Revision Worksheet on Sequences and Series

A. Σ (Sigma) notation

Given $\sum_{r=m}^{n} u_r$,

First term = u_m

Last term = u_n

No. of terms = $n-m+1$

r: index of sum

m: lower limit

n: upper limit

Note: Lower limit of a sum **does not** always need to be 1.

B. Basic properties/rules of summation

No.	Property	Express the following examples in a similar form to the property given (*the final answer is not required*)
1	$\sum_{r=m}^{n} k = (n-m+1)k$, where $m<n$ *Note: r is the index and k is treated as a constant* $n-m+1$ *is the number of terms in the series*	$\sum_{r=3}^{10} 4 = 8\times 4$ *(Note : can use GC to check the answer if the limits are finite known values)*
2	$\sum_{r=1}^{n} ku_r = k\sum_{r=1}^{n} u_r$	$\sum_{r=1}^{8}\left(\frac{5}{r}\right) = \sum_{r=1}^{8}\left(5\times\frac{1}{r}\right) = 5\sum_{r=1}^{8}\left(\frac{1}{r}\right)$
3	$\sum_{r=1}^{n}(u_r \pm v_r) = \sum_{r=1}^{n} u_r \pm \sum_{r=1}^{n} v_r$	$\sum_{r=1}^{n}(r+2^r) = \sum_{r=1}^{n} r + \sum_{r=1}^{n} 2^r$
4	$\sum_{r=m}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{m-1} u_r$ where $m<n$	$\sum_{r=5}^{13} r! = \sum_{r=1}^{13} r! - \sum_{r=1}^{4} r!$
5	Given the formula for S_n, i.e. $\sum_{r=1}^{n} u_r$, the n^{th} term of the series, $u_n = \sum_{r=1}^{n} u_r - \sum_{r=1}^{n-1} u_r$ $= S_n - S_{n-1}$	Given $S_n = 6 - \frac{3}{2^n}$, $u_n = S_n - S_{n-1}$ $= \left(6-\frac{3}{2^n}\right) - \left(6-\frac{3}{2^{n-1}}\right)$ $= \frac{3}{2^{n-1}} - \frac{3}{2^n}$ $= \frac{6}{2^n} - \frac{3}{2^n} = \frac{3}{2^n}$

C. Formula for some standard series

No.	Formula	Example	Remark
1	$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$	$\sum_{r=1}^{20} 4r$ $= 4\sum_{r=1}^{20} r$ $= 4\times\frac{(20)(21)}{2}$ $= 840$	*Using result of* $\sum_{r=1}^{n} ku_r = k\sum_{r=1}^{n} u_r$
2	$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ *Formula will be given in the question when required*	$\sum_{1}^{n}[r(3r-1)]$ $= \sum_{r=1}^{n}(3r^2 - r)$ $= 3\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$ $= 3\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{n(n+1)}{2}$ $= \frac{n(n+1)[(2n+1)-1]}{2}$ $= n^2(n+1)$	*Using result of* $\sum_{r=1}^{n} ku_r = k\sum_{r=1}^{n} u_r$
3	$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$ *Formula will be given in the question when required*	$\sum_{6}^{15}(r^3+2)$ $= \sum_{r=6}^{15} r^3 + \sum_{r=6}^{15} 2$ $= \left(\sum_{r=1}^{15} r^3 - \sum_{r=1}^{5} r^3\right) + \sum_{r=6}^{15} 2$ $= \left[\frac{(15)(16)}{2}\right]^2 - \left[\frac{(5)(6)}{2}\right]^2 + (15-6+1)(2)$ $= 14400 - 225 + 20$ $= 14195$	*change lower limit to 1*

D. Limit and Convergence of Series

Let $S_n = \sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + ... + u_n$.

When we write $\lim_{n\to\infty} S_n = S$, where S is a unique finite value , it means that S_n approaches the finite value S as $n \to \infty$.

- If $\lim_{n\to\infty} S_n = S$, where S is a finite value, the series S_n **<u>converges</u>.**
- If $\lim_{n\to\infty} S_n$ does not exist or is infinite, the series S_n **<u>diverges</u>.**

E. Recurrence Relations

Recurrence relations are when the nth term is expressed in terms of the previous terms.
Eg. $u_{n+1} = 3u_n$; $u_1 = 3$; . To find the terms, we do the following:

When n =1, $u_2 = 3u_1 = 3(3) = 9$

When n =2, $u_3 = 3u_2 = 3(9) = 27$

When n =3, $u_4 = 3u_3 = 3(27) = 81\ldots$and so on.

Do remember how to use the graphing calculator to find the *n*th term of a recurrence relation.

