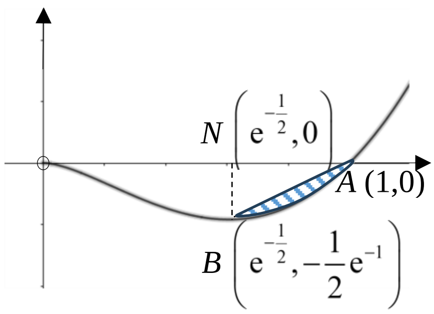
</table>

	For $2x+1+\dfrac{1}{x-2} > \sqrt{16-(x-2)^2}+5$ **(upper semicircle only)** $2<x<2.29$ or $3.52<x\le 6$ (the circle is only defined for [–2, 6])

Q9	Suggested Answers
(a)	Coordinates of A are $(1,0)$
(b)	$y = x^2 \ln x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x\ln x + x^2\left(\dfrac{1}{x}\right) = x(2\ln x+1)$ $x(2\ln x+1)=0$ $x=0$ or $\ln x = -\dfrac{1}{2}$ Since $x>0$, $x=\mathrm{e}^{-\frac{1}{2}}$ Coordinates of B are $\left(\mathrm{e}^{-\frac{1}{2}}, -\dfrac{1}{2}\mathrm{e}^{-1}\right)$
(c)	 $u=\ln x \qquad \dfrac{\mathrm{d}v}{\mathrm{d}x}=x^2$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{x} \qquad v=\dfrac{1}{3}x^3$ **Method 1:** Area required $=\left(-\int_{\mathrm{e}^{-\frac{1}{2}}}^{1}\left(x^2\ln x\right)\mathrm{d}x\right)-$ area of triangle ABN $=-\left(\left[\dfrac{1}{3}x^3\ln x\right]_{\mathrm{e}^{-\frac{1}{2}}}^{1}-\dfrac{1}{3}\int_{\mathrm{e}^{-\frac{1}{2}}}^{1}x^2\,\mathrm{d}x\right)-\dfrac{1}{4}\mathrm{e}^{-1}\left(1-\mathrm{e}^{-\frac{1}{2}}\right)$ using integration by parts $=-\left(\left[0-\dfrac{1}{3}\mathrm{e}^{-\frac{3}{2}}\left(-\dfrac{1}{2}\right)\right]-\dfrac{1}{9}\left[x^3\right]_{\mathrm{e}^{-\frac{1}{2}}}^{1}\right)-\dfrac{1}{4}\mathrm{e}^{-1}\left(1-\mathrm{e}^{-\frac{1}{2}}\right)$ $=-\dfrac{1}{6}\mathrm{e}^{-\frac{3}{2}}+\dfrac{1}{9}\left(1-\mathrm{e}^{-\frac{3}{2}}\right)-\dfrac{1}{4}\mathrm{e}^{-1}\left(1-\mathrm{e}^{-\frac{1}{2}}\right)$ $=\dfrac{1}{9}-\dfrac{1}{36}\mathrm{e}^{-\frac{3}{2}}-\dfrac{1}{4}\mathrm{e}^{-1}$
	Method 2: (area bounded between line and curve) Equation of line joining A and B is $y=\dfrac{\frac{1}{2}\mathrm{e}^{-1}}{1-\mathrm{e}^{-\frac{1}{2}}}(x-1)=\dfrac{1}{2\mathrm{e}\left(1-\mathrm{e}^{-\frac{1}{2}}\right)}(x-1)=\dfrac{1}{2\left(\mathrm{e}-\mathrm{e}^{\frac{1}{2}}\right)}(x-1)$ Area required

	$=\int_{e^{-\frac{1}{2}}}^{1}\frac{1}{2\left(e-e^{\frac{1}{2}}\right)}(x-1)\,dx-\int_{e^{-\frac{1}{2}}}^{1}\left(x^2\ln x\right)dx$ $=\frac{1}{2\left(e-e^{\frac{1}{2}}\right)}\left[\frac{x^2}{2}-x\right]_{e^{-\frac{1}{2}}}^{1}-\left\{\left[\frac{1}{3}x^3\ln x\right]_{e^{-\frac{1}{2}}}^{1}-\frac{1}{3}\int_{e^{-\frac{1}{2}}}^{1}x^2\,dx\right\}$ using integration by parts $=\frac{1}{2\left(e-e^{\frac{1}{2}}\right)}\left[\left(\frac{1}{2}-1\right)-\left(\frac{e^{-1}}{2}-e^{-\frac{1}{2}}\right)\right]-\left\{\left[0-\frac{1}{3}e^{-\frac{3}{2}}\left(-\frac{1}{2}\right)\right]-\frac{1}{9}\left[x^3\right]_{e^{-\frac{1}{2}}}^{1}\right\}$ $=\frac{1}{2\left(e-e^{\frac{1}{2}}\right)}\left[e^{-\frac{1}{2}}-\frac{1}{2}-\frac{e^{-1}}{2}\right]-\frac{1}{6}e^{-\frac{3}{2}}+\frac{1}{9}\left(1-e^{-\frac{3}{2}}\right)$ $=\frac{1}{4\left(e-e^{\frac{1}{2}}\right)}\left[2e^{-\frac{1}{2}}-1-e^{-1}\right]-\frac{5}{18}e^{-\frac{3}{2}}+\frac{1}{9}$

