Name:	Class:	Class Register Number:

中正中学

CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2023
SECONDARY 4

ADDITIONAL MATHEMATICS **4049/02**

Paper 2 **Tuesday 29 August 2023**

Candidates answer on the Question Paper. **2 hours 15 minutes**

MARKS SCHEME

This document consists of **20** printed pages and **2** blank pages.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 **(a)** The line $4x = 3y + 2$ intersects the curve $x^2 - xy + 5 = 0$ at the points A and B. Find the midpoint of AB. [5]

$4x = 3y + 2 \quad ...(1)$ $x^2 - xy + 5 = 0 \quad ...(2)$	
From (1): $3y = 4x - 2$ $y = \dfrac{4x-2}{3} \quad ...(3)$	
Sub. (3) into (2): $x^2 - x\left(\dfrac{4x-2}{3}\right) + 5 = 0$	M1 f.t. – substitution
$3x^2 - 4x^2 + 2x + 15 = 0$ $-x^2 + 2x + 15 = 0$ $x^2 - 2x - 15 = 0$ $(x-5)(x+3) = 0$	M1 f.t. – solving quadratic
$x = 5$ or -3 sub. into (3): $y = 6$ or $-\dfrac{14}{3}$	M1 – either correct x or y values
$\therefore A\left(-3, -\dfrac{14}{3}\right)$ **and** $B(5, 6)$	
Midpoint of $AB = \left(\dfrac{-3+5}{2}, \dfrac{-\frac{14}{3}+6}{2}\right)$	M1 f.t. – midpoint formula
$= \left(1, \dfrac{2}{3}\right)$	A1

(b) Find the least value of the integer h for which hx^2+5x+h is positive for all real values of x. [3]

For $hx^2+5x+h>0$, discriminant <0 $25-4(h)h<0$	B1 – discriminant
$25-4h^2<0$ $(5-2h)(5+2h)<0$	M1 f.t. – factorising quadratic
$h<-\frac{5}{2}$ or $h>\frac{5}{2}$	
Since $h>0$, $h>\frac{5}{2}$. **∴ Least integer value of h = 3.**	A1

(c) Given that the line $y=3x+p$ is tangent to the curve $y=x^2+5x+q$, where p and q are integers, prove that p and q are consecutive numbers. [4]

$y=3x+p$ ... (1) $y=x^2+5x+q$ ... (2)	
Sub. (1) into (2): $3x+p=x^2+5x+q$	M1 – substitution
$x^2+5x+q-3x-p=0$ $x^2+2x+q-p=0$	M1 – forming quadratic
a = 1, b = 2, c = k – c	
Line is tangent to curve $\rightarrow$ $b^2-4ac=0$ $2^2-4(1)(q-p)=0$	M1 f.t. – any use of discriminant
$4-4q+4p=0$ $4q=4+4p$ $q=1+p$ Since q = 1 + p, q will always be the next number after p. Hence, p and q are consecutive numbers (proved).	A1 – with explanation

Alternative:

$y = x^2 + 5x + q$

$\frac{dy}{dx} = 2x + 5$

Since line is tangent to curve and gradient of line = 3,

$2x + 5 = 3$

$2x = -2$

$x = -1$

M1 – equate $\frac{dy}{dx}$ to 3

M1 f.t. – finding x

$y = 3x + p$ ... (1)

$y = x^2 + 5x + q$ ... (2)

Sub. (1) into (2) and $x = -1$:

$3x + p = x^2 + 5x + q$

$3(-1) + p = (-1)^2 + 5(-1) + q$

$-3 + p = -4 + q$

$q - p = 1$

M1 – substitution

Since the difference between q and p is 1, q will always be the next number after p.
Hence, p and q are consecutive numbers (proved).

