Statistics 3 Tutorial: Discrete Random Variables

Additional Practice Questions

1. An unbiased cubical dice has three faces numbered '1', two faces numbered '2' and one face numbered '3'. The random variable X is the number showing on the top face of the dice when it is thrown. Show that $\mathrm{E}(X)=\frac{5}{3}$ and find $\mathrm{Var}(X)$.

Solution:

$X = x$	1	2	3
$\mathrm{P}(X = x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$$\mathrm{E}(X)=\frac{3}{6}+\frac{4}{6}+\frac{3}{6}=\frac{10}{6}=\frac{5}{3}$$

$$\mathrm{E}(X^2)=1^2\frac{3}{6}+2^2\frac{2}{6}+3^2\frac{1}{6}=\frac{20}{6}$$

$$\mathrm{Var}(X)=\frac{20}{6}-\left(\frac{5}{3}\right)^2=\frac{5}{9}$$

2. [2001/II/6]

A random variable X has the probability distribution given in the following table.

x	2	3	4	5
$\mathrm{P}(X = x)$	p	$\frac{2}{10}$	$\frac{3}{10}$	q

(i) Given that E(X) = 4, find p and q.
(ii) Show that Var(X) = 1.
(iii) Find E(|X – 4|).
(iv) Ten independent observations of X are taken. Find the probability that the value 3 is obtained at most three times.

Solutions:

(i)

$$\mathrm{E}(X)=4 \Rightarrow 2p+3\left(\frac{2}{10}\right)+4\left(\frac{3}{10}\right)+5q=4$$

$$\Rightarrow 2p+5q=\frac{11}{5} \quad ---(1)$$

Also, $p+\frac{2}{10}+\frac{3}{10}+q=1\Rightarrow q=\frac{1}{2}-p \quad ---(2)$

Subs. (2) into (1): $2p+5\left(\frac{1}{2}-p\right)=\frac{11}{5}$

$$p=\frac{1}{10}\Rightarrow q=\frac{2}{5}$$

(ii) $\mathrm{E}(X^2)=2^2\left(\frac{1}{10}\right)+3^2\left(\frac{2}{10}\right)+4^2\left(\frac{3}{10}\right)+5^2\left(\frac{2}{5}\right)=17$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2=17-4^2=1$ (shown)

(iii) $\mathrm{E}(|X-4|)=|-2|\left(\frac{1}{10}\right)+|-1|\left(\frac{2}{10}\right)+|0|\left(\frac{3}{10}\right)+|1|\left(\frac{2}{5}\right)=\frac{4}{5}$

(iv) Let Y denote the no. of times (out of 10) '3' is obtained.

Then $Y\sim \mathrm{B}\left(10,\frac{2}{5}\right)$

Required probability = $P(Y\le 3)=0.879$ (to 3 sf)

3. [2017 PJC MYE/ P2/ Q7]

In a game, 3 red balls and 7 white balls are placed in a bag. A player draws 3 balls at random and without replacement. The player scores 2 points for every red ball that is drawn, and 1 point for every white ball that is drawn. The total score X is obtained by adding up the scores of the 3 balls. Find the probability distribution of X. [3]

Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=0.49$. [2]

x	3 (3W)	4 (2W1R)	5 (1W2R)	6 (3R)
$\mathrm{P}(X=x)$	$\frac{7}{10}\times\frac{6}{9}\times\frac{5}{8}=\frac{7}{24}$ OR $\frac{{}^{7}C_{3}}{{}^{10}C_{3}}=\frac{7}{24}$	$\frac{7}{10}\times\frac{6}{9}\times\frac{3}{8}\times\frac{3!}{2!}=\frac{21}{40}$ OR $\frac{{}^{7}C_{2}\,{}^{3}C_{1}}{{}^{10}C_{3}}=\frac{21}{40}$	$\frac{7}{10}\times\frac{3}{9}\times\frac{2}{8}\times\frac{3!}{2!}=\frac{7}{40}$ OR $\frac{{}^{7}C_{1}\,{}^{3}C_{2}}{{}^{10}C_{3}}=\frac{7}{40}$	$\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}=\frac{1}{120}$ OR $\frac{{}^{3}C_{3}}{{}^{10}C_{3}}=\frac{1}{120}$

