2021 SH2 H3 Mathematics Preliminary Examinations Marking Scheme

Qn	Solution
1	Consider the set of remainders of integers when divided by $2n$, $X=\{0,1,\cdots,2n-1\}$. For $p,q\in X$, consider the pairs (p,q) such that $p+q\equiv 0(\bmod 2n)$.. Note that $(1,2n-1),(2,2n-2),\cdots(n-1,n+1)$,$(n,n),(0,0)$ are such pairs and there are precisely $(n-1)+1+1=n+1$ such pairs. As there are $n+2$ integers in S and only $n+1$ of such pairings possible, there must be (at least) two integers s_1,s_2 in S that belong the same pairing. Then either **Case 1** s_1,s_2 are not both congruent to 0 or n $\bmod 2n$, so $s_1+s_2\equiv 0(\bmod 2n)$ **Case 2** s_1,s_2 are actually both congruent to 0 or n $\bmod 2n$, so $s_1-s_2\equiv 0(\bmod 2n)$..

Qn	Solution
2 (i)	Suppose that $a+jb \equiv a+kb \pmod{m}$ for some $j,k \in \{0,1,2,\ldots,m-1\}$. Then, $$a+jb \equiv a+kb \pmod{m}$$ $$jb \equiv kb \pmod{m}$$ $$j \equiv k \pmod{m} \quad (\because \gcd(b,m)=1)$$ which implies that $j=k$ since $j,k \in \{0,1,2,\ldots,m-1\}$. This means that $a+jb=a+kb$. Thus, $a,\ a+b,\ a+2b,\ \ldots,\ a+(m-1)b$ are all incongruent modulo *m*.
2 (ii)	Suppose there exists a prime $q<n$ such that $q \nmid d$. Now, consider the *q* terms $$p,\ p+d,\ p+2d,\ \ldots,\ p+(q-1)d.$$ By **(i)**, and since $\gcd(q,d)=1$, all the *q* terms above are incongruent modulo *q*. So, each of the terms is congruent to exactly one of 0, 1, 2, …, *q* – 1 modulo *q*. In particular, $$p+rd \equiv 0 \pmod{q}$$ for some $r \in \{0,1,2,\ldots,q-1\}$. Thus, $q \mid p+rd$. The next goal is to show that $p+rd$ is composite, i.e., $q<p+rd$. Note that $n \le p$ because otherwise, if $p<n$, then one of the terms in $$p,\ p+d,\ p+2d,\ \ldots,\ p+(n-1)d$$ would be $p+pd=p(1+d)$ which is a product of two integers greater than 1 and hence composite. By the inequalities $q<n\le p\le p+rd$, we conclude that $p+rd$ is composite ($\because q \mid p+rd$ and $q<p+rd$), which is a contradiction to all the terms in the arithmetic progression being prime.

Qn	Suggest solutions
3	By AM-GM inequality, $$\frac{a^2+b^2}{2} \ge \sqrt{a^2b^2} = \lvert ab \rvert \ge ab.$$ Similarly, $\frac{b^2+c^2}{2} \ge bc$ and $\frac{c^2+a^2}{2} \ge ca$. Summing them up, we have $$a^2+b^2+c^2 \ge ab+bc+ca.$$

<table>
<tr><td>2(i)</td><td>
From the previous part, we know $x^2+y^2+z^2 \ge S$, so

$$\begin{aligned} S^2 &= (x+y+z)^2 \\ &= x^2+y^2+z^2+2(xy+yz+zx) \\ &\ge S+2S \\ &= 3S \end{aligned}$$