Q10	Suggested Answers
(a)	$\mathrm{fg}(1)=\mathrm{f}\left(\frac{a-1}{b-a}\right)$ $=2\left(\frac{a-1}{b-a}\right)+1$
(b)	$R_f=\mathbb{R}$ and $D_g=\mathbb{R}\setminus\left\{\frac{a}{b}\right\}$ Since $R_f\not\subseteq D_g$, gf does not exist.
(c)	Let $y=\frac{ax-1}{bx-a}$ $bxy-ay=ax-1$ $bxy-ax=ay-1$ $x=\frac{ay-1}{by-a}$ $g^{-1}(x)=\frac{ax-1}{bx-a}$
(d)	Hence method: Since $g(x)=g^{-1}(x)$ $gg(x)=gg^{-1}(x)$ $g^2(x)=x$

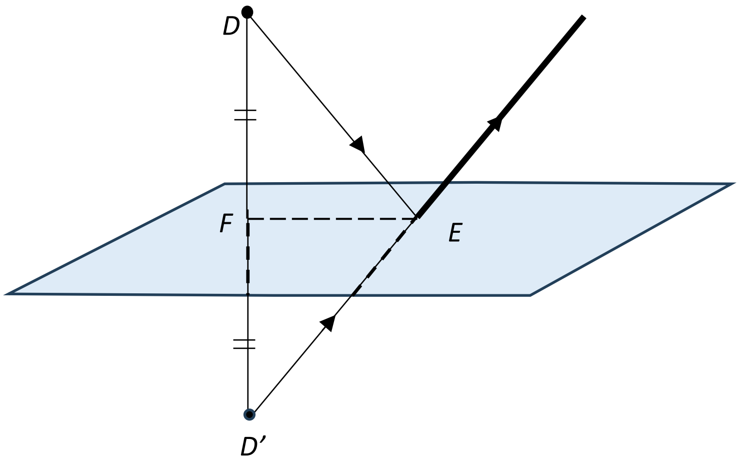
	Otherwise method: $\mathrm{g}^2(x)=\mathrm{gg}(x)$ $=\mathrm{g}\left(\dfrac{ax-1}{bx-a}\right)$ $=\dfrac{a\left(\dfrac{ax-1}{bx-a}\right)-1}{b\left(\dfrac{ax-1}{bx-a}\right)-a}$ $=\dfrac{a(ax-1)-(bx-a)}{b(ax-1)-a(bx-a)}$ $=\dfrac{a^2x-a-bx+a}{abx-b-abx+a^2}$ $=\dfrac{a^2x-bx}{a^2-b}$ $=x$
(e)	$y=\mathrm{g}(x)=\dfrac{ax-1}{x-a}=a+\dfrac{a^2-1}{x-a}$ where $a>0$ Replace y by $y+a$: $\quad y=\dfrac{a^2-1}{x-a}$ Replace y by $\dfrac{y}{\left(\dfrac{1}{a^2-1}\right)}$: $\quad y=\dfrac{1}{x-a}$ Replace x by $x+a$: $\quad y=\dfrac{1}{x}$ (1) **Translate** the graph by a units **in the negative direction** of the y-axis. (2) **Scaling** by a **factor** of $\dfrac{1}{a^2-1}$ **parallel** to the y-axis. (3) **Translate** a units **in the negative direction** of the x-axis. Note: For this qn, the following order are accepted also (1), (3), (2) (3), (1), (2) Transformation (1) must come before (2) if the above descriptions are used.

