2012 RI H2 Mathematics Preliminary Examination Paper 2

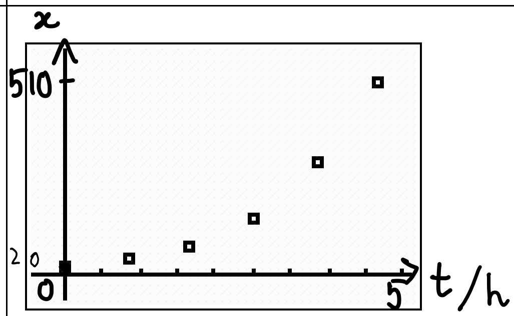
Qn	Solution
1 **[5]**	Let P_n be the statement $\sum_{r=1}^{n} \cos r\theta = \dfrac{\sin\left(n+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}, \quad n \in \mathbb{Z}^+$. When $n=1$, $\text{LHS} = \cos\theta$ $\text{RHS} = \dfrac{\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$ $= \dfrac{2\cos\left[\frac{1}{2}\left(\frac{3\theta}{2}+\frac{\theta}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{3\theta}{2}-\frac{\theta}{2}\right)\right]}{2\sin\frac{\theta}{2}}$ $= \dfrac{2\cos\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \cos\theta = \text{LHS}$ Hence P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} \cos r\theta = \dfrac{\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$. To prove P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} \cos r\theta = \dfrac{\sin\left(k+\frac{3}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$.

	$\text{LHS} = \sum_{r=1}^{k} \cos r\theta \ + \ \cos(k+1)\theta$ $= \dfrac{\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \ + \ \cos(k+1)\theta$ $= \dfrac{\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2} + 2\cos(k+1)\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$ $= \dfrac{\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2} + \sin\left[(k+1)\theta+\frac{\theta}{2}\right] - \sin\left[(k+1)\theta-\frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$ $= \dfrac{\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{\theta}{2} + \sin\left(k+\frac{3}{2}\right)\theta - \sin\left(k+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$ $= \dfrac{\sin\left(k+\frac{3}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \text{RHS}$ Hence P_k is true implies P_{k+1} is true. Since P_1 is true, and P_k is true implies P_{k+1} is true, by Mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$
2(i) **[6]**	$\dfrac{dx}{dt} = -kx, k > 0$ $\Rightarrow \int \frac{1}{x}\,dx = \int -k\,dt$ $\Rightarrow \ln x = -kt + c, \because x > 0$ $\Rightarrow x = Ae^{-kt}$ When $t = 0$, $x = 80$, thus $A = 80$. When $t = 3$, $x = 20$, $20 = 80e^{-3k}$ $\Rightarrow k = -\frac{1}{3}\ln\frac{1}{4} = \frac{1}{3}\ln 4$. Thus $x = 80e^{-\frac{t}{3}\ln 4}$
2(ii) **[3]**	When $t = 6$, $x = 80e^{-2\ln 4} = \frac{1}{16}(80)$ Just before the (n+1)th injection, the amount of drug present in the blood stream = $\frac{1}{16}u_n$

	Immediately after the (*n*+1)th injection, amount of drug present, $u_{n+1}=80+\frac{1}{16}u_n$
[1]	In the long run, the amount present approaches the value 85.3 (3 s.f.).
3a **[3]**	$\begin{pmatrix}-3\\1\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}0\\9\\c\end{pmatrix}+\mu\begin{pmatrix}22/7\\-4\\-2-c\end{pmatrix}$ $-3-\lambda=\frac{22}{7}\mu \quad ---(1)$ $-8+4\lambda=-4\mu \quad ---(2)$ $5+3\lambda=c-2\mu-c\mu \quad ---(3)$ From (1) and (2) we have $\lambda+\frac{22}{7}\mu=-3 \quad ---(4)$ $4\lambda+4\mu=8 \quad ---(5)$ Solve (4) and (5) by GC $\lambda=\frac{13}{3},\mu=-\frac{7}{3}$ Sub into (5) $\quad 5+3\left(\frac{13}{3}\right)=c-(2+c)\left(-\frac{7}{3}\right)\Rightarrow \quad c=4$.
[3]	Let the foot of perpendicular be F . Then $\overrightarrow{AF}\cdot\begin{pmatrix}-1\\4\\3\end{pmatrix}=0$ $\left(\begin{pmatrix}-3\\1\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\3\end{pmatrix}-\begin{pmatrix}0\\9\\4\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\4\\3\end{pmatrix}=0$ $3+4+15+\lambda(1+16+9)-(36+12)=0$ $\lambda=1$ The coordinates of F are $(-4,5,8)$.
3b **[2]**	$c=1=d$
3bi **[3]**	Shortest distance is