Since $S>0$, $S \ge 3$. (proven)
</td></tr>
<tr><td>2(ii)</td><td>
The Cauchy-Schwartz inequality

$$(a_1b_1+a_2b_2+a_3b_3)^2 \le \left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right),$$

yields $(x+y+z)^2 \le \left(1+y+z^2\right)\left(x^2+y+1\right)$
when $a_1=1, b_1=\sqrt{y}, c_1=z, a_2=x, b_2=\sqrt{y}, c_2=1$, so

$$\frac{1}{x^2+y+1} \le \frac{1+y+z^2}{(x+y+z)^2}.$$

Similarly,

$$\frac{1}{y^2+z+1} \le \frac{1+z+x^2}{(x+y+z)^2} \text{ and } \frac{1}{z^2+x+1} \le \frac{1+x+y^2}{(x+y+z)^2}.$$

Summing them up,

$$\begin{aligned} &\frac{1}{x^2+y+1}+\frac{1}{y^2+z+1}+\frac{1}{z^2+x+1} \\ &\le \frac{1+y+z^2}{(x+y+z)^2}+\frac{1+z+x^2}{(x+y+z)^2}+\frac{1+x+y^2}{(x+y+z)^2} \\ &= \frac{3+S+x^2+y^2+z^2}{(x+y+z)^2} \\ &\le \frac{2S+x^2+y^2+z^2}{(x+y+z)^2} \\ &= \frac{2(xy+yz+zx)+x^2+y^2+z^2}{(x+y+z)^2} \\ &= \frac{(x+y+z)^2}{(x+y+z)^2} \\ &= 1 \text{ (proven)} \end{aligned}$$
</td></tr>
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Qn	Solution
4(a)	Consider the remainders of squares modulo 4. By the division algorithm, every integer is congruent to exactly one of 0, 1, 2 or 3 modulo 4.

	$0^2 \equiv 0 \pmod 4$ $\quad 2^2 \equiv 0 \pmod 4$ $1^2 \equiv 1 \pmod 4$ $\quad 3^2 \equiv 1 \pmod 4$ $\therefore$ Any square is congruent to only 0 or 1 modulo 4. $\therefore$ The sum of two squares will be congruent to only 0, 1 or 2 modulo 4. However, each term in the sequence can be generically written as $\underbrace{999...999}_{k\ 9\text{'s}} = 10^k - 1 = 100\left(10^{k-2}\right) - 1$ for some integer $k \geq 2$. Since $100\left(10^{k-2}\right) - 1 = 4(25)\left(10^{k-2}\right) - 1 \equiv 3 \pmod 4$, each term in the sequence is congruent to 3 modulo 4. Therefore, the sequence does not contain term which is the sum of 2 squares.
4(b) (i)	We attempt to generate an arithmetic progression (AP) with terms ending with *n* 9's, where $n \in \mathbb{Z}^+$. For any positive integer *n*, consider the AP with $a = 10^n - 1 = \underbrace{999...999}_{n\ 9\text{'s}}$ and $b = 10^n$, so that each term in this AP ends with *n* consecutive 9's. Since $b - a = 1$, $\gcd(a, b) = \gcd\left(10^n - 1, 10^n\right) = 1$. Thus, for any positive integer *n*, by Dirichlet's Theorem, the AP with $a = 10^n - 1$ and $b = 10^n$ contains infinitely many primes. Thus, there exists a prime ending with *n* consecutive 9's for any positive integer *n*.
4(b) (ii)	Let *a* and *b* be some given coprime positive integers. Let *p* be a prime in the given arithmetic progression (AP). Such a prime exists due to Dirichlet's Theorem. So, $p = a + mb$ for some integer $m \geq 0$. <u>Claim</u>: For any integer $r \geq 1$, $p + rpb$ will be a composite number in the given AP. <u>Proof of claim</u>: $p + rpb = a + mb + rpb = a + (m + rp)b$ which is in the given AP as $m + rp > 0$. Also, $p + rpb = p(1 + rb)$ is composite as $p > 1$ and $1 + rb > 1$. Thus, the given AP contains infinitely many composite numbers, i.e., $p + pb,\ p + 2pb,\ p + 3pb,\ p + 4pb, ...$ (In fact, these composite numbers themselves form an arithmetic progression) -- <u>Alternative</u>:

	Note that since a and b are positive integers, $a+b\geq 2$. Let $s=a+b$. Then, the AP contains the terms $$s, s+b, \dots, s+sb, \dots, s+2sb, \dots, s+3sb, \dots$$ because for each $t\in\mathbb{Z}^+$, $s+tsb=a+b+tsb=a+(1+ts)b$ and $1+ts\geq 2$. Also, $s+tsb=s(1+tb)$. Since $s\geq 2$ and $1+tb\geq 2$, $s+tsb$ is composite for all $t\in\mathbb{Z}^+$. Therefore, the AP contains infinitely many composite numbers.