A1 – with explanation

2 **(a)** By considering the general term in the binomial expansion of $\left(x^3-\dfrac{2}{x}\right)^7$, explain why there are only odd powers of x in this expansion. [3]

$T_{r+1}=\binom{7}{r}\left(x^3\right)^{7-r}\left(-\dfrac{2}{x}\right)^r$ $=\binom{7}{r}\left(x^{21-3r}\right)(-2)^r(x)^{-r}$ $=\binom{7}{r}(-2)^r x^{21-4r}$	B1 – general formula
Power of x = 21 – 4r	M1 f.t. – finding powers of x
Since **4*r* is an even number** for all non-negative integer values of r and 21 is an odd number, then **21 – 4*r* is always an odd number**. Therefore, there are only odd powers of x in this expansion.	A1 – conclusion

(b) Find the term independent of x in the expansion of $\left(x^3-\dfrac{2}{x}\right)^7\left(\dfrac{5}{x}-2x^2\right)$. [3]

Consider $21-4r=1$, $4r=20$ $r=5$	M1 f.t. – identifying x term
$\left(x^3-\dfrac{2}{x}\right)^7\left(\dfrac{5}{x}-2x^2\right)=\left[\ldots+\binom{7}{5}(-2)^5x^{21-4(5)}+\ldots\right]\left(\dfrac{5}{x}-2x^2\right)$ $=(\ldots-672x+\ldots)\left(\dfrac{5}{x}-2x^2\right)$	M1 f.t. – expansion
Term independent of $x=-672\times5$ $=-3360$	A1

3 The expression $7\sin\theta + 3\cos\theta$ is defined for $0° \le \theta \le 360°$.

(a) Using $R\sin(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, solve the equation

$$7\sin\theta = 5 - 3\cos\theta .$$

[5]

$7\sin\theta + 3\cos\theta = R\sin(\theta+\alpha)$	
$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$	
Comparing,	
$7 = R\cos\alpha$ and $3 = R\sin\alpha$	
$R = \sqrt{7^2+3^2}$	M1 – attempt to find *R*
$= \sqrt{58}$	
$\tan\alpha = \dfrac{3}{7}$	
$\alpha = \tan^{-1}\left(\dfrac{3}{7}\right)$	M1 – attempt to find α
$= 23.1985...°$	[only accept $\tan^{-1}\left(\dfrac{3}{7}\right)$ or $\tan^{-1}\left(\dfrac{7}{3}\right)$]
$\therefore 7\sin\theta + 3\cos\theta = \sqrt{58}\sin(\theta + 23.1985...°)$	
$7\sin\theta = 5 - 3\cos\theta$	
$7\sin\theta + 3\cos\theta = 5$	
$\sqrt{58}\sin(\theta + 23.1985...°) = 5$	M1 f.t. – substitute *R*-form
$\sin(\theta + 23.1985...°) = \dfrac{5}{\sqrt{58}}$	
basic angle $= \sin^{-1}\left(\dfrac{5}{\sqrt{58}}\right)$	M1 f.t. – basic angle
$= 41.0359...°$	
$\theta + 23.1985...° = 41.0359...°$ or $180° - 41.0359...°$	
$\theta = 17.8°$ or $115.8°$ (1 dec. pl.)	A1

(b) State the largest and smallest values of $(7\sin\theta+3\cos\theta)^2-12$ and find the corresponding values of θ. [4]

largest value of $(7\sin\theta+3\cos\theta)^2-12=\left(\sqrt{58}\right)^2-12$ or $\left(-\sqrt{58}\right)^2-12$

$=46$ B1

occurs when $\theta+23.1985...^\circ=90^\circ$ or 270°

$\theta=66.8^\circ$ or 246.8° (1 dec. pl.) B1

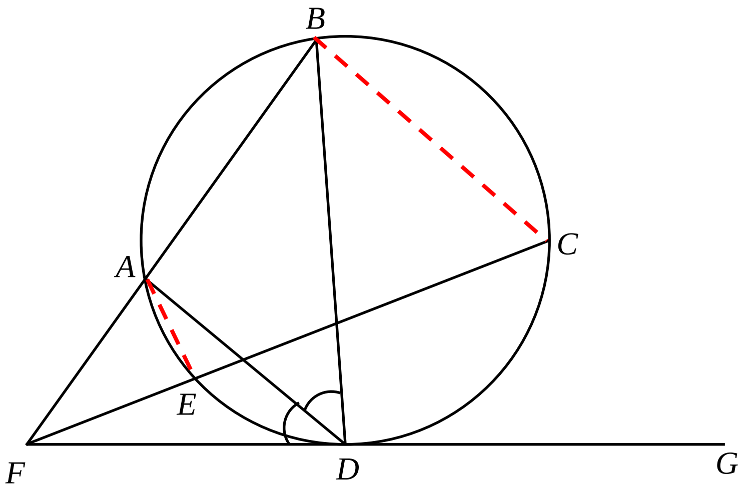
smallest value of $(7\sin\theta+3\cos\theta)^2-12=(0)^2-12$

