Chapter 6 (Pure Mathematics) Applications of Differentiation

Objectives

At the end of the chapter, you should be able to:

(a) interpret $f'(x) > 0$, $f'(x) = 0$ and $f'(x) < 0$ graphically;

(b) Determine the maximum and minimum points (local maxima and minima) and stationary points of inflexion analytically, in simple cases, using the first derivative test;
[Note: The use of the second derivative test is not required.]

(c) locate the maximum and minimum points with the help of a graphic calculator;

(d) find and use the equations of tangents to a curve;

(e) interpret the derivative as an instantaneous rate of change of a physical quantity, and use the concept to solve a variety of maxima and minima problems and problems involving connected rates of change.

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6.4 Stationary Points
- 6.4.1 Tests for types of Stationary Point

6.5 Maxima and Minima Problems

6.6 Rate of Change

References

New Syllabus Additional Mathematics (8th Edition), Shinglee Publishers Pte Ltd.

Relevant Resources

- http://mathinsite.bmth.ac.uk/applet/difffns/difffns.html (Applet to explore graphically the differentiation (1st and 2nd) of various functions)

Introduction

There are many applications of differentiation in science and engineering. Differentiation is also used in analysis of finance and economics. One important application of differentiation is in the area of optimisation, which means finding the condition for a maximum (or minimum) to occur. This is important in business e.g cost reduction, profit increase and engineering e.g maximum strength, minimum cost.

Example : **Solar Two** sustainable energy project in California

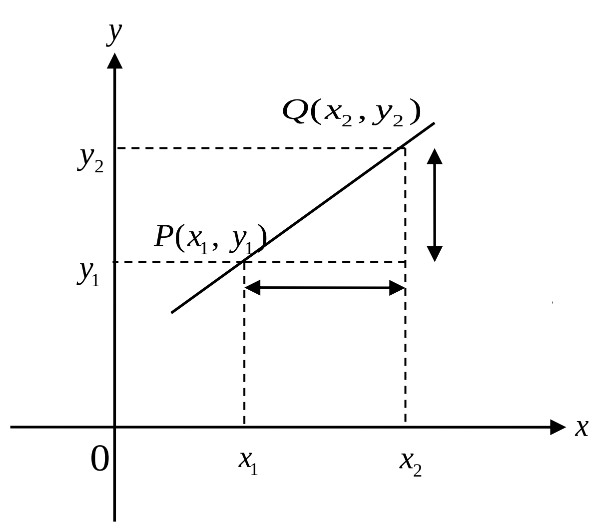
A power tower produces electricity from sunlight by focusing thousands of sun-tracking mirrors, called heliostats, on a single receiver sitting on top of a tower. The receiver captures the thermal energy of the sun and stores it in tanks of molten salt (to the right of the tower) at temperatures greater than 500 degrees centigrade.

When electricity is needed, the energy in the molten salt is used to create steam, which drives a conventional electricity-generating turbine (to the left of the tower).

Differentiation is used to **maximise** the efficiency of the process.

Source: http://www.intmath.com/Calculus/Calculus-intro.php

6.1 Some Notation and Terminology



<u>6.1.1 Gradient of a straight line</u>

The gradient of a straight line from point $P(x_1, y_1)$ to $Q(x_2, y_2)$ is defined as $\dfrac{y_2 - y_1}{x_2 - x_1}$.
i.e. The gradient of a straight line is the ratio of the change in *y*-coordinate to that of the *x*-coordinate between two points on the line.
Note:
1. The gradient of a straight line is constant throughout.

2. There are 4 different types of straight lines.

gradient $= 0$ | positive gradient | gradient is undefined | negative gradient

<u>6.1.2 Equation of a straight line</u>

There are two methods of forming an equation of a straight line.

<u>Method 1</u>

$$y = mx + c$$

where *m* is the gradient of the line and *c* is the *y*-intercept of the line.

<u>Method 2</u>

$$y - y_1 = m\left(x - x_1\right)$$

where *m* is the gradient of the line and $\left(x_1, y_1\right)$ is a point on the line.

Note:
Method 1 should be used when the gradient and *y*-intercept of the line are known and method 2 should be used when the gradient and one point on the line are known.

Example 1

Find the equation of line *l* that has

(a) a gradient of 2 and passes the point $(0,3)$

(b) a gradient of 3 and passes the point $(3,5)$

Solution:

(a) Equation of line *l*,

$$y = 2x + 3$$

(b) Equation of line *l*,

$$\begin{aligned} y - 5 &= 3(x-3) \\ y &= 3x - 9 + 5 \\ y &= 3x - 4 \end{aligned}$$

6.1.3 Gradient of perpendicular lines

Product of the gradients of two lines that are perpendicular to each other is -1.
i.e. Lines $y = m_1 x + c_1$ and line $y = m_2 x + c_2$ are perpendicular to each other if and only if $m_1 m_2 = -1$

Note:

$m_1 = -\dfrac{1}{m_2}$ and $m_2 = -\dfrac{1}{m_1}$

Hence two lines are perpendicular to each other $\Leftrightarrow$ the gradient of one is negative reciprocal of the other.

Example 2

Find the equation of the line that is perpendicular to the line $y = 2x + 3$ and passes through the point $(4,2)$.

Solution:

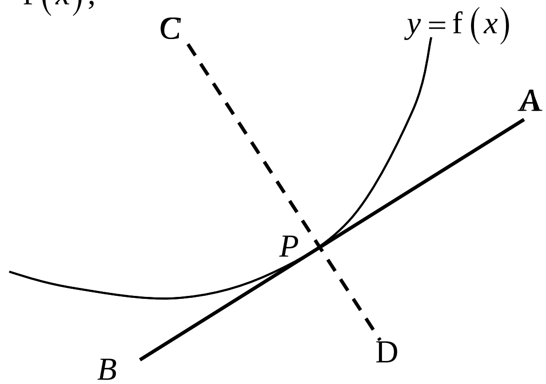
Gradient of the line $= -\dfrac{1}{2}$

Equation of line *l*, $y - 2 = -\dfrac{1}{2}(x-4)$

$$\begin{aligned} y &= -\frac{1}{2}x + 2 + 2 \\ y &= -\frac{1}{2}x + 4 \end{aligned}$$

6.1.4 Gradient of a curve

Given any curve $y = \mathrm{f}(x)$,



The line touching the curve at point P is called the tangent to the curve at point P.
i.e. line AB.
The line passing through P and perpendicular to the tangent at P is known as the normal to the curve at point P. i.e. line CD.

Note:

1. The gradient of a curve is not constant throughout.

2. The gradient of the curve at any point is defined as the gradient of the tangent to the curve at that point.

Example 3

The curve $y = ax^2 + bx + 2$ has gradient of 2 at the point $(1, 2)$, find the value of a and b.

Solution:

$y = ax^2 + bx + 2 \cdots (1)$

substitute $(1, 2)$ into (1),

$2 = a + b + 2$

$a + b = 0 \cdots (2)$

$\dfrac{dy}{dx} = 2ax + b$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 2$

$2a(1) + b = 2$

$2a + b = 2 \cdots (3)$

(3) – (2),

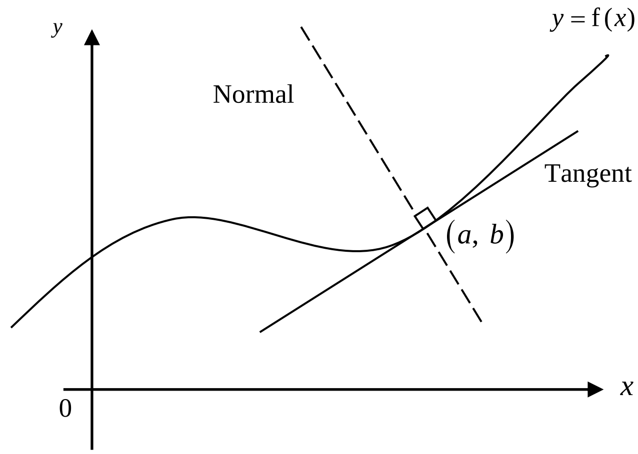
$a = 2$

$\therefore b = -2$

6.2 Equation of Tangents to a curve

For any curve $y=\mathrm{f}(x)$, gradient at point (a, b) is $\mathrm{f}'(a)$ or $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$.