Qn	Solution
5	$u^2=\tan x \Rightarrow x=\tan^{-1}\left(u^2\right) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=\frac{2u}{1+u^4}$ Thus, $\int \sqrt{\tan x}\ \mathrm{d}x=\int \sqrt{u^2}\cdot\frac{2u}{1+u^4}\ \mathrm{d}u=\int \frac{2u^2}{u^4+1}\ \mathrm{d}u.$ Factorising the denominator, $u^4+1=\left(u^2-au+1\right)\left(u^2+au+1\right)$ Comparing coefficients of u^2, $a^2+2=0 \Rightarrow a=\pm\sqrt{2}$. Thus, $u^4+1=\left(u^2-\sqrt{2}u+1\right)\left(u^2+\sqrt{2}u+1\right)$ Hence, $\frac{2u^2}{u^4+1}=\frac{Au+B}{u^2-\sqrt{2}u+1}+\frac{Cu+D}{u^2+\sqrt{2}u+1}$. Solving, $A=\frac{1}{\sqrt{2}}$, $C=-\frac{1}{\sqrt{2}}$ and $B=D=0$. So, $\int \frac{2u^2}{u^4+1}\ \mathrm{d}u=\frac{1}{\sqrt{2}}\int \frac{u}{u^2-\sqrt{2}u+1}-\frac{u}{u^2+\sqrt{2}u+1}\ \mathrm{d}u$ $=\frac{1}{2\sqrt{2}}\int \frac{2u-\sqrt{2}+\sqrt{2}}{\left(u-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}-\frac{2u+\sqrt{2}-\sqrt{2}}{\left(u+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\ \mathrm{d}u$ $=\frac{1}{2\sqrt{2}}\left\{\begin{array}{l}\ln\left(u^2-\sqrt{2}u+1\right)+2\tan^{-1}\left(\frac{u-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)\\ -\ln\left(u^2+\sqrt{2}u+1\right)+2\tan^{-1}\left(\frac{u+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)\end{array}\right\}+c$ $=\frac{1}{2\sqrt{2}}\ln\left(\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right)+\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}u-1\right)+\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}u+1\right)+c$ $=\frac{1}{2\sqrt{2}}\ln\left(\frac{\tan x-\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right)$ $+\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2\tan x}-1\right)+\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2\tan x}+1\right)+c$

Qn	Solution

6 (a)	We see that $$\frac{x+14}{(x-1)(x-2)(x+4)}=\frac{Ax+B}{(x-1)(x-2)}+\frac{C}{x+4}$$. $$x+14=(Ax+B)(x+4)+C(x-1)(x-2).$$ When $x=1$, $$15=(A+B)(5)$$ $$3=A+B$$ When $x=2$, $$16=(2A+B)(6)$$ $$\frac{16}{6}=A+3$$ $$A=-\frac{1}{3}$$ Thus, $B=\frac{10}{3}$. Furthermore, when $x=0$, $$14=4B+2C$$ $$14-\frac{40}{3}=2C$$ $$C=\frac{1}{3}$$ So, $\frac{x+14}{(x-1)(x-2)(x+4)}=\frac{-\frac{1}{3}x+\frac{10}{3}}{(x-1)(x-2)}-\frac{\frac{1}{3}}{x+4}$.
6 (b)	Let $P(n)$ be the statement ' $\frac{p(x)}{q(x)}=\sum_{i=1}^{n}\frac{c_i}{x-x_i}$ for polynomials $p(x)$, $q(x)$ such that $\deg(p(x))<\deg(q(x))$ and $q(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$ '. When $n=1$, $q(x)=x-x_1$, and since $\deg(p(x))<\deg(q(x))$, $p(x)$ is a constant function, let's call it c. Thus, $$\frac{p(x)}{q(x)}=\frac{c}{x-x_1}.$$ Suppose that the statement is true up to $n=k$, i.e.,' $\frac{p_0(x)}{q_0(x)}=\sum_{i=1}^{k}\frac{c_i}{x-x_i}$ for polynomials $p_0(x),q_0(x)$,such that $\deg(p_0(x))<\deg(q_0(x))$ and $q_0(x)=(x-x_1)(x-x_2)\ldots(x-x_k)$'