	$\dfrac{\left\lvert\left(\begin{pmatrix}0\\9\\1\end{pmatrix}-\begin{pmatrix}-3\\1\\5\end{pmatrix}\right)\times\begin{pmatrix}-1\\4\\3\end{pmatrix}\right\rvert}{\sqrt{1+4^2+3^2}}=\dfrac{\left\lvert\begin{pmatrix}3\\8\\-4\end{pmatrix}\times\begin{pmatrix}-1\\4\\3\end{pmatrix}\right\rvert}{\sqrt{26}}$ $=\dfrac{1}{\sqrt{26}}\left\lvert\begin{pmatrix}40\\-5\\20\end{pmatrix}\right\rvert$ $=\dfrac{5\sqrt{8^2+1+4^2}}{\sqrt{26}}=\dfrac{45}{\sqrt{26}}$
3bii [2]	$\mathbf{r}\cdot\begin{pmatrix}8\\-1\\4\end{pmatrix}=\begin{pmatrix}0\\9\\1\end{pmatrix}\cdot\begin{pmatrix}8\\-1\\4\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}8\\-1\\4\end{pmatrix}=-5$
3biii [2]	$p_1:\quad 8x-y+4z=-5$ $p_2:\quad 8x-(-y)+4(z-2)=-5$ $8x+y+4z=3$ $\mathbf{r}\cdot\begin{pmatrix}8\\1\\4\end{pmatrix}=3$
4(a) [2]	$z^6=1$ $z=e^{k\left(\frac{\pi}{3}\right)i},k=-2,-1,0,1,2,3$
4bi [2]	$\angle P_kOP_{k+1}=\dfrac{2\pi}{n}$. $OQ_2=\cos\left(\dfrac{2\pi}{n}\right)\quad P_1Q_2=\sin\left(\dfrac{2\pi}{n}\right)$
4bii [2]	For k = 2, 3, …,n-1 $OQ_{k+1}=OQ_k\cos\left(\dfrac{2\pi}{n}\right)\Rightarrow\dfrac{OQ_{k+1}}{OQ_k}=\cos\left(\dfrac{2\pi}{n}\right)$ (constant) Since the ratio of consecutive terms is a constant, $OQ_2,OQ_3,...,OQ_n$ are in GP.

	For k = 3, 4, …,n-1 $Q_kQ_{k+1}=OQ_k\sin\left(\frac{2\pi}{n}\right)$ $\frac{Q_kQ_{k+1}}{Q_{k-1}Q_k}=\frac{OQ_k\sin\left(\frac{2\pi}{n}\right)}{OQ_{k-1}\sin\left(\frac{2\pi}{n}\right)}=\cos\left(\frac{2\pi}{n}\right)$ Since the ratio of consecutive terms is a constant, Q_2Q_3, Q_3Q_4, ..., $Q_{n-1}Q_n$ are in GP.
4bii i [4]	$P_1Q_2+Q_2Q_3+\cdots+Q_{n-1}Q_n$ $=\sin\left(\frac{2\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)+\cdots+\left(\cos\left(\frac{2\pi}{n}\right)\right)^{n-2}\sin\left(\frac{2\pi}{n}\right)$ $=\sin\left(\frac{2\pi}{n}\right)\left[\frac{1-\left(\cos\left(\frac{2\pi}{n}\right)\right)^{n-1}}{1-\cos\left(\frac{2\pi}{n}\right)}\right]$ $<\frac{\sin\left(\frac{2\pi}{n}\right)}{1-\cos\left(\frac{2\pi}{n}\right)}\because\left(1-\cos\left(\frac{2\pi}{n}\right)^{n-1}\right)<1$ $=\frac{2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}{1-\left(1-2\sin^2\left(\frac{\pi}{n}\right)\right)}=\cot\left(\frac{\pi}{n}\right)$
5 [3]	Let X be the monthly rainfall in a region. Since n = 60 is large, by Central Limit Theorem, $\overline{X}\sim N(159,\frac{45^2}{60})$ approximately $P(\overline{X}<150)=0.0607$ (3s.f) Assume that the rainfall in each month is independent and identically distributed.
6 [4]	Consider users from the following 5 strata: below 13 year-olds; 13 – 21 year-olds; 22 – 40 year-olds; 41 – 60 year-olds; above 60 year-olds. Conduct the survey on the first 24 people from each stratum who exits a particular train station (eg City Hall MRT at 12 noon on Weekday). Quota sampling may not give a fair representation of the perception of the general population of MRT users. For example, users who exit from other stations (say along Circle line or NE line) or who travel during peak rush hour in the morning or early evening may have a different travel experience. OR as the surveyor may survey people who appear more approachable and willing to

