RAFFLES INSTITUTION
2023 YEAR 5 PROMOTION EXAMINATION

CANDIDATE NAME

CLASS 24

MATHEMATICS 9758

3 hours

Candidates answer on the Question Paper.

Note that the list of formulae can be found on page 2.

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank page on page 24 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only						
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	
/ 4	**/ 8**	**/ 8**	**/ 7**	**/ 8**	**/ 8**	
Q7	**Q8**	**Q9**	**Q10**	**Q11**		**TOTAL**
/ 12	**/ 9**	**/ 12**	**/ 12**	**/ 12**		**/ 100**

This document consists of **23** printed pages and **1** blank page.

PURE MATHEMATICS

Algebraic series

Binomial expansion:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Maclaurin expansion:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^n}{n!}\mathrm{f}^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x|<1)$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Trigonometry

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

$$\sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$

$$\sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

$$\cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$

$$\cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

Principal values:

$$-\tfrac{1}{2}\pi \le \sin^{-1} x \le \tfrac{1}{2}\pi \qquad (|x| \le 1)$$

$$0 \le \cos^{-1} x \le \pi \qquad (|x| \le 1)$$

$$-\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Derivatives:

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\sec x$	$\sec x\tan x$

Integrals

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \tfrac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \tfrac{1}{2}\pi)$

Vectors

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

Vector product:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

1 The first 3 terms of a sequence are given by $u_1 = 1823$, $u_2 = 200$ and $u_3 = 2023$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

2 **Do not use a calculator in answering this question.**

Solve the inequality $x^2 + 6x + 5 \leq \dfrac{x+5}{3x+1}$. [4]

Hence solve

(a) $x^4 + 6x^2 + 5 \le \dfrac{x^2+5}{3x^2+1}$, [2]

(b) $(\ln x)^2 + 6\ln x + 5 \le \dfrac{5+\ln x}{1+3\ln x}$. [2]

3 Vectors $\mathbf{a}$ and $\mathbf{b}$ are such that the magnitude of $\mathbf{a}$ is 2 and $\mathbf{b}$ is a unit vector perpendicular to $\mathbf{a}$.

(a) Find the area of the parallelogram with adjacent sides formed by the vectors $\mathbf{a}+3\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$. [3]

A vector $\mathbf{c}$ is such that $\mathbf{b}\times\mathbf{c}=21(\mathbf{a}\times\mathbf{b})$.

(b) Show that $\mathbf{c}=\lambda\mathbf{b}-21\mathbf{a}$, where λ is a constant. [2]

(c) Give the geometrical meaning of $|\mathbf{b}\,.\,\mathbf{c}|$ and find the possible values of λ if $|\mathbf{b}\,.\,\mathbf{c}| = 5$. [3]

4 **(a)** Write $\dfrac{1}{r(r+1)}$ in partial fractions. [1]

(b) Using your answer to part **(a)**, find $\displaystyle\sum_{r=1}^{n} \frac{1}{r^2 + r}$. [2]

Hence find

(i) $\displaystyle\sum_{r=n+1}^{2n} \frac{1}{r^2+r}$, [2]

(ii) $\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+\dfrac{1}{4\times 5}+\cdots$ [2]

5 In this question you may use expansions from the List of Formulae (MF26).

(a) Find the first four non-zero terms of the Maclaurin series of $e^{-x}\left(1+\cos 3x\right)$, in ascending powers of x. [4]

(b) It is given that the first three terms of this series are equal to the first three terms in the series expansion, in ascending powers of x, of $\frac{1}{a+bx}+cx^2$. Find the values of a, b and c. [4]

6 **(a)** Let $y = a^x$, where a is a positive constant. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$. [2]

(b) A curve has equation

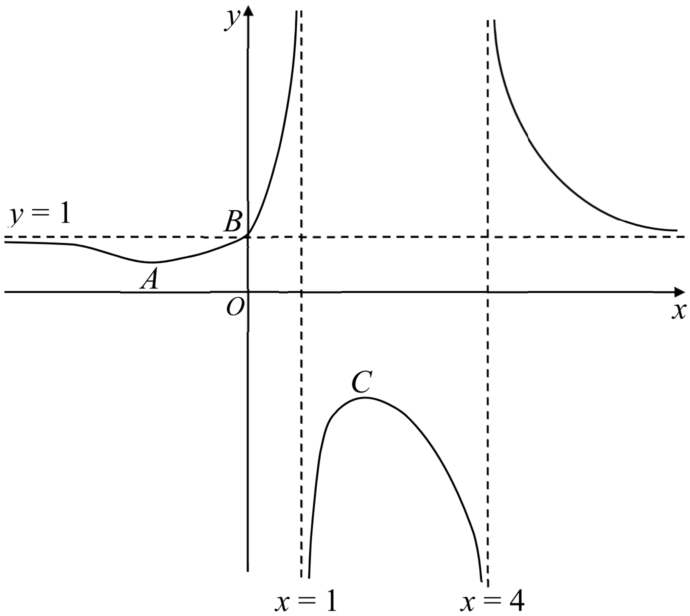
$$y\left(3^x\right) + 2^{y-1} = 2.$$

Find the equation of the normal to the curve at the point $(0, 1)$. Give your answer in the form $y = Ax + B$, where A and B are constants in the exact form. [6]