	Let $p(x), q(x) = (x-x_1)(x-x_2)\ldots(x-x_k)(x-x_{k+1})$ such that $\deg(p(x)) < \deg(q(x)) = k+1$. Let $q_1(x) = (x-x_1)(x-x_2)\ldots(x-x_k)$. Consider the polynomial $$p(x) - \frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x).$$ Note that when $x = x_{k+1}$, $p(x) - \frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x) = 0$. Thus, by quotient-remainder theorem, $$p(x) - \frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x) = a(x)(x - x_{k+1})$$ for some polynomial $a(x)$. As $\deg(p(x)) \le \deg(q_1(x))$, $\deg(a(x)) < \deg(p(x)) \le \deg(q_1(x))$. Consequently, $$p(x) - \frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x) = a(x)(x - x_{k+1})$$ $$p(x) = \frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x) + a(x)(x - x_{k+1})$$ $$\frac{p(x)}{q(x)} = \frac{a(x)(x-x_{k+1})}{q(x)} + \frac{\frac{p(x_{k+1})}{q_1(x_{k+1})} q_1(x)}{q(x)}$$ $$= \frac{a(x)}{q_1(x)} + \frac{\frac{p(x_{k+1})}{q_1(x_{k+1})}}{x - x_{k+1}}$$ $$= \sum_{i=1}^{k} \frac{c_i}{x - x_i} + \frac{\frac{p(x_{k+1})}{q_1(x_{k+1})}}{x - x_{k+1}}$$ $$= \sum_{i=1}^{k+1} \frac{c_i}{x - x_i},$$, where the 2nd last line follows from our induction hypothesis and $c_{k+1} = \frac{p(x_{k+1})}{q_1(x_{k+1})}$. Consequently, if P_k is true, then P_{k+1} is true as well. Since P_1 is true and if P_k is true, then P_{k+1} is true , by mathematical induction P_n is true for $n \ge 1$. In other words, for any $n \ge 1$, $\frac{p(x)}{q(x)}$ can be written in the form $\sum_{i=1}^{n} \frac{c_i}{x - x_i}$ for suitable constants c_i.

Qn	Solution

<table>
<tr><td>7 (a) (i)</td><td>
<table>
<tr><td>Configuration</td><td>No. of ways</td></tr>
<tr><td>All 1s</td><td>1</td></tr>
<tr><td>2, 1….1</td><td>6</td></tr>
<tr><td>3, 1….1</td><td>6</td></tr>
<tr><td>2,2,1,1</td><td>$\frac{4!}{2!2!}$</td></tr>
<tr><td>2,3,1</td><td>3!</td></tr>
<tr><td>2,2,2</td><td>1</td></tr>
<tr><td>3,3</td><td>1</td></tr>
</table>