$\mathrm{E}(X)=3\times\frac{7}{24}+4\times\frac{21}{40}+5\times\frac{7}{40}+6\times\frac{1}{120}=3.9$

$\mathrm{E}(X^2)=3^2\times\frac{7}{24}+4^2\times\frac{21}{40}+5^2\times\frac{7}{40}+6^2\times\frac{1}{120}=15.7$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=15.7-(3.9)^2=0.49$

4. [2017 SRJC MYE/ P2/ Q10 (part of)]

(a) It is known that X is a discrete random variable such that

$$4\mathrm{E}(X^2)=\mathrm{Var}(3+2X)+1.$$

Find E(X). [2]

(b) To promote the sales of a snack for children from March to August, one game card is inserted into each box of the snack. For each month, the game cards feature a different series of comic book characters and there are twelve distinct game cards in each series. Andy buys four boxes of the snack in March.

The random variable X denotes the number of distinct games cards that Andy will get from his purchases.

(i) Show that $\mathrm{P}(X = 3) = \frac{55}{144}$ and $\mathrm{P}(X = 2) = \frac{77}{1728}$. Hence, tabulate the probability distribution of X. [4]

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

(a)

$$4\mathrm{E}(X^2) = \mathrm{Var}(3 + 2X) + 1.$$

$$\Rightarrow 4\mathrm{E}(X^2) = 4\mathrm{Var}(X) + 1$$

$$= 4\mathrm{E}(X^2) - 4[\mathrm{E}(X)]^2 + 1$$

$$\Rightarrow 4[\mathrm{E}(X)]^2 = 1$$

$$\therefore\ \mathrm{E}(X) = \pm\frac{1}{2}$$

(b)(i)

$$\mathrm{P}(X = 3) = \left(\frac{1}{12}\right)\left(\frac{11}{12}\right)\left(\frac{10}{12}\right) + \left(\frac{11}{12}\right)\left(\frac{2}{12}\right)\left(\frac{10}{12}\right) + \left(\frac{11}{12}\right)\left(\frac{10}{12}\right)\left(\frac{3}{12}\right)$$

$$\text{OR } \frac{{}^{12}C_1\,{}^{11}C_2 \times \frac{4!}{2!}}{12^4}$$

$$= \frac{55}{144}$$

$$\mathrm{P}(X = 2) = \frac{{}^{12}C_2 \times \frac{4!}{2!2!}}{12^4} + \frac{{}^{12}C_1\,{}^{11}C_1 \times \frac{4!}{3!}}{12^4} = \frac{77}{1728}$$

X	1	2	3	4
$\mathrm{P}(X = x)$	$\left(\frac{1}{12}\right)^3 = \frac{1}{1728}$ OR: $\frac{{}^{12}C_1}{12^4}$	$\frac{77}{1728}$	$\frac{55}{144}$	$\left(\frac{11}{12}\right)\left(\frac{10}{12}\right)\left(\frac{9}{12}\right) = \frac{55}{96}$ OR: $\frac{{}^{12}P_4}{12^4}$

(ii)

$$\mathrm{E}(X) = \frac{1}{1728} + 2\left(\frac{77}{1728}\right) + 3\left(\frac{55}{144}\right) + 4\left(\frac{55}{96}\right)$$

$$= \frac{6095}{1728}$$

OR 3.527199 = 3.53 (to 3 s.f.) (By GC)

$$\mathrm{E}(X^2) = \frac{1}{1728} + 2^2\left(\frac{77}{1728}\right) + 3^2\left(\frac{55}{144}\right) + 4^2\left(\frac{55}{96}\right)$$

$$= \frac{7363}{576}$$

Var(X) = $\frac{7363}{576} - \left(\frac{6095}{1728}\right)^2$ OR Using GC, $\left(0.5846817963\right)^2$

$= \frac{1020767}{2985984}$ or 0.342 (3sf)

5. [2018 VJC H2 MYE Qn10]

A board of directors consists n men and $(10-n)$ women, where $3 \le n \le 7$. A committee consisting of 3 randomly chosen people is to be formed. Let W be the number of women in the committee.