Q11	Suggested Answers
(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(160 - x), \quad \text{where } k > 0$ $\frac{1}{160-x}\frac{\mathrm{d}x}{\mathrm{d}t} = k$ $\int \frac{1}{160-x}\,\mathrm{d}x = k\int 1\,\mathrm{d}t$ $-\ln\lvert 160-x\rvert = kt + C$ $160 - x = A\mathrm{e}^{-kt} \quad \text{where } A = \pm\mathrm{e}^{-C}$ $t=0, x=0 \Rightarrow A=160$ $160 - x = 160\mathrm{e}^{-kt}$ $t=12, x=40 \Rightarrow 120 = 160\mathrm{e}^{-12k}$ $\frac{3}{4} = \mathrm{e}^{-12k}$ $k = -\frac{1}{12}\ln\frac{3}{4}$ $\therefore 160 - x = 160\mathrm{e}^{\left(\frac{1}{12}\ln\frac{3}{4}\right)t} = 160\left(\mathrm{e}^{\ln\frac{3}{4}}\right)^{\frac{t}{12}} = 160\left(\frac{3}{4}\right)^{\frac{t}{12}}$ When $x=100$, $\quad 60 = 160\left(\frac{3}{4}\right)^{\frac{t}{12}} \Rightarrow t = 40.9$ (3 s.f.) The time taken is 40.9 h.
(b)	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(160-x) - d = (160k - d) - kx$ $\frac{1}{(160k-d)-kx}\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ $-\frac{1}{k}\ln\lvert(160k-d)-kx\rvert = t + C$ $\ln\lvert(160k-d)-kx\rvert = -kt - kC$ $(160k-d)-kx = B\mathrm{e}^{-kt} \quad \text{where } B = \pm\mathrm{e}^{-kC}$ $t=0, x=0 \Rightarrow B = 160k - d$ $(160k-d)-kx = (160k-d)\mathrm{e}^{-kt}$ $kx = (160k-d)\left(1-\mathrm{e}^{-kt}\right)$ $x = \left(160 - \frac{d}{k}\right)\left(1-\mathrm{e}^{-kt}\right)$ $= \left(160 + \frac{12d}{\ln\frac{3}{4}}\right)\left(1 - \mathrm{e}^{\left(\frac{1}{12}\ln\frac{3}{4}\right)t}\right) = \left(160 + \frac{12d}{\ln\frac{3}{4}}\right)\left(1 - \left(\frac{3}{4}\right)^{\frac{t}{12}}\right)$

(c)	As $t \to \infty, \left(\frac{3}{4}\right)^{\frac{t}{12}} \to 0$ Thus $x \to 160 + \frac{12d}{\ln \frac{3}{4}}$ (limit in the long run) Let $160 + \frac{12d}{\ln \frac{3}{4}} = 10$ (*x* is increasing as *t* increases and approaches the limit) $d = 3.5960$ (5 s.f.) $= 3.60$ (3 s.f.)

Q12	**Suggested Answers**
(a)	$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}$ A normal to plane is $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ Equation of plane *ABC* : $\mathbf{r} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = 15$ $3x - 3y + 2z = 15$
(b)	$\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}, \ \mu \in \mathbb{R}$ Equation of plane *ABC* : $\mathbf{r} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = 15$ At intersection, $\begin{pmatrix} 5+3\mu \\ -1-3\mu \\ 8+2\mu \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = 15$ $34 + 22\mu = 15 \Rightarrow \mu = -\frac{19}{22}$ Position of the foot of perpendicular is $\overrightarrow{OF} = \frac{1}{22}\begin{pmatrix} 53 \\ 35 \\ 138 \end{pmatrix}$

(c)	Let ϕ be the angle between the light ray and the normal of the mirror. Then $\cos\phi = \dfrac{\left\lvert \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \right\rvert}$ $\phi = 55.9^o$ Hence angle between the mirror and light ray $= 90^\circ - 55.9^\circ = 34.1^\circ$ **Alternative method:** Let θ be the acute angle between the light ray and the mirror. $\sin\theta = \dfrac{\left\lvert \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \right\rvert}$ $\sin\theta = \dfrac{\lvert -38 \rvert}{\sqrt{209 \times 22}} \quad \Rightarrow \theta = 34.1^o$ (or 0.595 radian)
(d)	Equation of plane ABC : $\mathbf{r} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = 15$ Equation of line l : $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix}, \ \lambda \in \mathbb{R}$ At intersection, $\begin{pmatrix} 5-6\lambda \\ -1-2\lambda \\ 8-13\lambda \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = 15$ $34 - 38\lambda = 15 \Rightarrow \lambda = \dfrac{1}{2}$ $\overrightarrow{OE} = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix} + \dfrac{1}{2} \begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ \frac{3}{2} \end{pmatrix}$ Hence coordinates of point E is $\left(2, -2, \dfrac{3}{2}\right)$

(e)	 Let D’ be the point of reflection of D in the mirror. Using part (b), $\dfrac{\overrightarrow{OD'}+\overrightarrow{OD}}{2}=\overrightarrow{OF}=\dfrac{1}{22}\begin{pmatrix}53\\35\\138\end{pmatrix}$ $\overrightarrow{OD'}=\dfrac{1}{11}\begin{pmatrix}53\\35\\138\end{pmatrix}-\begin{pmatrix}5\\-1\\8\end{pmatrix}=\dfrac{1}{11}\begin{pmatrix}-2\\46\\50\end{pmatrix}$ $\overrightarrow{ED'}=\dfrac{1}{11}\begin{pmatrix}-2\\46\\50\end{pmatrix}-\begin{pmatrix}2\\-2\\3/2\end{pmatrix}=\begin{pmatrix}-24/11\\68/11\\67/22\end{pmatrix}=\dfrac{1}{22}\begin{pmatrix}-48\\136\\67\end{pmatrix}$ Vector equation of line is $\mathbf{r}=\begin{pmatrix}2\\-2\\3/2\end{pmatrix}+\gamma\begin{pmatrix}-48\\136\\67\end{pmatrix},\ \gamma\in\mathbb{R}$