$=-12$ B1

occurs when $\theta+23.1985...^\circ=180^\circ$ or 360°

$\theta=156.8^\circ$ or 336.8° (1 dec. pl.) B1

4



The diagram shows a circle passing through the points *A*, *B*, *C*, *D* and *E*. The straight line *FDG* is tangent to the circle at *D* while *FAB* and *FEC* are secant lines.
Given that angle *FDA* = angle *ADB*,

(a) show that triangle *ABD* is an isosceles triangle, [2]

$\angle ABD = \angle FDA$ (**alternate segment theorem**) B1
$= \angle ADB$

Since $\angle ABD = \angle ADB$, they form **base angles of isosceles triangle**. B1
Thus, triangle *ABD* is an isosceles triangle. (shown)

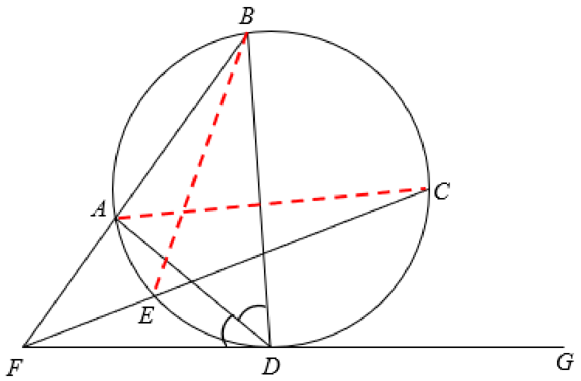
(b) prove that $AF \times BF = EF \times CF$. [3]

$\angle FAE = \angle FCB$ (**exterior $\angle$ of cyclic quadrilateral**) B1

$\angle AFE = \angle CFB$ (**common $\angle$**) B1

Thus, triangle *AFE* is similar to triangle *CFB*.

$\frac{AF}{CF} = \frac{EF}{BF}$ (**ratio of corresponding sides are equal**) B1
$AF \times BF = EF \times CF$ (proved)



Alternative:

$\angle FBE = \angle FCA$ (**$\angle$s in same segment**) B1

$\angle EFB = \angle AFC$ (**common $\angle$**) B1

Thus, triangle *EFB* is similar to triangle *AFC*.

$$\frac{AF}{EF} = \frac{CF}{BF}$$ (**ratio of corresponding sides are equal**) B1

$AF \times BF = EF \times CF$ (proved)

5 An object is heated in an oven until it reaches a temperature of X °C. It is then allowed to cool under room temperature. Its temperature, T °C, can be modelled by $T = 18 + 62e^{-kt}$, where t is the time in minutes since the object starts cooling.

(a) Find the value of X. [1]

When $t = 0$, $T = X$.

$$X = 18 + 62e^{-k(0)}$$
$$= 18 + 62$$
$$= 80$$

B1

When $t = 10$, the temperature of the object is 65 °C.

(b) Find the temperature of the object an hour later, giving your answer to one decimal place. [5]

When $t = 10$, $T = 65$.

$$65 = 18 + 62e^{-k(10)}$$

B1 – sub. 10 and 65 correctly

$$\frac{47}{62} = e^{-10k}$$
$$\ln\frac{47}{62} = \ln e^{-10k}$$

M1 – Take ln on both sides

$$\ln\frac{47}{62} = -10k$$
$$k = -\frac{1}{10}\ln\frac{47}{62}$$
$$= 0.0276986...$$

M1 – finding k

When $t = 60$,

$$T = 18 + 62e^{-\left(-\frac{1}{10}\ln\frac{47}{62}\right)(60)}$$
$$= 29.765...$$
$$= 29.8 \text{ °C (1 dec. pl.)}$$