Total number of ways
= 27</td></tr>
<tr><td>7 (b) (i)</td><td>$r=6$
$\left(x^0+x^1+x^2+x^3+x^4+x^5+x^6\right)$
$\times\left(x^0+x^2+x^4+x^6\right)$
$\times\left(x^0+x^3+x^6\right)$
$=...+7x^6+...$
7 ways.</td></tr>
<tr><td>7 (b) (ii)</td><td>$\left(x^0+x^1+x^2+x^3+...+x^9+x^{10}\right)$
$\times\left(x^0+x^2+x^4+x^6+x^8+x^{10}\right)$
$\times\left(x^0+x^3+x^6+x^9\right)$</td></tr>
<tr><td>7(c) (i)</td><td>5</td></tr>
<tr><td>7(c) (ii)</td><td>There are 5 partitions of 24 where each part is distinct and is a prime number.</td></tr>
<tr><td>7(d) (i)</td><td>$(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)...$
$=1+x+x^2+x^3+...$</td></tr>
<tr><td>7(d) (ii)</td><td>Every positive integer can be uniquely written as a sum of distinct powers of 2.</td></tr>
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Qn	Solution
8 (i)	$\beta_k = k\pi$
8 (ii)	$\sin x = x\prod_{k=-\infty}^{\infty}\left(1-\frac{x}{k\pi}\right)$ $\frac{\sin x}{x} = \prod_{k=-\infty}^{\infty}\left(1-\frac{x}{k\pi}\right)$ $= \prod_{k=1}^{\infty}\left(1-\frac{x}{k\pi}\right)\left(1+\frac{x}{k\pi}\right)$ $= \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$
8 (iii)	From the Maclaurin series of $\sin x$, $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ $\frac{\sin x}{x} = 1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots$ On the other hand, by collecting the x^2 terms from the factorised form of $\frac{\sin x}{x}$, coefficient of x^2 in the expansion of $\frac{\sin x}{x}$ $=-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots$ $=-\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}$ Therefore, by comparing the coefficients of x^2, $-\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}=-\frac{1}{6}$ $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$
8 (iv)	The coefficient of x^{2m} in the Maclaurin series of $\frac{\sin x}{x}$ is $-\frac{1}{(2m+1)!}$. The coefficient of x^{2m} from the factorised form of $\frac{\sin x}{x}$ is $=-\frac{1}{\pi^{2m}}-\frac{1}{2^{2m}\pi^{2m}}-\frac{1}{3^{2m}\pi^{2m}}-\frac{1}{4^{2m}\pi^{2m}}-\cdots$ $=-\sum_{n=1}^{\infty}\frac{1}{n^{2m}\pi^{2m}}$ Therefore, by comparing the coefficients of x^{2m}, $-\sum_{n=1}^{\infty}\frac{1}{n^{2m}\pi^{2m}}=-\frac{1}{(2m+1)!}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=\frac{\pi^{2m}}{(2m+1)!}$

Qn	Solution
8 (l. p.)	Let S_k denote the sum of the first *k* terms in the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$. Observe that $$\begin{aligned} S_{2^{k+1}}-S_{2^k} &= \frac{1}{\left(2^k+1\right)^p}+\frac{1}{\left(2^k+2\right)^p}+\cdots\frac{1}{\left(2^{k+1}\right)^p} \\ &< \underbrace{\frac{1}{\left(2^k\right)^p}+\frac{1}{\left(2^k\right)^p}+\cdots\frac{1}{\left(2^k\right)^p}}_{2^k \text{ times}} \\ &= \frac{2^k}{\left(2^k\right)^p}=\frac{1}{\left(2^{p-1}\right)^k} \end{aligned}$$ Hence $$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^p} &= 1+\sum_{k=0}^{\infty}\left(S_{2^{k+1}}-S_{2^k}\right) \\ &\le 1+\sum_{k=0}^{\infty}\frac{1}{\left(2^{p-1}\right)^k} \\ &= 1+\frac{1}{\left(1-\frac{1}{2^{p-1}}\right)} \\ &= 1+\frac{2^{p-1}}{2^{p-1}-1}, \end{aligned}$$ which is finite. Thus, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges. **OR** The series $\sum_{k=0}^{\infty}\frac{1}{\left(2^{p-1}\right)^k}$ is a convergent geometric series since its common ratio, $\frac{1}{2^{p-1}}$, has an absolute value that is smaller than 1. Thus, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Qn | **Working**

9 (i)

g_n counts the number of calves born in the n^{th} year.