	participate in the survey such as tourists. No, as the sampling frame, a list of all users of the train system is not easily obtainable.
7i **[2]**	Number of ways in which 1 man and 2 women are chosen $= {}^{6}C_{1}\,{}^{n}C_{2} = \dfrac{6n(n-1)}{2} = 3n(n-1)$ Number of ways in which 3 women are chosen $= {}^{n}C_{3} = \dfrac{n(n-1)(n-2)}{3!} = \dfrac{n(n-1)(n-2)}{6}$ Total number of ways = $\dfrac{n(n-1)(n-2)}{6} + 3n(n-1) = \dfrac{n(n-1)(n+16)}{6}$ So, $a = \dfrac{1}{6}$
7ii **[2]**	$\mathrm{f}(n) = \dfrac{n(n-1)(n+16)}{6}$ is an increasing function for $n \geq 1$. From the GC, when n = 9, f(n) = 300 when n = 10, f(n) = 390 Thus, $n \in \square$, $n > 9$ (or $n \geq 10$)
[1]	X does not follow a binomial distribution. Reason: The event that a committee member is a man is not independent of the event another committee member is a man. OR The gender of a committee member is not independent of that of another.
8i **[2]**	$r = 0.929$ (3 s.f) r is close to 1, which indicates a strong linear correlation between x and t. So it suggests that a linear model is appropriate.
8ii **[2]**	 The scatter diagram shows that as t increases, x increases at an increasing rate, which suggests that the data points may fit better with $\ln x = a + bt$.
8iii **[2]**	$\ln x = 3.02 + 0.654t$ (3 s.f) r = 0.999
8iv **[2]**	When $x = 200$, $\ln 200 = 3.0210 + 0.65432t$ $t = 3.5$ (1 d.p) Since x = 200 lies within the given range of data of x between 20 to 510 and there is a strong linear correlation between ln x and t as the product moment correlation coefficient is 0.999 (close to 1), the estimate is reliable.
9i	Let R be the mass of a box of red grapes in grams.