Equation of tangent at (a, b) is $y-b=\mathrm{f}'(a)(x-a)$



Example 4
Find the equation of the tangent of the curve $y=2x^2-5x+6$ at the point where it crosses the y-axis.
Solution:

$y=2x^2-5x+6$

$\frac{\mathrm{d}y}{\mathrm{d}x}=4x-5$

When the curve crosses the y-axis, $x=0$. $\Rightarrow y=6$.

At point $(0,6)$,

Gradient of tangent $=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}$

$=4(0)-5$

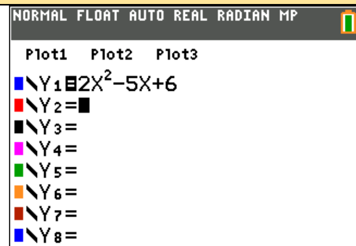
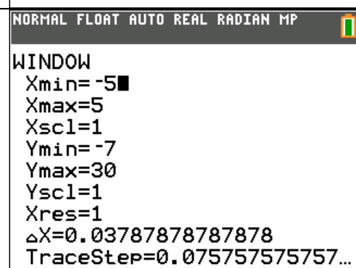
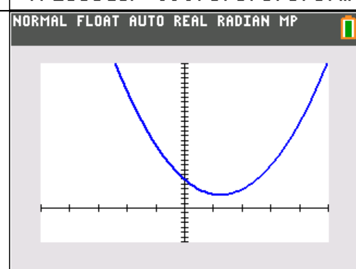
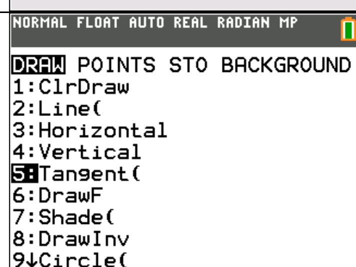
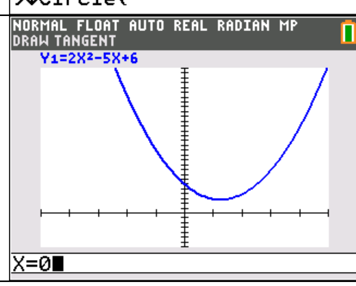
$=-5$

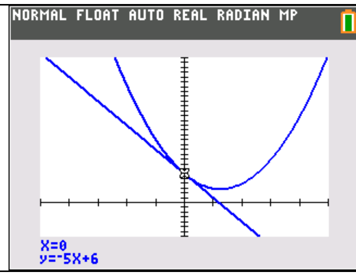
Equation of tangent at $(0,6)$,

$y=-5x+6$

6.2.1 Using GC to Draw a Tangent Line in the Function Mode

We can verify the answer to Example 4 using GC as shown below.

Step	Screenshot	Note
1. Enter the equation of graph of $Y_1 = 2x^2 - 5x + 6$		
2. Press $\boxed{\text{window}}$ and change window setting		
3. Press $\boxed{\text{graph}}$		
4. Press $\boxed{\text{2nd}}\boxed{\text{prgm}}[\text{draw}]$		
5. Key in ‘0’ (since 0 is the value of x where tangent is to be drawn)		By default, the value of x is 0. You can enter other value of your choice.

6. Press $\boxed{\text{enter}}$		The tangent line and its equation are displayed. From GC, the equation of tangent at $(0,6)$ is $y=-5x+6$.

Exercise 1

1. The equation of a curve is $y = 2x + \dfrac{1}{x}$. Find the equation of the tangent to the curve at $x = 2$. [Ans: $y = \dfrac{7}{4}x + 1$]

   Solution:

$\dfrac{dy}{dx} = 2 - \dfrac{1}{x^2}$ At $x=2$, $y = 2(2) + \dfrac{1}{2} = \dfrac{9}{2}$, $\dfrac{dy}{dx} = 2 - \dfrac{1}{2^2} = \dfrac{7}{4}$. Gradient of tangent when $x=2$ is $\dfrac{7}{4}$	Equation of tangent when $x=2$, $y - \dfrac{9}{2} = \dfrac{7}{4}(x-2)$ $y = \dfrac{7}{4}x - \dfrac{7}{2} + \dfrac{9}{2}$ $= \dfrac{7}{4}x + 1$

2. Find the equation of the tangent to the curve $y = (x-2)^3$ at the point $(3,1)$. Calculate the coordinates of the point where this tangent meets the curve again.

   [Ans: $y = 3x - 8$, $(0,-8)$]

   Solution:

$\frac{dy}{dx}=3(x-2)^2$. At $(3,1)$, $\frac{dy}{dx}=3(3-2)^2=3$. Equation of the tangent at $(3,1)$, $y-1=3(x-3)$ $y=3x-8$	When tangent meets curve, $3x-8=(x-2)^3$ $x^3-6x^2+9x=0$ $x(x^2-6x+9)=0$ $x(x-3)^2=0$ $x=0$ or $x=3$ (rej) When $x=0$, $y=(0-2)^3=-8$ Coordinates of point where tangent meets curve again = $(0,-8)$

3. The tangent to the curve $y=2x^2-7x+3$ at a certain point is parallel to the straight line $y=x+2$. Find the equation of this tangent and the point where it cuts the *y*-axis.

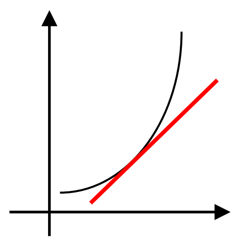
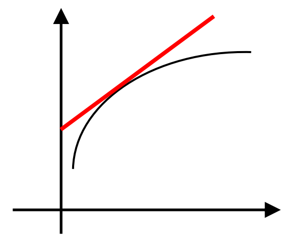
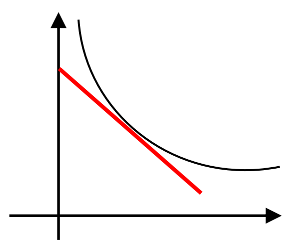
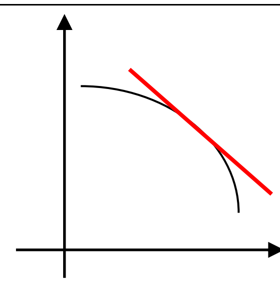
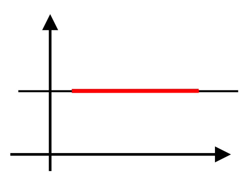
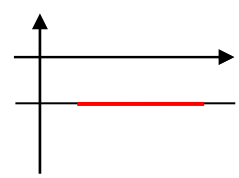
[Ans: $y=x-5$, $(0,\ -5)$]

Solution:

Gradient of tangent = 1. $\frac{dy}{dx}=4x-7=1$ $x=2$ $y=2(2)^2-7(2)+3=-3$	Equation of tangent when *x* = 2, $y-(-3)=x-2$ $y=x-5$ When $x=0$, $y=-5$. Therefore, the line cuts the y-axis at (0,-5).

<u>6.3 Increasing and Decreasing Functions</u>

To determine if the function is an increasing or decreasing function, we consider the sign of the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$.

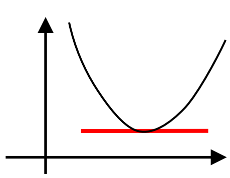
$\frac{\mathrm{d}y}{\mathrm{d}x}$	Description	Graphical Representation	
		Case 1	Case 2
>0	Gradient at any point in these curves is positive. i.e. y increases as x increases $y=\mathrm{f}(x)$ is a (strictly) increasing function.		
<0	Gradient at any point in these curves is negative. i.e. y decreases as x increases $y=\mathrm{f}(x)$ is a (strictly) decreasing function.		
$=0$	$y=\mathrm{f}(x)$ is a constant function. i.e. y remains constant as x increases		

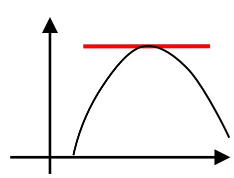
Applet: https://www.geogebra.org/m/nRT7Ktd8

Note:

If $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ or $\mathrm{f}'(x)=0$ at a particular point on the curve, then the tangent to the curve at that point will be parallel to x-axis.