Calves are born by two groups of cows: cows which reach their fourth year, and cows which reach at least their fifth year.

The number of cows in these two groups are mutually exclusive and exhaustive if we look at only cows which can bear calves.

In the n^{th} year:

The number of cows which reach their fourth year is given by g_{n-3}.

Since there are g_{n-1} number of calves born in $(n-1)^{\text{th}}$ year, it means that there are g_{n-1} cows which are in their fourth year. Consequently, in the n^{th} year, there are g_{n-1} cows which have reached at least their fifth year.

Since each cow which reaches at least their fourth year gives birth to exactly one calf, we have

$g_n = g_{n-1} + g_{n-3}$ for $n \geq 4$.

(ii)

Method 1: Recurrence

$$\begin{aligned} g_{10} &= g_9 + g_7 \\ &= (g_8 + g_6) + (g_6 + g_4) \\ &= (g_7 + g_5) + 2g_6 + g_4 \\ &= (g_6 + g_4) + g_5 + 2g_6 + g_4 \\ &= g_5 + 3g_6 + 2g_4 \\ &= 3 + 3\times 4 + 2\times 2 \\ &= 19 \end{aligned}$$

$g_4 = g_3 + g_1 = 1 + 1 = 2$

$g_5 = g_4 + g_2 = 2 + 1 = 3$

$g_6 = g_5 + g_3 = 3 + 1 = 4$

Method 2: Enumeration

n	1	2	3	4	5	6	7	8	9	10
No. of calves born in n^{th} year	1	1	1	2	3	4	6	9	13	19

(iii)

$h_2 = 1$

$h_3 = 2$

(iv)

h_{n-1} counts the number of compositions of n consisting of parts of sizes 1 or 3 (or both).

	For each composition counted by h_{n-1}, we add a part of size 1 at the end. In this way, h_{n-1} would have counted all compositions of n ending with part of size 1 . For each composition counted by h_{n-3}, we add a part of size 3 to the end. In this way, h_{n-3} would have counted all compositions of n ending with part of size 3 . Since compositions ending with part of size 1 and part of size 3 are mutually exclusive and exhaustive, $h_n = h_{n-1} + h_{n-3}$.
(v)	$2n-2r$ is the smallest numerator r is the greatest denominator We need $2n-2r \geq r$ So we need the greatest integer r such that $2n-2r \geq r$ $2n \geq 3r$ $n \geq 1.5r$ which is equivalent to solving for $r = \left\lfloor \frac{n}{1.5} \right\rfloor$. $\sum_{r=0}^{\left\lfloor \frac{n}{1.5} \right\rfloor} \left\lceil \frac{2n-2r}{r} \right\rceil$

Qn	Suggest solutions
10(i)	Let the rectangle has vertices (a,b), $(a+k,b)$, $(a+k,b+h)$ and $(a,b+h)$, where $a,b,k,h \in \mathbb{Z}$, $k>0$ and $h>0$. In this case, $A = kh$. Interior to the rectangle, any lattice point has x-coordinate between $(a+1)$ and $(a+k-1)$ inclusive, and y-coordinate between $(b+1)$ and $(b+h-1)$ inclusive. By MP, $I = [(a+k-1)-(a+1)+1][(b+h-1)-(b+1)+1]$ $= (k-1)(h-1)$ On each of its horizontal boundary (excluding the vertices), there are $(a+k-1)-(a+1)+1 = k-1$ lattice points. On each of its vertical boundary (excluding the vertices), there are $(b+h-1)-(b+1)+1 = h-1$ lattice points. Thus, $B = 2(k-1)+2(h-1)+4$ $= 2k+2h$ Therefore, for any rectangle,