(i) Show that $\mathrm{P}(W=1) = \frac{n(n-1)(10-n)}{240}$ and find, in terms of n, the probability distribution of W. [4]

(ii) Given $\mathrm{E}(W) = 2.1$, find n. [2]

(iii) Hence, find Var(W). [1]

Solutions:

<table>
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<td>i</td>
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$\mathrm{P}(W=1) = \mathrm{P}(MMW) + \mathrm{P}(MWM) + \mathrm{P}(WMM)$

$= \frac{10-n}{10} \times \frac{n}{9} \times \frac{n-1}{8} \times 3$

$= \frac{n(n-1)(10-n)}{240}$

Alternative Method:

$\mathrm{P}(W=1) = \frac{{}^{n}C_2 \times {}^{10-n}C_1}{{}^{10}C_3} = \frac{\frac{n(n-1)}{2!} \times \frac{10-n}{1!}}{120}$

$= \frac{n(n-1)(10-n)}{240}$

<table>
<tr><td>w</td><td>$\mathrm{P}(W=w)$</td></tr>
<tr><td>0</td><td>$\frac{n(n-1)(n-2)}{720}$ $\left(\text{or } \frac{{}^{n}C_3}{120}\right)$</td></tr>
<tr><td>1</td><td>$\frac{n(n-1)(10-n)}{240}$</td></tr>
<tr><td>2</td><td>$\frac{n(10-n)(9-n)}{240}$ $\left(\text{or } \frac{n\left({}^{10-n}C_2\right)}{120}\right)$</td></tr>
<tr><td>3</td><td>$\frac{(10-n)(9-n)(8-n)}{720}$ $\left(\text{or } \frac{{}^{10-n}C_3}{120}\right)$</td></tr>
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ii

$$\mathrm{E}(W)=\frac{n(n-1)(10-n)}{240}+2\left(\frac{n(10-n)(9-n)}{240}\right)$$

$$+3\left(\frac{(10-n)(9-n)(8-n)}{720}\right)$$

$$\begin{aligned}2.1&=\frac{10-n}{720}\left(3n(n-1)+6n(9-n)+3(9-n)(8-n)\right)\\&=\frac{10-n}{240}\left(n(n-1)+2n(9-n)+(9-n)(8-n)\right)\\&=\frac{10-n}{240}\left(n^2-n+18n-2n^2+72-17n+n^2\right)\\&=0.3(10-n)\end{aligned}$$

$$0.3n=0.9$$

$$n=3$$

Or By GC, n = 3

iii

w	0	1	2	3
$\mathrm{P}(W=w)$	$\frac{1}{120}$	$\frac{7}{40}$	$\frac{21}{40}$	$\frac{7}{24}$

By GC, Var(W) = 0.49

6 [2018 SRJC H2 MYE P2 Qn8]

An unbiased 4-sided die has its faces numbered 1, 2, 3 and 4. When the die is thrown, the number on the face in contact with the table is noted. Two such dice are thrown and the score X is found by multiplying these numbers together.