Examples:





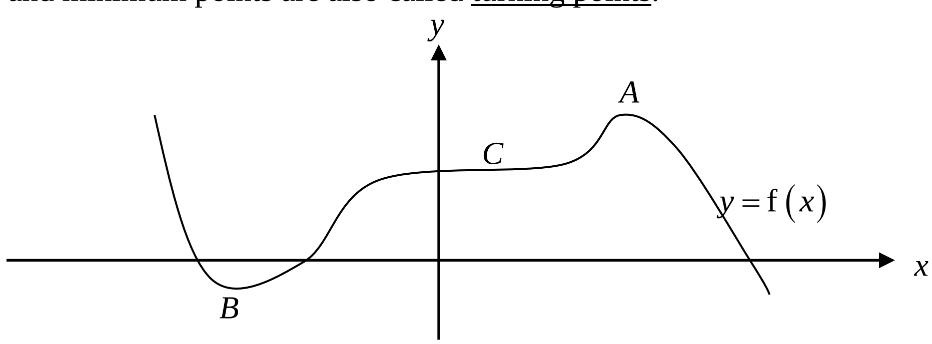
6.4 Stationary Points

A point on a curve $y = \mathrm{f}(x)$ is called a **stationary point** if the gradient, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ or $\mathrm{f}'(x) = 0$.

At a stationary point, the curve is flat and the tangent drawn is parallel to the *x*-axis.
The values of the *y*-coordinates of the stationary points are called **stationary values**.

There are three types of stationary points:
(a) maximum point,
(b) minimum point and
(c) point of inflexion.

Maximum and minimum points are also called turning points.



In the diagram above, A is a maximum point, *B* is a minimum point and *C* is a stationary point of inflexion. *A*, *B* and *C* are all stationary points.

6.4.1 Tests for types of Stationary Point

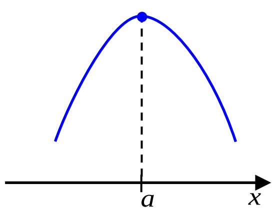
There are two different methods of testing the nature of stationary points:
(a) First Derivative Test,
(b) Second Derivative Test and

First Derivative Test

Suppose $y = \mathrm{f}(x)$ has a stationary point at $x = a$ i.e. $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = 0$ or $\mathrm{f}'(a) = 0$.

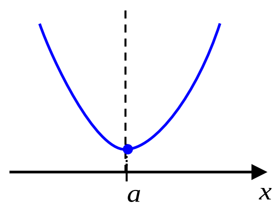
Check the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$ on either side of stationary points.

Maximum point



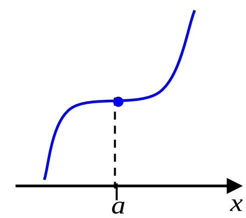
x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	–
Sketch of tangent	/	—	\

Minimum point

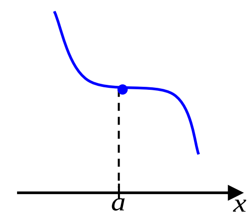


x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	–	0	+
Sketch of tangent	\	—	/

Stationary point of Inflexion



x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	+
Sketch of tangent	/	—	/



x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	–	0	–
Sketch of tangent	\	—	\

$\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change sign as x increases through a

Note:

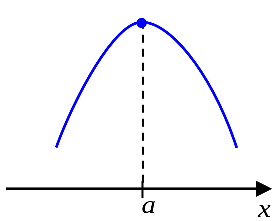
1. a^- denotes a value slightly less than a.
   a^+ denotes a value slightly more than a.

2. The values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ has to be evaluated for every value of x in the First Derivative Test.

Second Derivative Test

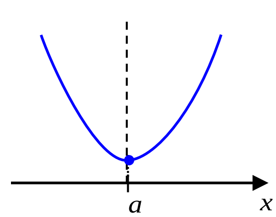
Suppose $y = \mathrm{f}(x)$ has a stationary point at $x = a$ i.e. $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = 0$ or $\mathrm{f}'(a) = 0$.

Maximum point



$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ at $x = a$

Minimum point



$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ at $x = a$

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = a$,

Second derivative test is inconclusive, we go back to the first derivative test.

<u>Example 5</u>

Find the stationary points of $y = x^3(x-1)^2$ and determine the nature of the stationary points.

Solution:

$$y = x^3(x-1)^2 = x^5 - 2x^4 + x^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 - 8x^3 + 3x^2$$

To find stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$5x^4 - 8x^3 + 3x^2 = 0$$

$$x^2(5x^2 - 8x + 3) = 0$$

$$x^2(x-1)(5x-3) = 0$$

$$x = 0 \text{ or } x = 1 \text{ or } x = \frac{3}{5}$$

When $x = 0$, $y = 0$.

When $x = 1$, $y = 1 - 2 + 1 = 0$

When $x = \frac{3}{5}$,

$$y = \left(\frac{3}{5}\right)^5 - 2\left(\frac{3}{5}\right)^4 + \left(\frac{3}{5}\right)^3 = \frac{108}{3125}$$

Therefore, $(0,0)$, $(1,0)$ and $\left(\frac{3}{5}, \frac{108}{3125}\right)$ are stationary points.

Next, we have to determine the nature of the stationary points.

<u>Method A</u>: Use of **First Derivative Test** to determine the nature of stationary points

x	-0.01	0	0.01
$\frac{\mathrm{d}y}{\mathrm{d}x}$	3.0805×10^{-4}	0	2.9205×10^{-4}
Sketch of tangent	/	—	/

x	0.99	1	1.01
$\frac{\mathrm{d}y}{\mathrm{d}x}$	-0.019112	0	0.02091
Sketch of tangent	\	—	/

x	0.599	$\frac{3}{5}$	0.601
$\frac{\mathrm{d}y}{\mathrm{d}x}$	7.1940×10^{-4}	0	-7.2060×10^{-4}
Sketch of tangent	/	—	\

Therefore, $(0,0)$ is a stationary point of inflexion, $(1,0)$ is a minimum point and $\left(\frac{3}{5}, \frac{108}{3125}\right)$ is a maximum point.

Method B: Use of **Second Derivative Test** to determine the nature of stationary points

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 20x^3 - 24x^2 + 6x$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=1} = 2 > 0 \quad \Rightarrow \quad (1,0) \text{ is a minimum point.}$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{3}{5}} = -\frac{18}{25} < 0 \quad \Rightarrow \quad \left(\frac{3}{5}, \frac{108}{3125}\right) \text{ is a maximum point.}$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = 0 \ \Rightarrow \text{Second Derivative Test inconclusive.}$$

We use the first derivative test:

x	-0.01	0	0.01
$\frac{\mathrm{d}y}{\mathrm{d}x}$	3.0805×10^{-4}	0	2.9205×10^{-4}
Sketch of tangent	/	—	/

Hence, $(0,0)$ is a stationary point of inflexion

Note:

1. In general, second derivative test is used when it is easy and fast to obtain the second derivative. If finding $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is tedious, first derivative test should be used even though it is still possible to find the second derivative.
2. When using the first derivative test, the numerical value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ needs to be calculated as shown in the table above. Do not just merely use + or – sign in the table.
3. Use GC to confirm the nature of the stationary points.

Exercise 2

1. Find the nature and coordinates of the turning points on the curve $y = 4x + \dfrac{9}{x}$.

[Ans: $\left(\dfrac{3}{2}, 12\right)$ min, $\left(-\dfrac{3}{2}, -12\right)$ max]

Solution:

At turning points,

$$\frac{dy}{dx} = 0$$

$$4 - \frac{9}{x^2} = 0$$

$$4x^2 - 9 = 0$$

$$(2x-3)(2x+3) = 0$$

$$x = \frac{3}{2} \text{ or } x = -\frac{3}{2}$$

$$y = 4\left(\frac{3}{2}\right) + \frac{9}{3/2} = 12 \quad \text{or}$$

$$y = 4\left(-\frac{3}{2}\right) + \frac{9}{-3/2} = -12$$

$$\frac{d^2y}{dx^2} = \frac{18}{x^3}$$

At $\left(\dfrac{3}{2}, 12\right)$, $\dfrac{d^2y}{dx^2} = \dfrac{18}{\left(3/2\right)^3} = \dfrac{16}{3} > 0$

$\left(\dfrac{3}{2}, 12\right)$ is a minimum point.