7 **(a)** A curve C has equation $y = \dfrac{ax-3}{(x-3)(x-1)}$ where a is a real constant such that $a \neq 1,\ 3$. Determine the range of values of a for which the curve C has no turning points. [4]

7 [Continued]

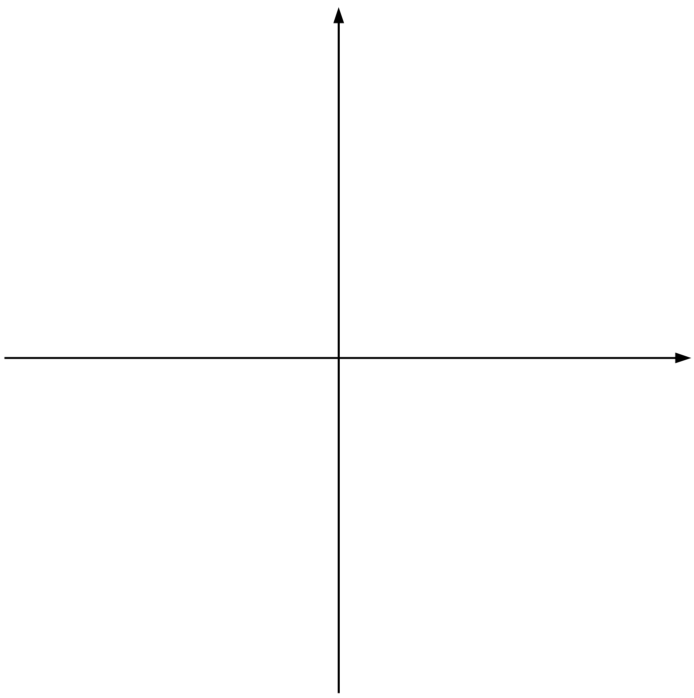
(b)



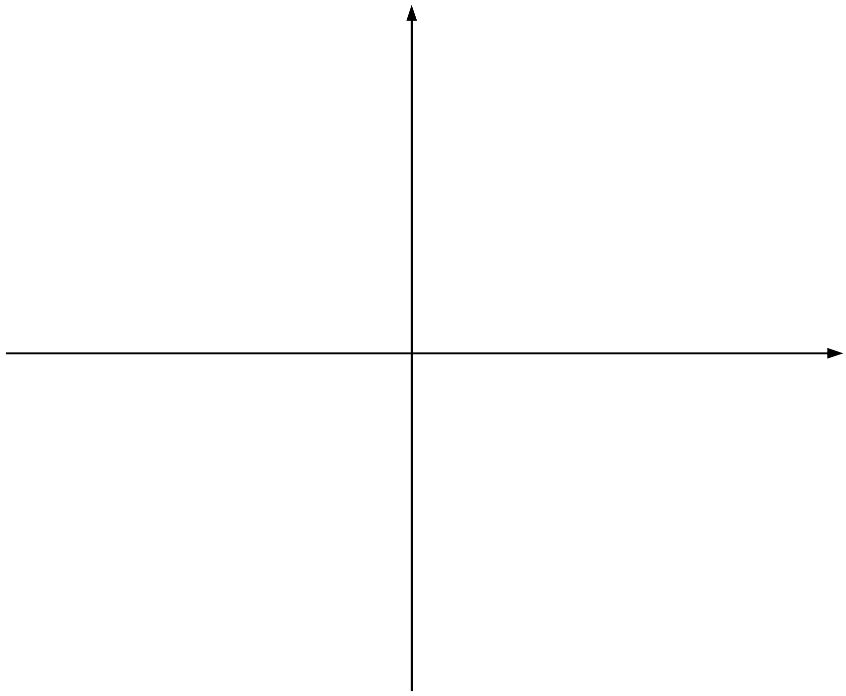
The diagram above shows the graph of $y = \mathrm{f}(x)$. It has asymptotes $y = 1$, $x = 1$ and $x = 4$. The curve cuts the y-axis at the point $B(0, 1)$, has a minimum at the point $A\left(-2, \dfrac{2}{3}\right)$ and a maximum at the point $C(2, -2)$.