When $t = 70$,

$$T = 18 + 62e^{-\left(-\frac{1}{10}\ln\frac{47}{62}\right)(70)}$$
$$= 26.9193...$$
$$= 26.9 \text{ °C (1 dec. pl.)}$$

M1 f.t.– sub. $t = 60$ or 70

A1

(c) What does the model suggest about the room temperature? Explain your answer. [2]

Since $e^{-kt} > 0$, then $62e^{-kt} > 0$.
Thus, **as t becomes larger**, **$62e^{-kt}$ approaches zero**.
Therefore, T will **approach 18 °C** as t becomes larger.
Hence, the room temperature is **18 °C.**

B1 – explanation

B1 – temperature

6 The equation of a curve is $y=\ln\left(\dfrac{2x-1}{3x-1}\right)$, where $x>\dfrac{1}{2}$.

(a) Find $\dfrac{dy}{dx}$, expressing it as a single fraction. [3]

$$y=\ln\left(\frac{2x-1}{3x-1}\right)$$
$$=\ln(2x-1)-\ln(3x-1)$$

B1 – quotient law of logarithms

$$\frac{dy}{dx}=\frac{2}{2x-1}-\frac{3}{3x+1}$$

M1 f.t. – differentiate either term

$$=\frac{2(3x-1)-3(2x-1)}{(2x-1)(3x-1)}$$
$$=\frac{1}{(2x-1)(3x+1)}$$

A1 – single fraction

<u>Alternative:</u>

$$\frac{dy}{dx}=\frac{\dfrac{2(3x-1)-3(2x-1)}{(3x-1)^2}}{\dfrac{2x-1}{3x-1}}$$

B1 – differentiate ln in the form $\dfrac{f'(x)}{f(x)}$

M1 – apply quotient rule to numerator.

$$=\frac{\dfrac{1}{(3x-1)^2}}{\dfrac{2x-1}{3x-1}}$$
$$=\frac{1}{(2x-1)(3x+1)}$$

A1 – single fraction

(b) Explain why the curve will be almost parallel to the x-axis as x becomes very large. [2]

Since $(2x-1)(3x+1)$ becomes a very large number as x becomes very large, $\dfrac{dy}{dx}$ approaches **<u>0</u>**, M1

the curve will **<u>almost horizontal</u>**, thus it will be almost parallel to the x-axis. A1

(c) Find the value of x at the instant when the rate of change of x is twice the rate of change of y. [3]

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$	
$\frac{dy}{dt}=\frac{1}{(2x-1)(3x-1)}\times 2\left(\frac{dy}{dt}\right)$	M1 f.t. – substitution
$1=\frac{2}{(2x-1)(3x-1)}$	B1 – substitute $\frac{dx}{dt}=2\frac{dy}{dt}$
$(2x-1)(3x-1)=2$	
$6x^2-5x+1-2=0$	
$6x^2-5x-1=0$	
$(x-1)(6x+1)=0$	
$x=1$ or $-\frac{1}{6}$ (rej. $\because x>\frac{1}{2}$)	A1

7 **(a)** Show that $\tan A + \cot A = 2\operatorname{cosec} 2A$. [3]

Alternative:

$$\begin{aligned}\text{LHS} &= \tan A + \cot A \\ &= \tan A + \frac{1}{\tan A} \\ &= \frac{\tan^2 A + 1}{\tan A} \\ &= \frac{\sec^2 A}{\tan A} \\ &= \frac{\frac{1}{\cos^2 A}}{\frac{\sin A}{\cos A}} \\ &= \frac{1}{\sin A \cos A} \\ &= \frac{2}{2\sin A \cos A} \\ &= 2\operatorname{cosec} 2A \text{ (shown)}\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \tan A + \cot A \\ &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} \\ &= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \\ &= \frac{1}{\sin A \cos A} \\ &= \frac{2}{2\sin A \cos A} \\ &= \frac{2}{\sin 2A} \\ &= 2\operatorname{cosec} 2A \text{ (shown)}\end{aligned}$$