At $\left(-\dfrac{3}{2}, -12\right)$,

$$\frac{d^2y}{dx^2} = \frac{18}{\left(-3/2\right)^3} = -\frac{16}{3} < 0$$

$\left(-\dfrac{3}{2}, -12\right)$ is a maximum point.

2. If k is a positive constant, show that there is one turning point on the graph of $y = \ln x - kx$, $x \in \mathbb{R}$ and determine whether it is a maximum or minimum point. Find the value of k for which the turning point is on the x-axis.

[Ans:max; $k = e^{-1}$]

Solution:

$y = \ln x - kx$ $\frac{dy}{dx} = \frac{1}{x} - k$ $\frac{dy}{dx} = 0 \Rightarrow \frac{1}{x} - k = 0$ $\frac{1}{x} = k$ $x = \frac{1}{k}$ Therefore, there is only one turning point.	$\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ Therefore, the turning point is a maximum one. When $x = \frac{1}{k}$, $y = \ln\left(\frac{1}{k}\right) - k\left(\frac{1}{k}\right)$ $= \ln 1 - \ln k - 1$ $= -\ln k - 1$ When the turning point is on the x – axis, $y = 0 \Rightarrow -\ln k - 1 = 0$ $\ln k = -1$ $k = e^{-1}$

3. **[TPJC/Prelims/2007]**

Let $\text{f}(x) = x + \ln\left(x^2 - 4\right),\ x \in \mathbb{R}, x < -2 \text{ or } x > 2$.

(i) Using differentiation, find the stationary point on the graph of $y = \text{f}(x)$.

Determine the nature of this stationary point.

(ii) By considering the graph of $y = \text{f}(x)$, find the range of values of x for which f is increasing.

[Ans: $\left(-1-\sqrt{5}, -1.39\right)$,max point; $x < -1-\sqrt{5}$ or $x > 2$]

Solution:

(i) $\text{f}(x) = x + \ln\left(x^2 - 4\right),\ x \in \mathbb{R}, x < -2 \text{ or } x > 2$

$$\frac{dy}{dx} = 1 + \frac{2x}{x^2 - 4}$$

$$\frac{dy}{dx} = 0 \Rightarrow 1 + \frac{2x}{x^2 - 4} = 0$$

$$1 = -\frac{2x}{x^2 - 4}$$

$$x^2 - 4 = -2x$$

$$x^2 + 2x - 4 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2}$$

$x = -1+\sqrt{5}$ or $x = -1-\sqrt{5}$

(reject since not in range)

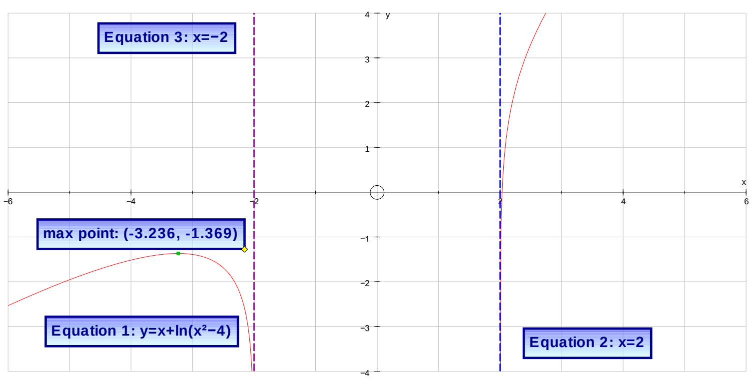
$y = -1.39$

Using first derivative test,

x	$\left(-1-\sqrt{5}\right)^-$	$\left(-1-\sqrt{5}\right)$	$\left(-1-\sqrt{5}\right)^+$
$\frac{dy}{dx}$	+	0	−
GI	/	—	\

Therefore, $\left(-1-\sqrt{5}, -1.39\right)$ is a maximum point.

(ii)



By f is increasing when

$x < -1-\sqrt{5}$ or $x > 2$

6.5 Maxima and Minima Problems

For optimization problems of functions involving a single variable such as $y = x^3(x-1)^2$, we can use the technique mentioned in Section 7.4 to determine the nature (maximum or minimum) of the stationary point and hence calculate the maximum or minimum value.

However, if the problem involves 2 or more variables, we will eliminate one or more variables by conditions given in the question and use the technique given in Section 7.4.

Method for Solving Maxima and Minima Problems

1. Formulate a function, say A which is to be maximised or minimised .

2. Express this function in terms of one variable only, say x by using given conditions. i.e. $A = \mathrm{f}(x)$.

3. Solve $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ to solve for x and find the **stationary values** of A.
   x is the value which will make A maximum or minimum.
   There may be more than one valid answer.

4. Use the first or second derivative tests to check for the nature of the stationary points.

Example 6

Find the minimum value of $x^2 + 2y^2$ if x and y are related by the equation $x + 2y = 1$.

Solution:

Note that x and y in this case varies so we have to ensure that the expression has only one variable before we differentiate.

Given $x + 2y = 1$

$\Rightarrow x = 1 - 2y \cdots (1)$

Let $u = x^2 + 2y^2 \cdots (2)$

Substitute (1) into (2),

$$u = (1-2y)^2 + 2y^2$$
$$= 1 - 4y + 6y^2$$
$$\frac{\mathrm{d}u}{\mathrm{d}y} = -4 + 12y$$

At stationary point,

$$\frac{\mathrm{d}u}{\mathrm{d}y} = 0, -4 + 12y = 0$$
$$y = \frac{1}{3}$$

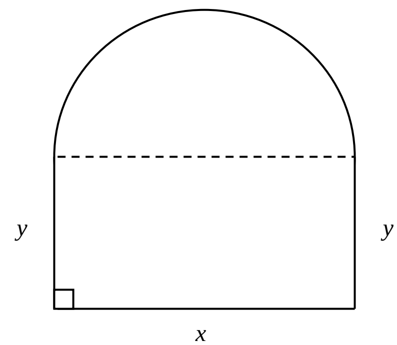
When $y = \frac{1}{3}$, $u = 1 - 4\left(\frac{1}{3}\right) + 6\left(\frac{1}{3}\right)^2$

$$= \frac{1}{3}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2} = 12 > 0$$

Therefore, minimum $u = \frac{1}{3}$

<u>Example 7</u> [2008 GCE A Level P1/Q7]



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area.

Solution:

Area of the flower bed,

$$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \longleftarrow$$

This is the function A which we wish to maximize.

$$A = xy + \frac{1}{8}\pi x^2 \,....(1) \longleftarrow$$

A is in terms of two variables x and y. We cannot differentiate A at this stage.

To express A in only **one** variable before differentiation, we use the information:

Total length of straight parts = $y + x + y = 2y + x$

Total length of semi-circular part = $\frac{1}{2}(2\pi)\left(\frac{x}{2}\right) = \frac{\pi}{2}x$

As total time taken is given to be 180 hours, we have

$$180 = 3(x+2y) + 9\left(\frac{\pi x}{2}\right) \longleftarrow$$

This is the condition given in the question which we can use it to eliminate one variable in equation (1) .

$$180 = 3x + 6y + \frac{9\pi x}{2}$$

$$60 = x + 2y + \frac{3\pi x}{2}$$

We can express y in terms of x, $y = 30 - \frac{x}{2} - \frac{3\pi x}{4}$ ....(2)

Substituting (2) into (1)

$$A = x\left(30 - \frac{x}{2} - \frac{3\pi x}{4}\right) + \frac{\pi x^2}{8}$$

$$A = 30x - \frac{x^2}{2} - \frac{3\pi x^2}{4} + \frac{\pi x^2}{8}$$

$$A = 30x - \frac{x^2}{2} - \frac{5\pi x^2}{8}$$

$$A = 30x - \left(\frac{1}{2} + \frac{5\pi}{8}\right)x^2$$

Now A involves only one variable, x, we can differentiate A with respect to x.

Thus $$\frac{\mathrm{d}A}{\mathrm{d}x} = 30 - \left(1 + \frac{5\pi}{4}\right)x$$

<u>To find stationary value of *A:*</u>

At stationary point,

$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, we have $30 = \left(1 + \frac{5\pi}{4}\right)x$

So $x = \frac{30}{1 + \frac{5\pi}{4}} = 6.0889 \approx 6.09$ (to 3 s.f.)