<u>Practice Problems</u>

1 HCI JC 1 Promo 9758/2024/Q2

(a) Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, find $\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right]$, in terms of *n*. [4]

(b) Explain why $\sum_{r=1}^{n}\left[\left(\frac{1}{2}\right)^r - r\left(r+\frac{1}{3}\right)\right] < 1$ for all positive integers *n*. [1]

2 NYJC JC2 Prelim 9758/2019/01/Q10

For this question, you may use the results $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

(i) Find $\displaystyle\sum_{r=1}^{n} r^2(2r-1)$ in terms of *n*. [2]

(ii) Find $\displaystyle\sum_{r=1}^{n} r^2(r-1)$ in terms of *n*. Hence find $\displaystyle\sum_{r=2}^{n-1} r(r+1)^2$ in terms of *n*. [5]

(iii) Without using a graphing calculator, find the sum of the series

$$4(25) - 5(36) + 6(49) - 7(64) + \cdots - 59(3600).$$ [3]

3 SAJC JC2 Prelim 9758/2019/02/Q4 (b)

(b) **(i)** Cauchy's root test states that a series of the form $\sum_{r=0}^{\infty} a_r$ (where $a_r > 0$ for all r) converges when $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, and diverges when $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$. When $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the test is inconclusive. Using the test and given that $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all positive p, explain why the series $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ converges for all positive values of x.

[3]

(ii) By considering $(1-y)^{-2} = 1+2y+3y^2+4y^3+\dots$, evaluate $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ for the case when $x=1$.

[2]

4 TJC JC2 Prelim 9758/2019/02/Q2 (modified)

It is given that $\sum_{r=1}^{N} \left(\left(\frac{1}{7}\right)^{r+1} \frac{6r+7}{r(r+1)} \right) = \frac{1}{7} - \frac{7^{-N-1}}{N+1}$

(i) Give a reason why the series $\sum_{r=1}^{\infty} \left(\left(\frac{1}{7}\right)^{r+1} \frac{6r+7}{r(r+1)} \right)$ converges, and write down its value.

[2]

(ii) Use your answer in part **(i)** to find $\sum_{r=1}^{N} \left(\left(\frac{1}{7}\right)^{r+1} \frac{6r+13}{(r+1)(r+2)} \right)$. [3]

5 CJC JC1 Promo 9758/2024/Q3

It is given that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$.

(a) Find $\sum_{r=5}^{n+1} \frac{1}{r(r+1)}$. [3]

(b) Give a reason why the series in part **(a)** is convergent and state the value of $\sum_{r=5}^{\infty} \frac{1}{r(r+1)}$. [2]

6. **CJC JC1 Promo 9758/2024/Q8 (b)**

(b) A sequence $u_0,\ u_1,\ u_2,\ \ldots$ is given by

$$u_0 = 400 \text{ and } u_n = 1.01u_{n-1} - x \text{ for } n \geq 1, \text{ where } x \text{ is an integer.}$$

(i) Show that $u_n = 1.01^n(400) - 100x(1.01^n - 1)$. [3]

(ii) Given that $x = 16$ and $y = 1.01u_k$, where $0 < y < 16$, find the value of k and y. [4]

7 DHS JC 1 Promo 9758/2024/Q3(a)

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, evaluate $\sum_{r=-n}^{n}(r+1)(r+3)$ in terms of n. [4]

8 ACJC JC 1 Promo 9758/2024/Q8

(a) It is given that $\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$.

(i) Explain why the series $\sum_{r=1}^{\infty} \frac{1}{4r^2-1}$ converges and write down the value to which it converges. [2]

(ii) Find $\sum_{r=6}^{N} \frac{1}{(2r+1)(2r+3)}$ in terms of N, express your answer in a single fraction. [3]

(b) The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_{n+1} = 1 - \frac{1}{u_n}$, $n \geq 1$.

Find the value of u_2, u_3 and u_4. Hence find the value of $\sum_{r=1}^{50} u_r$. [4]

9 RI JC 1 Promo 9758/2024/Q5

(a) Find $\sum_{r=0}^{n}[(n+2)r+n]$, giving your answer in terms of n. [3]

[You may use the result $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for the rest of this question.]

(b) By writing down the first two and the last two terms in the series, find $\sum_{r=1}^{n}(r+2)^3$, giving your answer in terms of n. [3]

(c) Find $1^3-2^3+3^3-4^3+5^3-6^3+\ldots+(2n-1)^3-(2n)^3$, giving your answer in terms of n. [3]

End Answers

1. (a) $1-\left(\frac{1}{2}\right)^n-\frac{n}{3}(n+1)^2$

2. (i) $\frac{1}{6}n(n+1)(3n^2+n-1)$ (ii) $\frac{1}{12}n(n+1)(3n+2)(n-1)$; $\frac{1}{12}n(n+1)(n-1)(3n+2)-4$

   (iii) -108836

3. (b)(ii) 6

4. $\frac{1}{7}$; $\frac{1}{14}-\frac{7^{-N-2}}{N+2}$

5. (a) $\frac{1}{5}-\frac{1}{n+2}$ (b) $\frac{1}{5}$

6. (b)(ii) $k=28$; $y=14.6$

7. a) $\frac{2n+1}{3}(n^2+n+9)$

8 (ai) $\frac{1}{2}$ (aii) $\frac{N-5}{13(2N+3)}$ (b) 26.5

9 (a) $\frac{n}{2}(n+1)(n+4)$; (b) $\frac{1}{4}(n+2)^2(n+3)^2-9$;(c) $-n^2(4n+3)$