M1 – $\frac{\sin A}{\cos A}$ or $\frac{1}{\tan A}$

M1 – for $\sin^2 A + \cos^2 A = 1$ or for $\tan^2 A + 1 = \sec^2 A$

A1 – award only if last 3 steps are shown

(b) Hence, solve the equation $\dfrac{1}{\tan A + \cot A} = \dfrac{1}{4}$ for $0 \le A \le 2\pi$. [3]

$$\begin{aligned}\frac{1}{\tan A + \cot A} &= \frac{1}{4} \\ \tan A + \cot A &= 4 \\ 2\operatorname{cosec} 2A &= 4 \\ \operatorname{cosec} 2A &= 2 \\ \sin 2A &= \frac{1}{2} \\ \text{basic } \angle &= \sin^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{6} \\ 2A &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\ A &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\end{aligned}$$

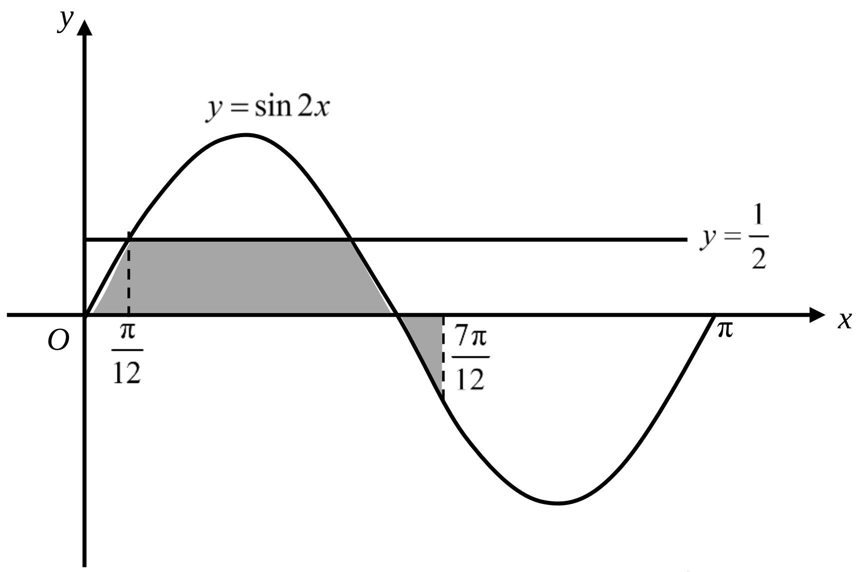
B1 – substitution of **(a)**

M1 – for finding $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

A1

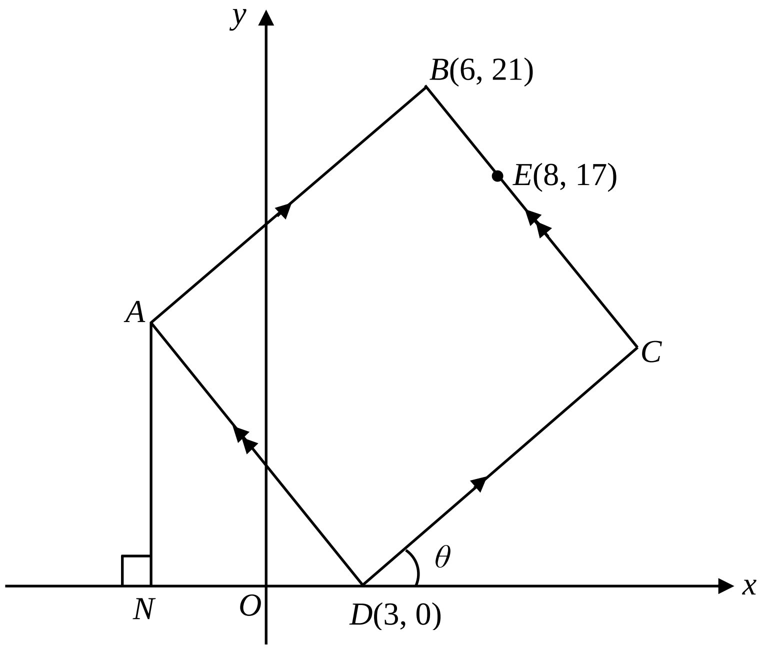
(c) The diagram shows, for $0 \le x \le \pi$, the curve $y = \sin 2x$ and the line $y = \frac{1}{2}$.