Substitute $x = 6.0889$ into (2), we have $y = 12.6$ (to 3 s.f.)

<u>To find the nature of stationary point:</u>

As $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -\left(1 + \frac{5\pi}{4}\right) < 0$, A is maximum when $x = \frac{30}{1 + \frac{5\pi}{4}}$.

That is, when $x = 6.09$ m, $y = 12.6$ m, the area of the flower bed is maximum.

Note:

This final step is necessary even the question has the phrase "maximum area". Do not assume that stationary value will give maximum area. You need to carry out the first/second derivative test to confirm that the stationary value gives the maximum area. You cannot omit this final step unless it is stated in the question.

<u>Exercise 3</u>

1. Given that the volume of a right solid cylinder of radius r is $250\pi \text{ cm}^3$, find the value of r for which the total surface area of the solid is a minimum.

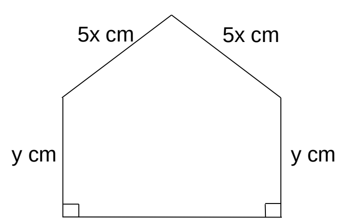
[Ans: $r = 5$]

Solution:

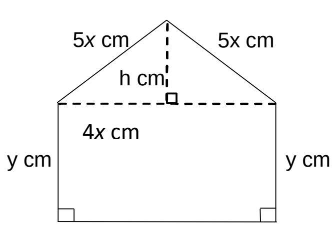
Volume of cylinder, $V = 250\pi$ $\pi r^2 h = 250\pi$ $h = \dfrac{250}{r^2}$ Total surface area, $A = 2\pi r^2 + 2\pi rh$ $= 2\pi r^2 + 2\pi r\left(\dfrac{250}{r^2}\right)$ $= 2\pi r^2 + \dfrac{500\pi}{r}$ $\dfrac{dA}{dr} = 4\pi r - \dfrac{500\pi}{r^2}$	At stationary point, $\dfrac{dA}{dr} = 0$, $4\pi r = \dfrac{500\pi}{r^2}$ $r^3 = 125$ $r = 5$ $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{1000\pi}{r^3}$ > 0 Therefore, when $r = 5$, the total surface area of the solid is a minimum.

2. A piece of wire, length 90 cm, is bent into the shape shown in the diagram. Show that the area $A \text{ cm}^2$, enclosed by the wire is given by $A = 360x - 60x^2$. Find the value of x and y for which A is a maximum.

[Ans: $x = 3,\ y = 18$]



Solution:

Let height of triangle be h cm. 	$A = \dfrac{1}{2}(h)(8x) + y(8x)$ $= \dfrac{1}{2}(3x)(8x) + (45 - 9x)(8x)$ $= 360x - 60x^2$ When A is max or min,

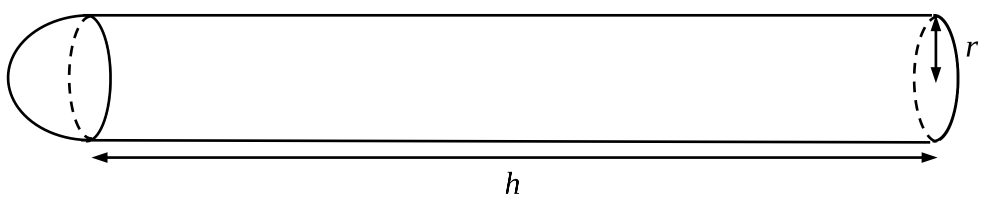
By Pythagoras' Theorem, $h^2 = (5x)^2 - (4x)^2 = 9x^2$ $h = 3x$ or $-3x$ (N.A.) Length of wire = $5x + 5x + y + y + 8x = 90$ $\therefore y = 45 - 9x$	$\frac{dA}{dx} = 360 - 120x$ At stationary point, $\frac{dA}{dx} = 0$ $x = 3$, $y = 45 - 9(3) = 18$ $\frac{d^2A}{dx^2} = -120 < 0$ A is maximum when $x = 3$, $y = 18$

3. A solid consists of a circular cylinder of radius r cm and height h cm, joined to a hemisphere whose radius is r cm, the flat base of the hemisphere being in contact with one of the circular ends of the cylinder. The total surface area of the solid is S.

(i) Prove that its volume is $\frac{r}{6}(3S - 5\pi r^2)$ cm^3.

(ii) Hence show that, for a given total surface area, the volume of the solid is greatest when $h = r$.

Solution:



(i) $S = 2\pi r^2 + 2\pi rh + \pi r^2$

$= 3\pi r^2 + 2\pi rh$

$h = \dfrac{S - 3\pi r^2}{2\pi r}$

Volume of solid, V

$= \dfrac{2}{3}\pi r^3 + \pi r^2 h$

$= \dfrac{2}{3}\pi r^3 + \pi r^2\left(\dfrac{S - 3\pi r^2}{2\pi r}\right)$

$= \dfrac{2}{3}\pi r^3 + \dfrac{1}{2}Sr - \dfrac{3}{2}\pi r^3$

$= \dfrac{1}{2}Sr - \dfrac{5}{6}\pi r^3$

$= \dfrac{r}{6}\left(3S - 5\pi r^2\right)$ cm^3

(ii) $\dfrac{dV}{dr} = \dfrac{1}{2}S - \dfrac{5}{2}\pi r^2$

For stationary point,

$\dfrac{dV}{dr} = 0 \Rightarrow \dfrac{5}{2}\pi r^2 = \dfrac{S}{2}$

$r = \sqrt{\dfrac{S}{5\pi}}$ or $-\sqrt{\dfrac{S}{5\pi}}$ (reject -ve value)

$$h = \frac{S - 3\pi\left(\sqrt{\dfrac{S}{5\pi}}\right)^2}{2\pi\left(\sqrt{\dfrac{S}{5\pi}}\right)}$$

$$= \frac{\dfrac{2}{5}S}{2\pi\left(\sqrt{\dfrac{S}{5\pi}}\right)}$$

$$= \sqrt{\frac{S}{5\pi}}$$

$= r$

$\dfrac{d^2v}{dr^2} = -5\pi r$

At $r = \sqrt{\dfrac{S}{5\pi}}$, $\dfrac{d^2v}{dr^2} = -5\pi\sqrt{\dfrac{S}{5\pi}} < 0$.

V is greatest when $h = r$. (shown)

6.6 Rate of Change

$\frac{\mathrm{d}y}{\mathrm{d}x}$ is defined as the rate of change of y with respect to x.
In general, any **rate of change** refers to a rate of change with respect to **time** unless otherwise stated. i.e. rate of change of y is $\frac{\mathrm{d}y}{\mathrm{d}t}$ or how fast y changes as t changes.

For example,

Statements	Mathematical Meaning
1. A balloon is being inflated such that its surface area, A, is **increasing** at a constant rate of $4\text{cm}^2/\text{s}$.	$\frac{\mathrm{d}A}{\mathrm{d}t} = 4\text{ cm}^2/\text{s}$.
2. Water is leaking out of a tank such that the volume of water, V, is **decreasing** at a constant rate of $5\text{cm}^3/\text{s}$	$\frac{\mathrm{d}V}{\mathrm{d}t} = -5\text{ cm}^3/\text{s}$

The chain rule is often used to solve "rate of change" problems:

$$\boxed{y = \mathrm{f}(x) \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}}$$

Method for Solving Rate of Change problems

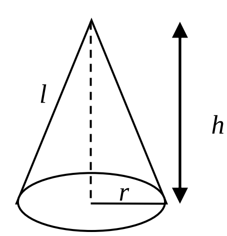
1. Understand what the rate of change required, e.g. $\frac{\mathrm{d}A}{\mathrm{d}t}$.
2. Identify the given rate of change, say $\frac{\mathrm{d}x}{\mathrm{d}t}$, any other relevant information.
3. Find an expression for A in terms of x.
4. Determine what you need using the Chain Rule: $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

Note:
Differentiation must take place before particular values of the variables are substituted in the equations.