By showing clearly the equations of asymptotes and the coordinates of the points corresponding to A, B and C where possible, sketch, on **separate diagrams**, the graphs of

(i) $y = 3\mathrm{f}(x+1)$, [4]



(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]



8 **Do not use a calculator in answering this question.**

(a) One root of the equation $2z^3 - 5z^2 + \alpha z - 5 = 0$, where $\alpha \in \mathbb{R}$, is $z = 1 - 2\text{i}$. Find the value of α and the other roots. [5]

(b) The complex number z is given by

$$z = \frac{\left(-2 + 2\sqrt{3}\,\mathrm{i}\right)^2}{\cos\frac{1}{12}\pi + \mathrm{i}\sin\frac{1}{12}\pi}.$$

Find $|z|$ and $\arg(z)$. [4]

9 It is given that

$$\mathrm{f} : x \mapsto \left| \frac{1}{x-4} \right| \text{, where } x \in \mathbb{R}, x \neq 4,$$

$$\mathrm{g} : x \mapsto \ln(x+2) \text{ , where } x \in \mathbb{R}, x > -2.$$

(a) Explain why the composite function gf exists and find gf in a similar form. [3]

(b) Find the range of gf. [1]

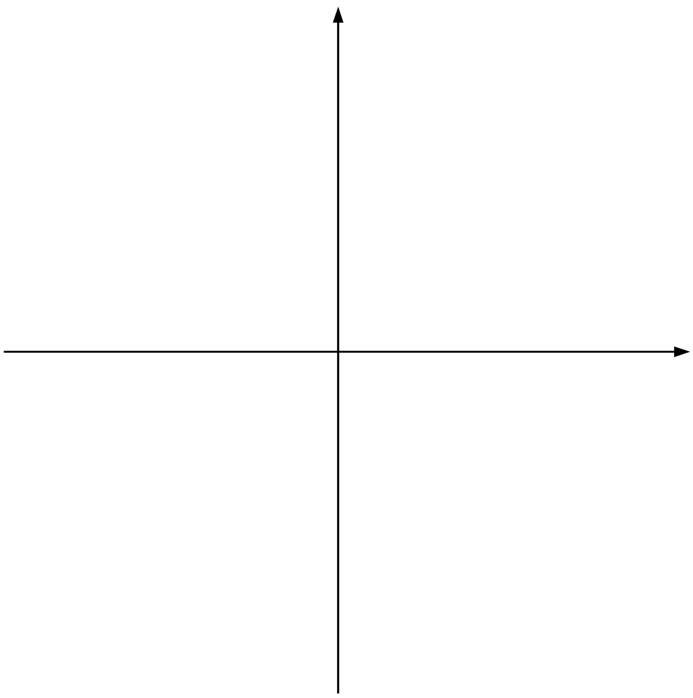
(c) Explain why f does not have an inverse. [1]

(d) If the domain of f is further restricted to $x < k$, state the maximum value of k such that the function f^{-1} exist. [1]

In the rest of this question, the domain of f is $x \in \mathbb{R},\ x < 3$.

(e) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(f) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram and state the geometrical relation between the two graphs. [3]



10 In long-track speed skating, races are run counter-clockwise on a 400-metre two-lane oval rink. A full 400-metre round the rink is known as a lap. The current world records for the 10000 metres men's race and 500 metres men's race are 12 minutes 30.74 seconds (11 Feb 2022) and 33.61 seconds (9 Mar 2019) respectively.

Abel and Caine are 2 skaters training for the 10000 metres men's race. During a particular training session for Abel, he completes the first lap in b seconds and he takes 10% longer to complete each succeeding lap than he does in the previous lap.

(a) Write down an expression for the time taken by Abel to complete n laps, in terms of b and n.

Hence find the value of b that will enable Abel to complete 10000 metres in 13 minutes. [3]

(b) Comment on the feasibility of this value of b in the context of the question. [1]

After training for 6 months, Abel and Caine both entered a 10000 metres men's race. During the race, Abel completes the first lap in k seconds and then he takes 1% longer to complete each succeeding lap than he does in the previous lap. Caine also completes the first lap in k seconds and on each subsequent lap he spends d seconds more than he spent on the previous lap. They arrive at the finish line at the same time.

(c) Find the value of $\frac{d}{k}$, correct to 5 decimal places. [4]

(d) Given that Abel completes his 25$^{\text{th}}$ lap in p seconds and Caine completes his 25$^{\text{th}}$ lap in q seconds, evaluate $\frac{q}{p}$, giving your answer correct to 3 decimal places.

Hence determine the skater who has the faster time for the first 24 laps. [4]

11 The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

respectively, where λ and μ are parameters.

(a) Without the use of a calculator, show that l_1 and l_2 are skew lines. [3]

(b) Find a vector, $\mathbf{n}$, that is perpendicular to both l_1 and l_2. [1]

Referred to the origin O, points P and Q have position vectors $\begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix}$ respectively.

(c) Find the exact length of projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$. [2]

A plane π has equation $3x + z = 11$.

(d) Find, in degrees, the acute angles between l_1 and π, and between l_2 and π. Hence, or otherwise, determine which one of l_1 or l_2 is not intersecting π. [4]

(e) Find the exact distance between the line determined in part **(d)** and π. [2]

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You may continue your working on this page if necessary, indicating the question number(s) clearly.