Showing all your working, find the area of the shaded region. [5]



Area bounded by curve, x-axis and lines $x = 0$ and $x = \frac{\pi}{12}$	
$= \int_0^{\frac{\pi}{12}} \sin 2x \, dx$	
$= \left[\frac{-\cos 2x}{2} \right]_0^{\frac{\pi}{12}}$	M1 – integrate sin $2x$
$= \left[-\frac{\cos \frac{\pi}{6}}{2} \right] - \left[-\frac{\cos 0}{2} \right]$	DM1 – evaluate with either limits 0 to $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ to $\frac{\pi}{2}$ or $\frac{\pi}{2}$ to $\frac{7\pi}{12}$
$= \left(-\frac{\sqrt{3}}{4} + \frac{1}{2} \right)$ units^2 or 0.066987... units^2	
Area of rectangle $= \left(\frac{5\pi}{12} - \frac{\pi}{12} \right) \times \frac{1}{2}$	M1 – area of rectangle
$= \frac{\pi}{6}$ units^2	
Area of shaded region $= 3 \times \left(-\frac{\sqrt{3}}{4} + \frac{1}{2} \right) + \frac{\pi}{6}$	M1 f.t. – area under curve × 3
$= 0.725$ units^2 (3 sig.fig)	A1

8



The diagram shows a parallelogram with vertices A, $B(6, 21)$, C and $D(3, 0)$. The point $E(8, 17)$ lies on BC. The line CD makes an angle θ with the positive x-axis such that $\tan\theta = 1$. A line is drawn, parallel to the y-axis, from A to meet the x-axis at N.

(a) Show that the coordinates of A are $(-3, 12)$. [5]

$m_{AB} = m_{CD}$ $= 1$	B1 – gradient = $\tan\theta$
Eq. of AB: $y - 21 = (1)(x-6)$ $y = x + 15$ ...(1)	
$m_{AD} = m_{BE}$ $= \dfrac{17-21}{8-6}$ $= -2$	B1 – gradient formula
Eq. of AD: $y - 0 = -2(x-3)$ $y = -2x + 6$ ...(2)	M1 f.t. – finding equation of AD or AB
Sub. (1) into (2): $x + 15 = -2x + 6$ $3x = -9$	M1 f.t. – substitution
$x = -3$ sub. into (1): $y = 12$	
$\therefore$ $A(-3, 12)$ (shown)	A1 (A.G.)

Alternative:

Eq. of CD: $y-0=(1)(x-3)$

$y=x-3$ ...(1)

$m_{BC}=m_{BE}$

$=\frac{17-21}{8-6}$

$=-2$

B1 – gradient formula

Eq. of BC: $y-21=-2(x-6)$

$y=-2x+33$ ...(2)

M1 f.t. – finding equation of BC

Sub. (1) into (2):

$x-3=-2x+33$

$3x=36$

$x=12$ sub. into (1):

$y=9$

M1 f.t. – substitution

$\therefore$ ***C*(12, 9)**

Midpoint of AC = Midpoint of BD

$$\left(\frac{x+12}{2}, \frac{y+9}{2}\right)=\left(\frac{3+6}{2}, \frac{0+21}{2}\right)$$

$$\left(\frac{x+12}{2}, \frac{y+9}{2}\right)=\left(\frac{9}{2}, \frac{21}{2}\right)$$

M1 f.t. – equating midpoints

Comparing,

$\frac{x+12}{2}=\frac{9}{2}$ $\frac{y+9}{2}=\frac{21}{2}$

$x+12=9$ and $y+9=21$

$x=-3$ $y=12$

A1 (A.G.)