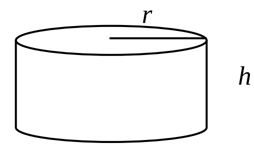
Useful formulae NOT in formula list:

Volume of a cone $= \frac{1}{3}\pi r^2 h$

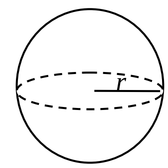
Curved surface area of a cone $= \pi r l$



Volume of a cylinder $= \pi r^2 h$ Curved surface area of cylinder $= 2\pi rh$



Volume of a sphere = $= \dfrac{4}{3}\pi r^3$ Curved surface area of sphere $= 4\pi r^2$

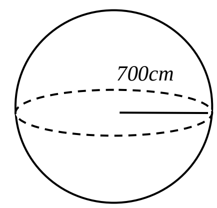


Example 8

Gas is escaping from a spherical hot air balloon at 10 000 cm^3/s.

(i) How fast is the radius decreasing when the radius is 700 cm?

(ii) At what rate is the surface area of the balloon decreasing then?



Solution:

Given: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -10000 \text{ cm}^3/\text{s}$

Volume of the balloon, $V = \dfrac{4}{3}\pi r^3$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

When radius, $r = 700$ cm,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V} \text{ x } \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{1}{4\pi(700)^2}(-10000)$$

$$= -\frac{1}{196\pi} \text{ cm/s or } -0.00162 \text{ cm/s}$$

The radius is decreasing at the rate of 0.00162 cm/s.

Find $\dfrac{\mathrm{d}r}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 700$ cm

----

Surface area of the balloon, $A = 4\pi r^2$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \text{ x } \frac{\mathrm{d}r}{\mathrm{d}t}$$

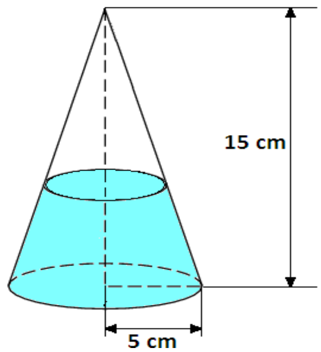
$$= 8\pi(700)\left(-\frac{1}{196\pi}\right)$$

$$= -28\frac{4}{7} \text{ cm}^2/\text{s or } -28.6 \text{ cm}^2/\text{s}$$

The surface area of the balloon is decreasing at the rate of $28.6 \text{ cm}^2/\text{s}$.

Example 9

A right circular cone with base radius 5 cm and height 15 cm is initially full of water. Water is leaking from the circular base of the cone at a rate of 10 cm^{3}s^{-1}. Find the exact rate of change of the depth of water when the depth of water is 12 cm.

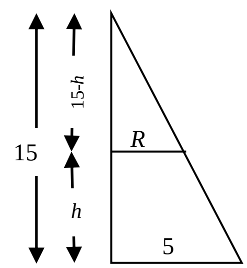


Solution:

Let the depth of the water be h and the radius of the water level at that point be R.

Given $\frac{dV}{dt} = -10cm^3s^{-1}$

Find $\frac{\mathrm{d}h}{\mathrm{d}t}$



Using similar triangles

$$\frac{15-h}{15} = \frac{R}{5} \Rightarrow R = \frac{15-h}{3}$$

$$V = \frac{1}{3}\pi(5)^2(15) - \frac{1}{3}\pi\left(\frac{15-h}{3}\right)^2(15-h)$$

$$V = \frac{1}{3}\pi(5)^2(15) - \frac{1}{27}\pi(15-h)^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = -\frac{3}{27}\pi(15-h)^2(-1) = \frac{\pi(15-h)^2}{9}$$

Given $\frac{\mathrm{d}V}{\mathrm{d}t} = -10cm^3s^{-1}$,

By Chain rule, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$

$$-10 = \frac{\pi(15-h)^2}{9} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

At h = 12 cm,

$$-10 = \frac{\pi(15-12)^2}{9} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$-10 = \pi \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{10}{\pi} cm\ s^{-1}$$

Exercise 4

1. A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm^2. Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm^{2}s^{-1}, find the rate of increase at this instant of
   (i) the length of the cylinder,
   (ii) the volume of the cylinder,
   (iii) the radius of the cylinder.

[Ans: 0.5 cm/s; $5 \text{ cm}^3\text{/s}$; 0.0315 cm/s]

Solution:

$\left.\dfrac{dx}{dt}\right\|_{x=5} = 0.025\,\text{cm}^2\text{s}^{-1}$ (i) $l = 20x$ $\dfrac{dl}{dt} = \dfrac{dl}{dx} \times \dfrac{dx}{dt}$ $\dfrac{dl}{dt} = 20\dfrac{dx}{dt}$ $\left.\dfrac{dl}{dt}\right\|_{x=5} = 20\left.\dfrac{dx}{dt}\right\|_{x=5}$ $= 20 \times 0.025$ cm/s $= 0.5$ cm/s (ii) $V = (20x)x$ $= 20x^2$ $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dt}$ $\dfrac{dV}{dt} = 40x\left(\dfrac{dx}{dt}\right)$ $\left.\dfrac{dV}{dt}\right\|_{x=5} = 40(5)\left.\dfrac{dx}{dt}\right\|_{x=5}$ $= 200 \times 0.025 \text{ cm}^3\text{/s}$ $= 5 \text{ cm}^3\text{/s}$	(iii) $\pi r^2 = x$ $r = \sqrt{\dfrac{x}{\pi}}$ $\dfrac{dr}{dt} = \dfrac{dr}{dx} \times \dfrac{dx}{dt}$ $\dfrac{dl}{dt} = \dfrac{1}{\sqrt{\pi}}\left(\dfrac{1}{2}x^{-\frac{1}{2}}\dfrac{dx}{dt}\right)$ $= \dfrac{1}{2\sqrt{\pi x}}\dfrac{dx}{dt}$ $\left.\dfrac{dl}{dt}\right\|_{x=5} = \dfrac{1}{2\sqrt{5\pi}}\left.\dfrac{dx}{dt}\right\|_{x=5}$ $= \dfrac{1}{2\sqrt{5\pi}}(0.025)\text{cm}/s$ $= 0.00315$cm/s

2. A cube is expanding in such a way that its sides are changing at a rate of 2 cm/s. Find the rate of change of the total surface area when its volume is $125\,\text{cm}^3$.

[Ans: $120\text{ cm}^2/s$]

Solution:

Let x be the length of each side of the cube. $\frac{dx}{dt} = 2$ cm/s Volume of cube, $V = x^3$ When $V = 125$, $x^3 = 125$ $x = 5$	Total surface area, $A = 6x^2$ $\frac{dA}{dx} = 12x$ When $x = 5$, $\frac{dA}{dt} = \frac{dA}{dx} . \frac{dx}{dt}$ $= 12(5)(2)$ $= 120\text{ cm}^2/s$

3. A gas in a container changes its volume according to the law, $PV = 3600$, where P is the number of units of pressure and V is the number of units of volume, Given that P is increasing at a rate of 20 units per second at the instant when P = 40, calculate the rate of change of volume at this instant.

[Ans: $-$ 45 units/s]

Solution:

When $P = 40$, $\frac{dP}{dt} = 20$ units/s. $V = \frac{3600}{P}$ $\frac{dV}{dP} = -\frac{3600}{P^2}$ When $P = 40$, $\frac{dV}{dP} = -\frac{3600}{40^2}$ $= -\frac{9}{4}$	$PV = 3600$ $\frac{dV}{dt} = \frac{dV}{dP} \cdot \frac{dP}{dt}$ $= -\frac{9}{4}(20)$ $= -$ 45 units/s

Practice Questions

1. [ACJC/2017/Prelim/Q5]

(a) The petrol consumption of a car, in millilitres per kilometre, is advertised to be $P(x) = \frac{2500}{x} + \frac{2x}{3}$ where x is the speed of the car in km/h.

(i) Find the exact speed of the car when petrol consumption is minimum.

(ii) Sketch the graph of $y = P(x)$ for $x > 0$.

Hence find the range of values of x for which the petrol consumption is at most 90 millilitres per kilometre.

(b) The tank is shaped such that when the petrol in it is at a height of h cm, the volume of petrol, V, is given by

$$V = \pi h^3.$$

Find the rate of change of h after 1 minute.