	$I+\frac{B}{2}-1=(k-1)(h-1)+\frac{2k+2h}{2}-1$ $=kh-k-h+1+k+h-1$ $=kh=A$ (proven)
10(ii)	Assemble two such identical triangles into one rectangle in part **(i)**, then each triangle has area $\frac{1}{2}kh$. Supposing there are l lattice points on the hypotenuse shared by the two triangles (excluding the vertices), these points are interior to the rectangle. For each triangle, $I=\frac{(k-1)(h-1)-l}{2}$ and $B=(k-1)+(h-1)+l+3=k+h+l+1$ There, for any such right-angled triangle $I+\frac{B}{2}-1=\frac{(k-1)(h-1)-l}{2}+\frac{(k+h+l+1)}{2}-1$ $=\frac{kh-k-h+1-l+k+h+l+1-2}{2}$ $=\frac{kh}{2}=A$ (proven)
10(iii)	Construct a rectangle subscribing the triangle, then rotate and/or reflect the figure into the following configure. The area and the number of lattice points inside/on the boundaries will not be affected. Let the numbers of lattice points interior to (I), (II), (III) and (IV) be I_1, I_2, I_3 and I_4 respectively. Let the number of lattice points on the boundaries of (I) be B_1 (including the vertices). Now, the number of lattice points interior to the rectangle is $(I_1+I_2+I_3+I_4+B_1-3)$ Let the numbers of lattice points on the boundaries of (II), (III) and (IIII) be B_2, B_3 and B_4 respectively (including the vertices). Now, the number of lattice points on the boundaries of the rectangle is $(B_2+B_3+B_4-B_1)$. From part **(i)**, the area of rectangle is $(I_1+I_2+I_3+I_4+B_1-3)+\frac{(B_2+B_3+B_4-B_1)}{2}-1$ $=\left(I_1+\frac{B_1}{2}-1\right)+\left(I_2+\frac{B_2}{2}-1\right)+\left(I_3+\frac{B_3}{2}-1\right)+\left(I_4+\frac{B_4}{2}-1\right)$ For part **(ii)**, the areas of (II), (III) and (IV) are $\left(I_2+\frac{B_2}{2}-1\right), \left(I_3+\frac{B_3}{2}-1\right)$ and $\left(I_4+\frac{B_4}{2}-1\right)$ respectively.

	Therefore, the area of (I) is $\left(I_1+\frac{B_1}{2}-1\right)$. (proven)
	Let $\mathrm{P_n}$ be the proposition that $A=I+\frac{B}{2}-1$ for any *n*-sided polygon, where $n\geq 3$. P_3 is true as shown in part **(iii)**. Assuming P_k is true for some $k\geq 3$. For P_{k+1}: We can partition a (*k* + 1)-sided polygon into a triangle and a *k*-sided polygon. Let the numbers of lattice points interior to the triangle and to the *k*-sided polygon be I_t and I_p respectively. Let the numbers of lattice points on the boundaries of the triangle and the *k*-sided polygon be B_t and B_p respectively. By the assumption made and part **(iii)** result, the area of the (*k* + 1)-sided polygon is $$A=\left(I_t+\frac{B_t}{2}-1\right)+\left(I_p+\frac{B_p}{2}-1\right)$$ Let the number of lattice points on edge shared by the triangle and the *k*-sided polygon be *L* (excluding the two end-points). The number of lattice points interior to the (*k* + 1)-sided polygon is $$I=I_t+I_p+L,$$ and on the boundaries is $B=B_t+B_p-2L-2$. $$\begin{aligned}I+\frac{B}{2}-1&=I_t+I_p+L+\frac{B_t+B_p-2L-2}{2}-1\\&=I_t+I_p+L+\frac{B_t}{2}+\frac{B_p}{2}-L-1-1\\&=\left(I_t+\frac{B_t}{2}-1\right)+\left(I_p+\frac{B_p}{2}-1\right)=A\end{aligned}$$ By mathematical induction, *Pick's Theorem* is true for any *n*-sided polygon.