$\therefore$ ***A*(−3, 12) (shown)**

(b) Hence, find the area of parallelogram *ABCD*. [2]

area of parallelogram $ABCD = 2 \times$ area of triangle ABD	
$= 2 \times \frac{1}{2}\begin{vmatrix} -3 & 3 & 6 & -3 \\ 12 & 0 & 21 & 12 \end{vmatrix}$	M1 – allow clockwise direction
$= 2 \times \frac{1}{2}\left[(63+72)-(-63+36)\right]$	
$= 135 + 27$	
$= 162 \text{ units}^2$	A1
Alternative: *If students have found the coordinates of C in part (a),*	
area of parallelogram $ABCD = \frac{1}{2}\begin{vmatrix} 3 & 12 & 6 & -3 & 3 \\ 0 & 9 & 21 & 12 & 0 \end{vmatrix}$	M1 f.t. – allow clockwise direction
$= \frac{1}{2}\left[(27+252+72)-(36-63+54)\right]$	
$= \frac{1}{2}(351-27)$	
$= 162 \text{ units}^2$	A1

(c) A point F with y-coordinate of 5 lies on the line CD.
Explain why $AEFN$ is a parallelogram. [5]

Eq. of CD: $y - 0 = (1)(x - 3)$

$y = x - 3 \quad ...(3)$ — B1

Let coordinates of F be $(x, 5)$.

Sub. $(8, y)$ into (3): — M1 f.t. – substitution

$y = 8 - 3$

$= 5$

$\therefore$ ***F*(8, 5)**

Coordinates of $N = (-3, 0)$

length of $AN = \sqrt{(-3+3)^2 + (12-0)^2}$

$= 12$ units — M1 f.t. – finding either lengths

length of $EF = \sqrt{(8-8)^2 + (17-5)^2}$

$= 12$ units

Since *F* lies directly below *E*, the line *EF* is a vertical line. Thus, the lines *EF* and *AN* are parallel. — A1 – vertical lines are parallel

And that the lengths of *AN* and *EF* are equal (i.e., 12 units),

they form a pair of parallel and equal opposite sides. Thus, *AEFN* is a parallelogram. — A1 – conclusion

Alternative 1:

gradient of $AE = \dfrac{17-12}{8-(-3)} = \dfrac{5}{11}$ gradient of $FN = \dfrac{5-0}{8-(-3)} = \dfrac{5}{11}$ — M1 f.t. – finding either gradients

Since gradients of lines *AE* and *FN* are the same, the lines *AE* and *FN* are parallel.

Since *F* lies directly below *E*, the line *EF* is a vertical line. Thus, the lines *EF* and *AN* are parallel. — A1 – vertical lines are parallel

Since there are 2 pairs of parallel lines, *AEFN* is a parallelogram. — A1 – conclusion

Alternative 2:

Midpoint of $AF = \left(\frac{-3+8}{2}, \frac{12+5}{2}\right)$ $= \left(\frac{5}{2}, \frac{17}{2}\right)$	M1 f.t. – midpoint of *AF*
Midpoint of $NE = \left(\frac{-3+8}{2}, \frac{0+17}{2}\right)$ $= \left(\frac{5}{2}, \frac{17}{2}\right)$	M1 f.t. – midpoint of *AE*
Since the diagonals intersect at the same point, *AEFN* is a parallelogram.	A1 – conclusion

9 **(a)** Differentiate $x\ln x$ with respect to x. [2]

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) = x\left(\frac{1}{x}\right) + (1)\ln x$$
$$= 1 + \ln x$$

B1, B1 – for each term

(b) A curve $y = \mathrm{f}(x)$ is such that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 24x^2 + \frac{16}{x}$, where $x > 0$. The line $y = 24x - 40$ is parallel to the tangent of the curve at $P(1, -16)$.

By using the result found in part **(a)**, find the equation of the curve. [6]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int 24x^2 + \frac{16}{x}\,\mathrm{d}x$$
$$= \frac{24x^3}{3} + 16\ln x + c$$
$$= 8x^3 + 16\ln x + c, \text{ where } c \text{ is a constant.}$$

B1, B1 – for each integral

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 24$.

$$24 = 8(1)^3 + 16\ln(1) + c$$
$$c = 16$$

M1 f.t. – finding c

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3 + 16\ln x + 16$$

$y = \int 8x^3 + 16\ln x + 16\,\mathrm{d}x$ **or** $y = \int 8x^3 + 16(1 + \ln x)\,\mathrm{d}x$

$$= \frac{8x^4}{4} + 16\left(x\ln x - \int 1\,\mathrm{d}x\right) + 16x + c_1$$
$$= \frac{8x^4}{4} + 16(x\ln x - x + c_2) + 16x + c_1$$
$$= 2x^4 + 16x\ln x - 16x + 16x + c_3$$
$$2x^4 + 16x\ln x + c_3, \text{ where } c_1, c_2 \text{ and } c_3 \text{ are constants.}$$

M1 f.t. – reverse

When $x = 1$, $y = -16$.

$$-16 = 2(1)^4 + 16(1)\ln(1) + c_3$$
$$-16 = 2 + c_3$$
$$c_3 = -18$$

M1 f.t. – finding c_3

$$\therefore y = 2x^4 + 16x\ln x - 18$$

A1

Continuation of working space for question **9(b)**.