(i) Obtain the probability distribution of X. [2]

(ii) Using an algebraic method, show that the value of $\mathrm{E}\left(|X-5|\right)$ is $\frac{27}{8}$. [1]

(iii) Two independent observations of X are taken. Find the probability that one of them is 2 and the other is at most 3. [2]

Solution

(i) Table of outcomes (Not a required working, but good to have)

Die 1 **Die 2**	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Probability distribution of X:

x	1	2	3	4	6	8	9	12	16
P(X=x)	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

– Correct values for all outcomes
– Correct values for all probabilities

(ii)

$\lvert x-5\rvert$	1	2	3	4	7	11
$\mathrm{P}\left(\lvert x-5\rvert=x\right)$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

$$\mathrm{E}\left(\lvert x-5\rvert\right)=1\left(\frac{5}{16}\right)+2\left(\frac{1}{8}\right)+3\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+7\left(\frac{1}{8}\right)+11\left(\frac{1}{16}\right)$$

$$=\frac{27}{8} \text{ (Shown)}$$

(iii) Req Prob $=\mathrm{P}(X_1=2, X_2=2)+2\left[\mathrm{P}(X_1=2, X_2=1)+\mathrm{P}(X_1=2, X_2=3)\right]$

$$=\left(\frac{1}{8}\right)^2+2\left[\left(\frac{1}{8}\right)\left(\frac{1}{16}\right)+\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)\right]$$

$$=\frac{1}{16}$$

7 [2018 HCI J2 MYE Q12]
A biased 6-sided die is numbered from 1 to 6. The number obtained when the die is thrown is denoted by X. The probability distribution of X is shown in the following table:

Number shown on die, x	1	2	3	4	5	6
Probability $\mathrm{P}(X=x)$	0.1	a	0.2	0.2	b	0.1

where $a > 0$ and $b > 0$.

(i) Write down an equation satisfied by both a and b. [1]

(ii) If $\mathrm{E}(X)=3.5$, what can you say about the relationship between a and b? [1]

It is given that $a > b$,

(iii) Find $\mathrm{E}(X)$ in terms of b. Using the result in **(i)**, find an inequality in the form $\alpha<\mathrm{E}(X)<\beta$, where α and β are constants to be determined. [3]

(iv) Find $\mathrm{Var}(X)$ in terms of b. [3]

The biased die is thrown two times, the first number obtained is denoted as X_1 and the second number obtained is denoted as X_2.

(v) Find the value of $\mathrm{P}(X_1-X_2<4)$. [3]

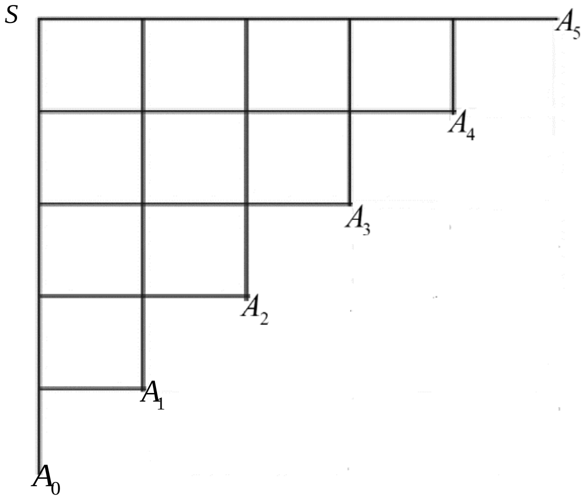
(vi) The random variable Q is the product of the two numbers obtained from the two throws. Find the probability that Q is even, leaving your answer in terms of *a*.

[2]

