Section A : Pure Mathematics [40 Marks]

1 It is given that $\mathrm{f}(x) = \ln\left(1+\mathrm{e}^{-x}\right)$.

(i) Using standard series from the List of Formulae (MF26) or otherwise, show that the first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ is $\ln a + bx + cx^2$, where a, b and c are constants to be determined. [4]

(ii) Hence find an expansion for $\dfrac{1}{1+\mathrm{e}^{x}}$, up to and including the term in x. [2]

2 In military exercises, parachutists jump from stationary helicopters and their motion is tracked by sensors tagged to their bodies. One parachutist jumps vertically from a helicopter. At time t seconds, the parachutist is x metres below the helicopter with velocity v ms^{-1}. It experiences a downward gravitational acceleration of 9.8 ms^{-2} and an upward acceleration R due to air resistance, where $R < 9.8$. R is directly proportional to v, with a constant of proportionality $k > 0$.

The rate of change of v for the parachutist is modelled as the difference between the downward gravitational acceleration and R. Write down a differential equation relating v, t and k, and solve this differential equation to find an expression for v in terms of t and k. [6]

The terminal velocity of a falling object is the value of its velocity after a long time. State the terminal velocity of the parachutist in terms of k. [1]

3 **(a)** Describe a sequence of two transformations which maps the graph of $y = \mathrm{e}^{x}$ onto the graph of $y = -\mathrm{e}^{x} + 2$. [2]

(b) The line with equation $y = x + 6$ undergoes the following transformations, in succession:

A: Scaling by a scale factor of $\dfrac{1}{3}$ parallel to the x-axis,
B: Translation of 2 units in the positive x-direction,
C: Scaling by a scale factor of $\dfrac{1}{3}$ parallel to the y-axis.

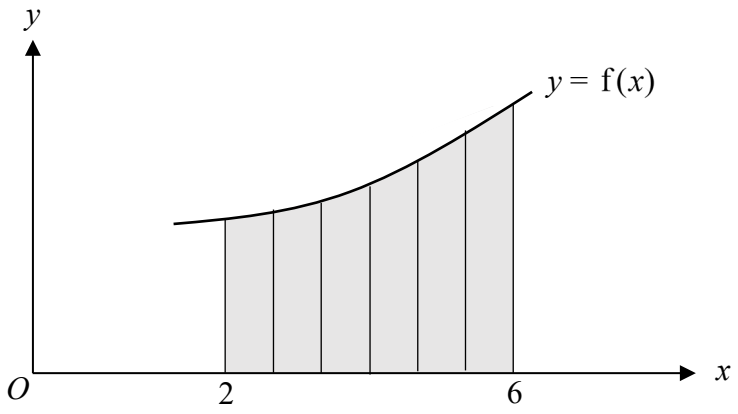
Find the equation of the resulting line. [3]

(c) The curve $y = \mathrm{f}(x)$ cuts the axes at $(a, 0)$ and $(0, b)$. $x = c$ and $y = d$ are asymptotes of $\mathrm{f}(x)$. It is given that $\mathrm{f}^{-1}(x)$ exists. State the equations of the asymptotes and the coordinates of the axial intercepts of the following curves.

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [2]

(ii) $y = \mathrm{f}^{-1}(x)$. [2]

4 The diagram shows a sketch of the curve $y = \mathrm{f}(x)$. The region under the curve between $x = 2$ and $x = 6$, shown shaded in the diagram, is A. This region is split into 6 vertical strips of equal width, h.



(i) State the value of h and show, using a sketch, that $\sum_{n=1}^{6}\big(\mathrm{f}(2+nh)\big)h$ is greater than the area of A. [3]

(ii) Find a similar expression that is less than the area of A. [1]

You are now given that $\mathrm{f}(x) = x\mathrm{e}^{x}$,

(iii) Use the expression given in part **(i)** and your expression from part **(ii)** to find the upper and lower bounds for the area of A, giving your answers to nearest integer value. [2]

(iv) Find the exact area of A. [3]

(v) Sketch the graph of a function $y = \mathrm{g}(x)$, between $x = 2$ and $x = 6$, for which the area between the curve, the x-axis and the lines $x = 2$ and $x = 6$ is greater than $\sum_{n=1}^{6}\big(\mathrm{g}(2+nh)\big)h$. [1]

5 **(a)** The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. P and Q are fixed points and R is a variable point. Describe geometrically the set of possible positions of R such that

(i) $\mathbf{r} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{q}$ [1]

(ii) $\mathbf{r} \times \mathbf{q} = \mathbf{p} \times \mathbf{q}$ [2]

(b) With respect to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. C lies on AB produced such that $AC : CB = 3 : 2$. Given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, the length of OC is $\sqrt{7}$ units and the angle AOB is 60°, find $|2\mathbf{a} - \mathbf{c}|$ by using a suitable scalar product. [5]

Section B : Statistics [60 Marks]

6 The production manager of a badminton racket company wishes to check that the mean mass of the badminton rackets produced is 800g as stated in the design specifications. The population standard deviation of the badminton rackets is 20g. He carries out a hypothesis test on a large sample of n badminton rackets. The sample mean mass is 807g and the null hypothesis is rejected at the 2% level of significance.

(i) Use an algebraic method to find the least value of n that will support this conclusion. [6]

(ii) Explain whether there is a need for the production manager to know anything about the population distribution of the masses of the badminton rackets. [1]

7 Evan has 15 toy bricks of which 5 are red, 4 are blue, 4 are yellow and 2 are green. The bricks of the same colour are identical.

(i) Find the number of ways in which Evan can choose 2 bricks of different colours. [1]

(ii) Find the number of ways in which Evan can choose 3 bricks. [3]

The table below shows the amount of money Evan paid for each type of bricks. The green bricks are given to him as free gifts.