(c) Explain why the condition $x > 0$ is necessary. [1]

ln *x* is defined for $x > 0$. | B1

or

ln *x* is undefined for $x < 0$.

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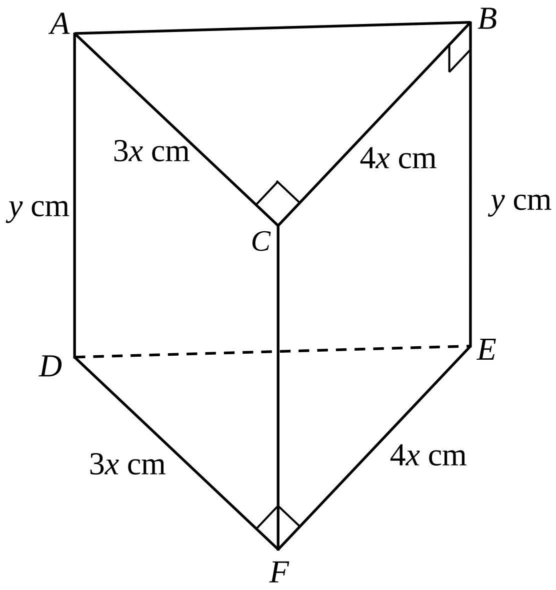

The diagram shows a solid prism with right-angled triangular ends that are perpendicular to the parallel sides *AD*, *BE* and *CF*, which are each *y* cm in length. The right-angled triangular ends have sides *AC* and *DF*, which are $3x$ cm, and sides *BC* and *EF*, which are $4x$ cm.

Given that the volume of the prism is 1200 cm^3,

(a) find an expression for y in terms of x, [2]

$\text{Vol. of prism} = \frac{1}{2}(3x)(4x)y$	B1 – volume formula
$1200 = 6x^2 y$	
$y = \frac{200}{x^2}$	B1

(b) show that the total surface area of the prism, S cm^2, is given by $S = 12x^2 + \dfrac{2400}{x}$. [3]

$AB = \sqrt{(3x)^2 + (4x)^2}$ $= \sqrt{25x^2}$ $= 5x$ cm	B1 – finding length of AB
Total surface area $= 2 \times \frac{1}{2}(3x)(4x) + 3xy + 4xy + 5xy$ $= 12x^2 + 12xy$ $= 12x^2 + 12x\left(\frac{200}{x^2}\right)$	M1 – at least 3 correct surfaces
$= 12x^2 + \frac{2400}{x}$ (shown)	A1 (A.G.)

(c) Given that x can vary, find the value of x for which the total surface area of the prism is a stationary value. [3]

$\frac{dS}{dx} = 24x - \frac{2400}{x^2}$	B1 – differentiation
For S to be stationary, $\frac{dS}{dx} = 0$ $24x - \frac{2400}{x^2} = 0$	M1 f.t.
$24x = \frac{2400}{x^2}$ $x^3 = 100$ $x = \sqrt[3]{100}$ $= 4.64$ (3 sig. fig.)	A1

(d) Explain why this value of x gives the smallest surface area possible. [2]

$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 24 + \frac{4800}{x^3}$	M1 f.t. – differentiation or 1st derivative test
When $x = \sqrt[3]{100}$, $\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 24 + \frac{4800}{\left(\sqrt[3]{100}\right)^3}$ $= 72 \quad > 0 \quad \because$ minimum	
Since $\frac{\mathrm{d}^2S}{\mathrm{d}x^2} > 0$, the surface area is the smallest when x = 4.64.	A1 – conclude '> 0' (must show '72')

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