[2]	Then $R \sim N(300, 40^2)$ Required probability $= \left[{}^9C_1 \times \mathrm{P}(R < 310)^8 \mathrm{P}(R \geq 310) \right] \mathrm{P}(R \geq 310) = 0.0239$ (3 s.f)
9ii **[3]**	Let G be the mass of a box of green grapes in grams. Then $G \sim N(150, 20^2)$ Let $X = R_1 + R_2 + R_3 - 3G$ $\mathrm{E}(X) = 3\mathrm{E}(R) - 3\mathrm{E}(G) = 450$ $\mathrm{Var}(X) = 3\mathrm{Var}(R) + 3^2\,\mathrm{Var}(G) = 8400$ Then $X \sim N(450, 8400)$ $P(\lvert X \rvert \geq 500) = 1 - P(\lvert X \rvert < 500) = 1 - P(-500 < X < 500) = 0.293$ (3 s.f)
9iii **[4]**	Let $Y = 0.005\left(R_1 + R_2 + ... + R_{10}\right) - 0.006\left(G_1 + G_2 + ... + G_{10}\right)$ $\mathrm{E}(Y) = 0.005 \times 10\mathrm{E}(R) - 0.006 \times 10\mathrm{E}(G) = 6$ $\mathrm{Var}(Y) = 0.005^2 \times 10\mathrm{Var}(R) + 0.006^2 \times 10\mathrm{Var}(G) = 0.544$ Then $Y \sim N(6, 0.544)$ $P(Y \leq 7) = 0.912$ (3 s.f)
10i **[2]**	$\mathrm{P}(A \cap B)$ = P(rise, rise, fall) + P(rise, fall,fall) $= (0.1 \times 0.7 \times 0.3) + (0.1 \times 0.3 \times 0.9)$ $= 0.048$
10ii **[3]**	$\mathrm{P}(B)$ $= \mathrm{P(A \cap B) + P(A' \cap B)}$ $= 0.048 +$ P(fall, fall, fall) + P(fall, rise, fall) $= 0.048 + (0.9 \times 0.9 \times 0.9) + (0.9 \times 0.1 \times 0.3) = 0.804$
10iii **[2]**	$\mathrm{P}(A' \cup B) = 1 - [\mathrm{P(A) - P(A \cap B)}]$ $= 1 - [0.1 - 0.048] = 0.948$
10iv **[2]**	$\mathrm{P}(B \mid A) = \dfrac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}} = \dfrac{0.048}{0.1}$ $= 0.48$
[1]	Since [P(B) = 0.804] ≠ [P($B \mid A$)= 0.48] $\therefore$ A and B are not independent.
11i **[2]**	Let X be the number of prizes given out in 2 days, out of 40. $X \sim B(40, 0.1)$ $P(X > 2) = 1 - P(X \leq 2)$ $= 1 - 0.22281$ $= 0.777$ (3s.f)
(ii) **[4]**	Let Y be the number of prizes given out in 7 days, out of 140 $Y \sim B(140, 0.1)$ Since n = 140 is large such that np = 14 > 5, n(1-p) = 126 > 5, $Y \sim N(14, 12.6)$ approximately

	$P(Y \le k) = P(Y \le k+0.5) < 0.4$ by continuity correction From GC, when k = 12, $P(Y \le 12.5) = 0.3363 < 0.4$ when k = 13, $P(Y \le 13.5) = 0.44399 > 0.4$ $\therefore$ largest $k = 12$
(iii) **[2]**	$P(X < 3) = P(X \le 2) = 0.223$ Let W be the number of restaurants, out of 10 that gives out less than 3 prizes in 2 days. $W \sim B(10, 0.223)$ $P(W \le 2) = 0.608$ (3s.f)
[3]	Let R be the number of prizes given out in a day, out of 50, by the restaurant. $R \sim B(50, 0.05)$ Since $n = 50$ is large and $p = 0.05$ is small such that $np = 2.5 < 5$, $R \sim \text{Po}(2.5)$ approximately Required probability $= (P(R \ge 1))^3$ $= (1 - 0.082085)^3$ $= 0.773$ (3sf)
12i **[4]**	To test $\text{H}_0 : \mu = 250$ vs $\text{H}_1 : \mu > 250$ Perform a 1–tail test at 5% level of significance. Under H_0, $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$ where $\mu_0 = 250$ From the sample, $\bar{x} = 251.25$, $s = 2.1213$, $n = 8$ Using a $t-$test, $p-$value$= 0.0698$ (3 s.f.) Since $p-$value = 0.0698 > 0.05, we do not reject H_0 and conclude that there is insufficient evidence, at 5% level of significance, that the mean quantity of coffee in a cup is more than 250 ml. In order to use a t-test., we need to assume that X, the quantity of coffee dispensed in a cup follows a normal distribution.
12ii **[1]**	A 5% significance level in this context means that the probability of concluding that the mean quantity of coffee in a cup is **more than** 250ml when it actually is 250ml is 0.05.
[5]	$\text{H}_0 : \mu = 250$ vs $\text{H}_1 : \mu \neq 250$ Under H₀, $\bar{X} \sim N(250, \frac{5^2}{8})$ Since H_0 is rejected, $\text{P}(\bar{X} > c) < 0.025$ or $\text{P}(\bar{X} < c) < 0.025$ $\text{P}(\bar{X} < c) > 0.975$ or $\text{P}(\bar{X} < c) < 0.025$ $c > 253$ (3s.f.) or $c < 247$ (3s.f.) Yes, we will need to assume that X is normally distributed as the sample size $n = 8$ is small.