[**Ans:** (a) (i) $25\sqrt{6}$ km/h (ii) $39.1 \le x \le 95.9$ (b) 0.0362 cm/s]

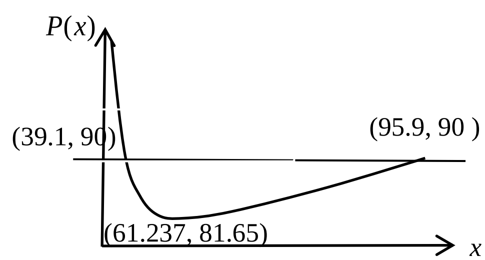
Solution:

(a)(i)

$P'(x) = -\frac{2500}{x^2} + \frac{2}{3} = 0$

$x^2 = 3750$

$\therefore x = 25\sqrt{6}$



(a)(ii) Min point (61.237, 81.65)

When $P(x) \le 90$, $39.1 \le x \le 95.9$.

(b)

Vol of fuel = 260.755 $\Rightarrow \pi h^3 = 260.755 \;\therefore h = 4.36208$

$V = \pi(h)^2 h = \pi h^3$

$\frac{\mathrm{d}V}{\mathrm{d}h} = 3\pi h^2$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

$= \frac{1}{3\pi h^2} \times 0.15\pi\sqrt{\pi(60)+1}$ when $t = 60$

$= 0.036173 \approx 0.0362$ cm/s

2.* [AJC/2017/Prelim/Q2]

The curve C has equation $y=\frac{1}{2}\mathrm{e}^{1-3x^2}$.

(i) Without using a calculator, find the equation of the tangent to C at the point P where $x=1$, giving your answer in the form where $y=mx+c$, where m and c are constants in exact terms to be found.

The tangent to C at P cuts x-axis at the point A and the y-axis at the point B.

(ii) Find the exact coordinates of the midpoint of AB.

(iii) Find the length of AB, giving your answer to 3 significant figures.

[**Ans:** $y=-3e^{-2}x+\frac{7}{2}e^{-2}$]

Solution:

(i) $y=\frac{1}{2}e^{1-3x^2} \Rightarrow \frac{dy}{dx}=\frac{1}{2}e^{1-3x^2}\left(-6x\right)=-3xe^{1-3x^2}$

At P, $x=1$, $y=\frac{1}{2}e^{1-3}=\frac{1}{2}e^{-2}$, $\frac{dy}{dx}=-3e^{1-3}=-3e^{-2}$

Equation of tangent is $y-\left(\frac{1}{2}e^{-2}\right)=(-3e^{-2})(x-1)$

$y-\left(\frac{1}{2}e^{-2}\right)=(-3e^{-2})(x-1)$

$y=-3e^{-2}x+3e^{-2}+\frac{1}{2}e^{-2}=-3e^{-2}x+\frac{7}{2}e^{-2}$

(ii) At B, $x=0$, $y=\frac{7}{2}e^{-2}$

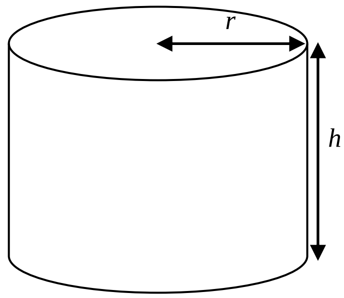
At A, $y=0$, $-3e^{-2}x+\frac{7}{2}e^{-2}=0 \Rightarrow x=\frac{\left(-\frac{7}{2}e^{-2}\right)}{-3e^{-2}}=\frac{7}{6}$

$A(\frac{7}{6},0)$ $B(0,\frac{7}{2}e^{-2})$

Midpoint of AB is $\left(\frac{\frac{7}{6}+0}{2},\frac{0+\frac{7}{2}e^{-2}}{2}\right)=\left(\frac{7}{12},\frac{7}{4}e^{-2}\right)$

(iii) $AB=\sqrt{\left(\frac{7}{6}-0\right)^2+\left(0-\frac{7}{2}e^{-2}\right)^2}$ =1.26 units

3*. [MJC/2017/Prelim/Q5]



A factory decides to design a closed cylindrical water tank of radius r cm and height h cm to hold a maximum of 16000π cm^3 of water. The outer surface area (including the base and lid) of the tank is to be coated with a layer of paint. It is assumed that the thickness of the cylindrical water tank is negligible.

(i) Show that the outer surface area of the tank, S cm^2 is given by $S = \dfrac{32000\pi}{r} + 2\pi r^2$.

(ii) In order to reduce the amount of paint used, the factory wishes to minimize the value of S. Using differentiation, find the value of r for which S is minimized. Hence find the minimum value of S.

(iii) Sketch, in this context, the graph of S against r.

(iv) The tank with minimum value of S is being manufactured. Water is being poured into this cylindrical tank at a constant rate of 1000 cm^3 per minute. Find the rate of change of the depth of water.

[**Ans:** (ii) $r = 20$; $S = 2400\pi$ cm^2 or 7540 cm^2 (iv) 0.796 cm/min]

Solution:

(i)

$16000\pi = \pi r^2 h$

$h = \dfrac{16000}{r^2}$ --- (1)

$S = 2\pi rh + 2\pi r^2$ --- (2)

Substitute (1) into (2),

$S = 2\pi r\left(\dfrac{16000}{r^2}\right) + 2\pi r^2$

$S = \dfrac{32000\pi}{r} + 2\pi r^2$ (shown)

(ii)

For minimum S, $\dfrac{\mathrm{d}S}{\mathrm{d}r} = 0$.

$\therefore \dfrac{\mathrm{d}S}{\mathrm{d}r} = -\dfrac{32000\pi}{r^2} + 4\pi r = 0$

$$4\pi r = \frac{32000\pi}{r^2}$$

$$r^3 = 8000$$

$$r = \sqrt[3]{8000}$$

$$r = 20$$

For $r = 20$,

r	19.9	20	20.1
$\dfrac{\mathrm{d}S}{\mathrm{d}r}$	-3.79	0	3.75
Sketch of tangent	$\backslash$	$-$	$/$

$\therefore S$ is minimum when $r = 20$.

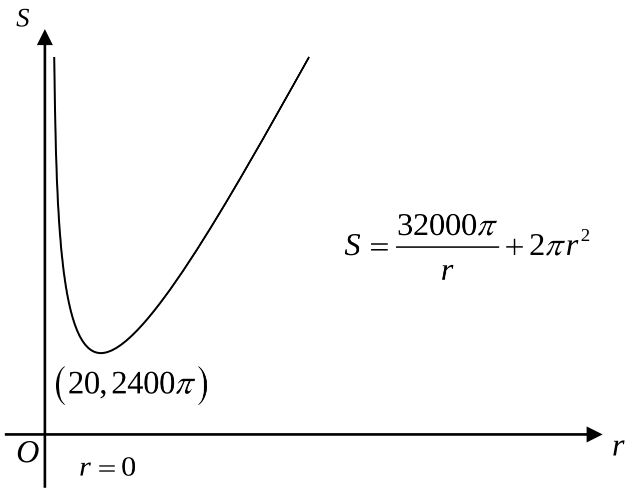
When $r = 20$,

$S = \dfrac{32000\pi}{20} + 2\pi(20)^2$

$= 1600\pi + 800\pi$

$= 2400\pi \text{ cm}^2$ or 7540 cm^2 (3s.f)

(iii)



(iv)

Given: $\frac{dV}{dt} = 1000 \text{ cm}^3/\text{min}$

$V = \pi r^2 h$

$V = \pi (20)^2 h$

$V = 400\pi h$

$\frac{dV}{dh} = 400\pi$

Chain rule:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= (1000) \times \frac{1}{400\pi}$$

$$= 0.796 \text{ cm}/\text{min} \quad (3 \text{ s.f})$$

4. [PJC/2017/Prelim/Q6]

The number of customers (in thousands), C, of a new company is believed to be modelled by the equation

$$C = 8(1 - e^{-kt}),$$

where t is the number of years from the time the company starts its operation and k is a positive constant.

(i) Given that the company has 7 thousand customers at the end of the 3rd year of operation, determine the exact value of k, giving your answer in the simplest form.

Using the value of k found in **(i)**,

(ii) sketch the graph of C against t, stating the equations of any asymptotes,

(iii) find the exact value of $\dfrac{\mathrm{d}C}{\mathrm{d}t}$ when $t = 2$, simplifying your answer. Give an interpretation of the value you have found, in the context of the question.