Colour of bricks	Red	Blue	Yellow	Green
Price paid per brick	\$3	\$2	\$1	Free

(iii) Evan randomly chooses 4 bricks from his 15 toy bricks and gives them to his friend. The bricks of the same colour are now considered to be non-identical. Find the number of ways in which he chooses \$8 worth of bricks. [3]

8 The events A and B are such that $\mathrm{P}(A) = a$ and $\mathrm{P}(B) = b$. A and B are mutually exclusive events.

(i) Find an expression for $\mathrm{P}(A' \cap B')$. [1]

(ii) Draw a Venn diagram to illustrate the case where A' and B' are also mutually exclusive events. Hence, state an equation relating a and b. [2]

For the rest of the question, A' and B' are **not** mutually exclusive events. The event C is such that $\mathrm{P}(C) = c$. A and C are independent events. It is also given that $a = 0.2$, $b = 0.3$ and $\mathrm{P}(B \cap C) = 0.14$.

(iii) Find an expression for $\mathrm{P}(A \cup B \cup C)$. [2]

(iv) Find an expression for $\mathrm{P}(A' \cap B' \cap C)$. [1]

(v) Use your answers to parts **(iii)** and **(iv)** to find the range of values of c. [2]

9 The shear strength of soil is the ability of the soil to resist tensional stresses and sustain loads. The water content of soil is the ratio of the weight of water to the weight of the solids in a mass of soil. A soil investigation laboratory collected soil from a reclaimed site to study the effect of water content on shear strength. The information gathered about the shear strength P, measured in KiloPascal (KPa), at different water content w, measured in percentage is presented in the table.

Water Content, w (%)	10.0	12.5	15.0	17.5	20.0	22.5	25.0	27.5
Shear Strength, P (KPa)	2500	2100	1820	1630	1500	1410	1320	1270

(i) Draw a scatter diagram of these data. Use your diagram to explain whether the relationship between P and w is likely to be well modelled by an equation of the form $P = aw + b$, where a and b are constants. [2]

(ii) By calculating the relevant product moment correlation coefficients, determine whether the relationship between P and w is modelled better by $P = aw + b$ or $P = a\sqrt{w} + b$. Explain how you decide which model is better, and state the equation in this case. [5]

(iii) Use your equation to estimate the shear strength when the water content is 40%. Explain whether your estimate is reliable. [2]

(iv) Subsequently, it is known that one data point is not presented in the information table and that the inclusion of this data point does not affect the appropriate equation in **(ii)**. State, the shear strength for the missing data point, giving your answer correct to the nearest integer. [1]

(v) Given that 1 MegaPascal = 1000 KiloPascal, re-write your equation in **(ii)** so that it can be used when the shear strength is measured in MegaPascal. Explain whether the corresponding product moment correlation coefficient would differ from that calculated **(ii)**. [2]

10 The table below shows the BMI-for-age for boys aged 18 years in Singapore.

Weight Indicator / Age (years)	**Underweight**	**Overweight**
	$\leq$ 5th percentile	**$\geq$ 90th percentile**
18	≤ 16.7	$\geq m$

It is given that the BMI for 18-year-old boys follows a normal distribution $\mathrm{N}\left(22.7, \sigma^2\right)$.

(i) Show that $\sigma = 3.65$. [2]

(ii) Hence, find the minimum value of BMI, m, such that a randomly chosen 18-year-old boy is overweight. [2]

The BMI for 15-year-old boys follows another normal distribution $\mathrm{N}\left(21.6, 3.49^2\right)$.

(iii) Find the probability that the average BMI of a randomly chosen 15-year-old boy and a randomly chosen 18-year-old boy is more than 20.1. [2]

(iv) State, in context, an assumption made in your calculation in **(iii)**. [1]

11 A fair cubical die has faces each labelled with one of the four numbers: 1, 2, 3 and 4. The die is thrown once and the number on the uppermost face is the score T. It is given that the mode of T is 2 and that $\mathrm{P}(T=1) = \mathrm{P}(T=4) = \frac{1}{6}$. Determine the probability distribution of T. [2]

In a game at a carnival, a player repeatedly throws the cubical die. If T is an odd number, the player loses and the game ends. Otherwise, the player continues to throw the die until a '4' is shown and the game ends. Let Y be the number of throws that it takes for the player to throw the first '4'. If the player loses the game, then $Y = 0$.

(i) Show that $\mathrm{P}(Y=0) = \frac{2}{3}$. [2]

It is given that for $|x| < 1$, $1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots = \frac{1+x}{(1-x)^3}$.

(ii) If $\mathrm{E}(Y) = \frac{2}{3}$, find $\mathrm{Var}(Y)$. [3]

12 A manufacturer produces screen protectors for handphones. On average, 4% of the screen protectors are cracked. The screen protectors are sold in boxes of n.

(i) State, in context, two assumptions needed for the number of cracked screen protectors in a box to be well modelled by a binomial distribution. [2]

Assume now that the number of cracked screen protectors in a box has a binomial distribution.

(ii) It is given that the most likely number of cracked screen protectors in a box is 2. Find the range of values of n, without using a calculator. [4]

For the rest of the question, the value of n is taken to be 50.

(iii) Find the probability that there are at most 3 cracked screen protectors in a box. [1]

(iv) Find the probability that there are more than 2 cracked screen protectors in a box, given that there are at most 6 cracked screen protectors in a box. [3]

(v) 10 boxes of screen protectors are randomly chosen. Find the probability that the seventh box is the fourth box with at most 3 cracked screen protectors. [2]