(i)	$a+b=0.4$, since total probability =1
(ii)	$\mathrm{E}(X)=3.5$ $3.5=0.1+2a+0.6+0.8+5b+0.6$ $1.4=2a+5b$ Solving **(i)** and **(ii)**, $a=b$
(iii)	$\mathrm{E}(X)=0.1+2a+0.6+0.8+5b+0.6$ $=2.1+2a+5b$ $=2.1+2(a+b)+3b$ $=2.1+2(0.4)+3b$ $=2.9+3b$ If $a>b$ and $a+b=0.4$, $\Rightarrow 0<b<0.2$ $\Rightarrow 0<3b<0.6$ $\Rightarrow 2.9<2.9+3b<3.5$ $\Rightarrow 2.9<\mathrm{E}(X)<3.5$
(iv)	$\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=0.1+4a+9\times0.2+16\times0.2+25b+36\times0.1-(2.9+3b)^2$ $=8.7+4a+25b-(8.41+17.4b+9b^2)$ $=8.7+4(a+b)+21b-(8.41+17.4b+9b^2)$ $=1.89+3.6b-9b^2$
(v)	$\mathrm{P}\left(X_1-X_2<4\right)$ $=1-\mathrm{P}\left(X_1-X_2\geq 4\right)$ $=1-\mathrm{P}\left(X_1=6,X_2=1\right)+\mathrm{P}\left(X_1=6,X_2=2\right)+\mathrm{P}\left(X_1=5,X_2=1\right)$ $=1-(0.1\times0.1+0.1\times a+0.1\times b)$ $=1-[0.01+0.1(a+b)]$ $=1-(0.01+0.1\times0.4)$ $=0.95$
(vi)	$\mathrm{P}(Q=even)=1-\mathrm{P}(\text{both odd})$ $=1-\left(0.3+b\right)^2$ $=1-\left(0.7-a\right)^2$ Alternatively,

	$\mathrm{P}(Q = even)$ $= \mathrm{P}(X_1 even) + \mathrm{P}(X_2\ even) - \mathrm{P}(X_1 even\ and\ X_2\ even)$ $=2(0.3+a)-(0.3+a)^2$ $=0.51+1.4a-a^2$

8. MI PU3 Prelim 9758/2019/02/Q10



A particle moves one step each time either to the right or downwards through a network of connected paths as shown above. The particle starts at *S*, and, at each junction, randomly moves one step to the right with probability *p*, or one step downwards with probability *q*, where $q = 1 - p$. The steps taken at each junction are independent. The particle finishes its journey at one of the 6 points labelled A_i, where $i = 0, 1, 2, 3, 4, 5$ (see diagram). Let $\{ X = i \}$ be the event that the particle arrives at point A_i.

(i) Show that $\mathrm{P}(X = 2) = 10p^2q^3$. [2]

(ii) After experimenting, it is found that the particle will end up at point A_2 most of the time. By considering the mode of *X* or otherwise, show that $\frac{1}{3} < p < \frac{1}{2}$. [4]

The above setup is a part of a two-stage computer game.

- If the particle lands on A_0, the game ends immediately and the player will not win any points.
- If the particle lands on A_i, where i = 2 or 4, then the player gains 2 points.
- If the particle lands on A_i, where i = 1, 3 or 5, then the player proceeds on to the next stage, where there is a probability of 0.4 of winning the stage. If he wins the stage, he gains 5 points; otherwise he gains 3 points.

Let Y be the number of points gained by the player when one game is played.

(iii) If $p = 0.4$, determine the probability distribution of Y. [4]

(iv) Hence find the expectation and variance of Y. [2]

(i)

$$\mathrm{P}(X = 2) = \mathrm{P}(\text{point } A_2)$$
$$= \mathrm{P}(\text{2 steps to the right and 3 steps downwards})$$
$$= {}^5C_2\, p^2 q^3$$
$$= 10p^2q^3 \text{ (Shown)}$$

(ii)

Note that $\mathrm{P}(X = 3) = \mathrm{P}(\text{point } A_3)$

$$= {}^5C_3\, p^3q^2 = 10p^3q^2$$

Since the particle will end up at point A_2 most of the time, then mode occurs at $X = 2$.

$$\therefore \mathrm{P}(X = 3) < \mathrm{P}(X = 2)$$
$$10p^3q^2 < 10p^2q^3$$
$$\frac{q}{p} > 1 \Rightarrow 1 - p > p$$
$$2p < 1 \Rightarrow p < 0.5$$