At the end of the 6th year of operation, the number of customers of the company is now believed to be modelled by the equation

$$C = -0.05t^2 + 0.7t + 5.475,$$

where $t \geq 6$.

(iv) Use differentiation to find the value of t, where $t \geq 6$, which gives the maximum value of C. Hence, find the maximum value of C.

Solution:

(i)

$$7 = 8(1 - e^{-3k})$$

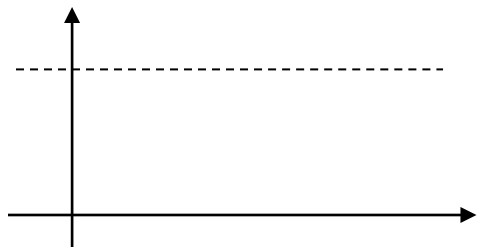
$$\frac{7}{8} = 1 - e^{-3k}$$

$$e^{-3k} = \frac{1}{8}$$

$$-3k = \ln\frac{1}{8} = -3\ln 2$$

$$k = \ln 2$$

(ii)



(iii) $C = 8(1-e^{-3k}) = 8 - 8e^{-t\ln 2}$

$\frac{dC}{dt} = 8\ln 2e^{-t\ln 2}$

$\left.\frac{dC}{dt}\right|_{t=2} = 8\ln 2e^{-2\ln 2}$

$= 8\ln 2e^{\ln\frac{1}{4}}$

$= 8\ln 2\left(\frac{1}{4}\right)$

$= 2\ln 2$

This value indicates that the number of customers is **increasing** at a rate of 2ln2 **thousands per year at the end of the second year of operation.**

(iv) $\frac{dC}{dt} = -0.1t + 0.7$

$-0.1t + 0.7 = 0$

$t = 7$

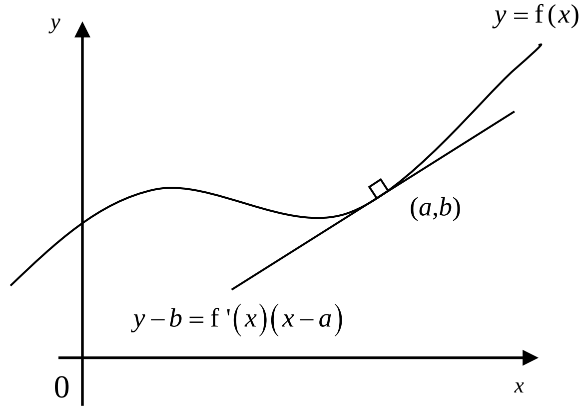
$C = -0.05(7)^2 + 0.7(7) + 5.475 = 7.925$

Hence, the maximum number of customers is 7.925 **thousands** customers or 7925 when $t = 7$.

Summary

1. Equation of Tangents

For any curve $y = \mathrm{f}(x)$,
Gradient at point (a, b) is $\mathrm{f}'(a)$
Equation of tangent at (a, b) is $y - b = \mathrm{f}'(x)(x - a)$



2. Stationary Points

Three types of stationary points:

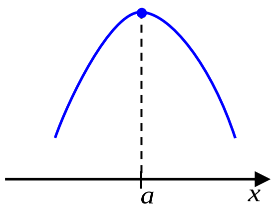
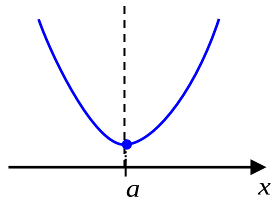
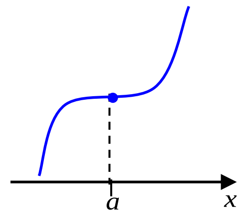
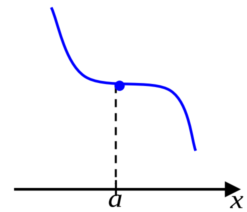
(a) Maximum point,
(b) Minimum point
(c) Point of inflexion.

Tests for the nature of stationary points:
(a) First Derivative Test
(b) Second Derivative Test

First Derivative Test

Suppose $y = \mathrm{f}(x)$ has a stationary point at $x = a$ i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=a} = 0$ or $\mathrm{f}'(a) = 0$.

Check the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$ on either side of stationary points.

<table>
<tr><th>Maximum point</th><th>Minimum point</th><th colspan="2">Stationary point of Inflexion</th></tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><table><tr><td>x</td><td>a^-</td><td>a</td><td>a^+</td></tr><tr><td>$\frac{dy}{dx}$</td><td>+</td><td>0</td><td>–</td></tr><tr><td>Sketch of tangent</td><td>/</td><td>—</td><td>\</td></tr></table></td>
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</tr>
<tr><td></td><td></td><td colspan="2">$\frac{dy}{dx}$ does not change sign as x increases through a</td></tr>
</table>

Note:

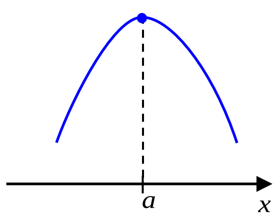
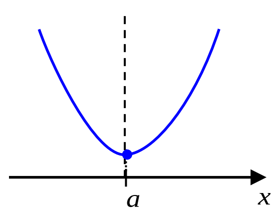
1. a^- denotes a value slightly less than a.
   a^+ denotes a value slightly more than a.

2. The values of $\frac{dy}{dx}$ has to be evaluated for every value of x in the First Derivative Test.

<u>Second Derivative Test</u>

Suppose $y = \mathrm{f}(x)$ has a stationary point at $x = a$ i.e. $\frac{dy}{dx}\big|_{x=a} = 0$ or $\mathrm{f}'(a) = 0$.

<table>
<tr><th>Maximum point</th><th>Minimum point</th><td rowspan="3">If $\frac{d^2y}{dx^2} = 0$ at $x = a$,

Second derivative test is inconclusive, we go back to the first derivative test.
.</td></tr>
<tr>
<td></td>
<td></td>
</tr>
<tr><td>$\frac{d^2y}{dx^2} < 0$ at $x = a$</td><td>$\frac{d^2y}{dx^2} > 0$ at $x = a$</td></tr>
</table>

Note:

1. In general, second derivative test is used when it is easy and fast to obtain the second derivative. If finding $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is tedious, first derivative test should be used even though it is still possible to find the second derivative.
2. In general, whenever $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ at the stationary point, first derivative test should be used.

3. <u>Maxima and Minima</u>

<u>Method for Solving Maxima and Minima Problems</u>

1. Formulate a function, say A which is to be maximised or minimised .

2. Express this function in terms of one variable only, say x by using given conditions. i.e. $A = \mathrm{f}\left(x\right)$.

3. Solve $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ to solve for x and find the **stationary values** of A.
   x is the value which will make A maximum or minimum.
   There may be more than one valid answer.

4. Use the first or second derivative tests to check for the nature of the stationary points.

4. <u>Rate of Change</u>

Rate of change of $y = \frac{\mathrm{d}y}{\mathrm{d}t}$

<u>Method for Solving Rate of Change problems</u>

1. Understand what the rate of change required, e.g. $\frac{\mathrm{d}A}{\mathrm{d}t}$.
2. Identify the given rate of change, say $\frac{\mathrm{d}x}{\mathrm{d}t}$, any other relevant information.
3. Find an expression for A in terms of x.
4. Determine what you need using the Chain Rule: $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

Checklist

I am able to:

Tests for types of Stationary Points

☐ Use First Derivative Test to determine minimum point, maximum point or stationary point of inflexion.

☐ Use Second Derivative Test to determine minimum point or maximum point.

Maxima and Minima

☐ Find and use the first derivative of a simple function defined implicitly.

Equations of Tangents to a curve

☐ Find and use the equations of tangents to a curve, including cases where the curve is defined implicitly.

Rate of change

☐ Find an equation relating the variables. Methods of finding the equation include using ‘similar triangles’, the Pythagoras’ Theorem, formulae for area, volume.

☐ Calculate the rate of change using Chain Rule to link up the variables.

Graphical Calculator Skills

☐ Draw a Tangent Line in Function Mode and finding the equation of tangent line of a given point.

☐ Finding the coordinates of maximum and minimum points.