In the same way, $\mathrm{P}(X = 2) > \mathrm{P}(X = 1)$

$$10p^2q^3 > 5pq^4$$
$$2p > q$$
$$2p > 1 - p$$
$$3p > 1 \Rightarrow p > \frac{1}{3}$$

Combining both inequalities, $\frac{1}{3} < p < \frac{1}{2}$ (Shown)

(iii)

Y	0	2	3	5
$\mathrm{P}(Y = y)$	0.07776	0.4224	0.299904	0.199936

$$\mathrm{P}(Y = 0) = \mathrm{P}(X = 0) = {}^5C_0\,(0.4)^0(0.6)^5 = 0.07776$$
$$\mathrm{P}(Y = 2) = \mathrm{P}(X = i, \text{ where } i \text{ is even})$$
$$= \mathrm{P}(X = 2) + \mathrm{P}(X = 4)$$
$$= 10(0.4)^2(0.6)^3 + {}^5C_4\,(0.4)^4(0.6)^1 = 0.4224$$

	$\mathrm{P}(Y=3)=\mathrm{P}\left(X=i,\ \text{where } i \text{ is odd and lose the game}\right)$ $=0.6\left(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5)\right)$ $=0.6\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$ $=0.299904$ $\mathrm{P}(Y=5)=\mathrm{P}\left(X=i,\ \text{where } i \text{ is odd and win the game}\right)$ $=0.4\left(\mathrm{P}(X=1)+\mathrm{P}(X=3)+\mathrm{P}(X=5)\right)$ $=0.4\left[{}^5C_1(0.4)^1(0.6)^4+{}^5C_3(0.4)^3(0.6)^2+{}^5C_5(0.4)^5(0.6)^0\right]$ $=0.199936$
(iv)	Using G.C. $\mathrm{E}(Y)=2.744192=2.74\ \ (3\text{ sf})$ $\mathrm{Var}(Y)=1.3626^2=1.8567=1.86\ \ (3\text{ sf})$

9 [2018 JJC J2 MYE P2 Q4]

A discrete random variable X takes values 1, 3, 5, 6 with probabilities as shown in the table.

x	1	3	5	6
$\mathrm{P}(X=x)$	$\frac{k}{3}$	k	$\frac{5k}{2}$	$\frac{3k}{4}$

(i) Find k, leaving your answer as a fraction. [2]

(ii) Find $\mathrm{E}\left(\lvert\sin X\rvert\right)$. [2]

X_1 and X_2 are two independent observations of X.

(iii) Show that $\mathrm{P}(X_1+X_2=7)=\dfrac{72}{3025}$. [1]

(iv) Find $\mathrm{P}(X_1-X_2=2)$. [3]

(i)	$\frac{k}{3}+k+\frac{5k}{2}+\frac{3k}{4}=1$ $\Rightarrow k=\frac{12}{55}$

(ii)	$\mathrm{E}\left(\lvert\sin X\rvert\right)=\dfrac{\lvert\sin 1\rvert\cdot 4+\lvert\sin 3\rvert\cdot 12+\lvert\sin 5\rvert\cdot 30+\lvert\sin 6\rvert\cdot 9}{55}$ = 0.661 (3 s.f.)
(iii)	$\mathrm{P}(X_1+X_2=7)=2\mathrm{P}(X_1=1,\ X_2=6)$ $=2\left(\dfrac{4}{55}\cdot\dfrac{9}{55}\right)$ $=\dfrac{72}{3025}$
(iv)	$\mathrm{P}(X_1-X_2=2)=\mathrm{P}(X_1=3,\ X_2=1)+\mathrm{P}(X_1=5,\ X_2=3)$ $=\left(\dfrac{12}{55}\cdot\dfrac{4}{55}\right)+\left(\dfrac{6}{11}\cdot\dfrac{12}{55}\right)$ $=\dfrac{408}{3025}$