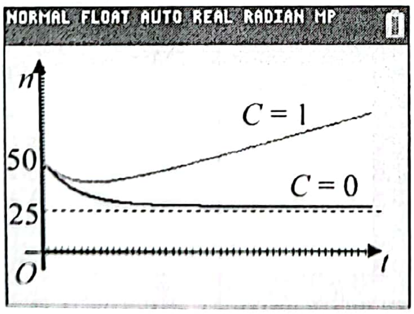
Jurong Junior College
2016 JC2 H2 Mathematics Prelim Paper 2 Solutions

Qn	Solution
1(a)	$\frac{dy}{dx}=\sec^2 y$ $\int \cos^2 y \, dy=\int 1 \, dx$ $\int \frac{\cos 2y+1}{2} \, dy=\int 1 \, dx$ $\frac{1}{2}\left[\frac{\sin 2y}{2}+y\right]=x+c$ When $y=0$, $x=1 \Rightarrow c=-1$ $\therefore \; x=\frac{1}{4}\sin 2y+\frac{1}{2}y+1$
1(b)(i)	$\frac{d^2n}{dt^2}=e^{-\frac{t}{5}}$ $\frac{dn}{dt}=\int e^{-\frac{t}{5}}dt=-5e^{-\frac{t}{5}}+C$ $n=25e^{-\frac{t}{5}}+Ct+D$
1(b)(ii)	$\frac{d^2n}{dt^2}=e^{-\frac{t}{5}}>0$ for all values of t. Solution curves are concave upwards.
1(b)(iii)	When $t=0, n=50$ $50=25e^0+C(0)+D$ $D=25$ $n=25e^{-\frac{t}{5}}+Ct+25$ When $C=0$, $n=25e^{-\frac{t}{5}}+25$. As $t\to\infty, n\to 25$ When $C=1$, $n=25e^{-\frac{t}{5}}+t+25$. As $t\to\infty, n\to\infty$ 

Qn	Solution
2(a)	Area of parallelogram $= \lvert \mathbf{a} \times (2\mathbf{a} + 3\mathbf{b}) \rvert$ $= \lvert 2(\mathbf{a} \times \mathbf{a}) + 3(\mathbf{a} \times \mathbf{b}) \rvert$ $= 3\lvert \mathbf{a} \times \mathbf{b} \rvert$ $= 3\lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin 30°$ $= 3(4)(5)\frac{1}{2}$ $= 30$
(b) (i)	A vector equation of l is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$. Let A be the point $(1,\ 3,\ 1)$ on l. Perpendicular distance from P to l $= \dfrac{\left\lvert \overrightarrow{AP} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert}$ $= \frac{1}{\sqrt{5}} \left\lvert \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert$ $= \frac{1}{\sqrt{5}} \left\lvert \begin{pmatrix} 4 \\ -5 \\ 8 \end{pmatrix} \right\rvert$ $= \frac{\sqrt{105}}{\sqrt{5}} = \sqrt{21}$
(ii)	Acute angle between l and L $= \cos^{-1} \dfrac{\left\lvert \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rvert}$ $= \cos^{-1} \frac{1}{\sqrt{5}}$ $= 63.4°$

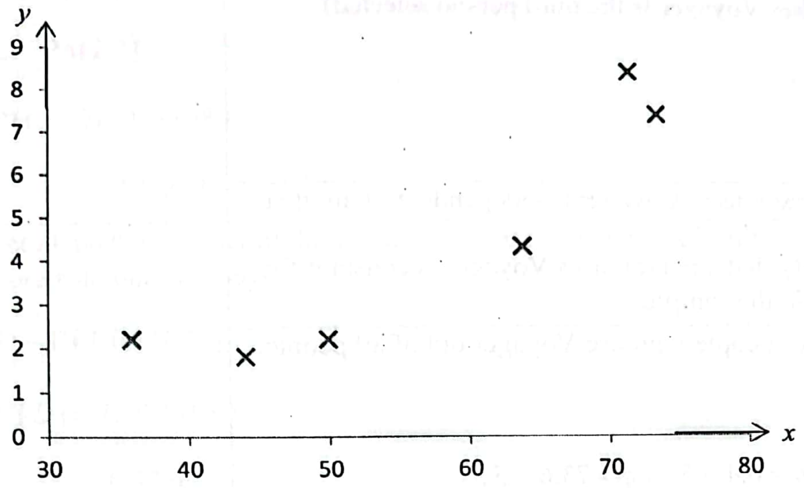
Qn	Solution
3(i)	Area of equilateral $\Delta = \frac{1}{2}x^2 \sin 60° = \frac{\sqrt{3}}{4}x^2$ Given that the volume of the box is 250 cm^3 $V = \frac{\sqrt{3}}{4}x^2 y = 250$ $y = \frac{1000}{\sqrt{3}x^2}$ Surface Area $A = 3xy + 3kxy + 2\left(\frac{\sqrt{3}}{4}x^2\right)$ $= 3xy(1+k) + \frac{\sqrt{3}}{2}x^2$ $= 3x(1+k)\frac{1000}{\sqrt{3}x^2} + \frac{\sqrt{3}}{2}x^2$ $= \frac{1000\sqrt{3}(1+k)}{x} + \frac{\sqrt{3}}{2}x^2$ (shown)
(ii)	For stationary points, $\frac{dA}{dx} = -\frac{1000\sqrt{3}(1+k)}{x^2} + \sqrt{3}x = 0$ $x^3 = 1000(1+k)$ $x = 10(1+k)^{\frac{1}{3}}$ $\frac{d^2A}{dx^2} = \frac{2000\sqrt{3}(1+k)}{x^3} + \sqrt{3} > 0$ Thus, $x = 10(1+k)^{\frac{1}{3}}$ gives a minimum surface area.
(iii)	Since $y = \frac{1000}{\sqrt{3}x^2}$ $\frac{y}{x} = \frac{1000}{\sqrt{3}x^3} = \frac{1000}{\sqrt{3}(1000)(1+k)}$ $= \frac{1}{\sqrt{3}(1+k)}$

(iv)	Since $0 < k \le 1$ $1 < 1 + k \le 2$ $\frac{1}{2} \le \frac{1}{1+k} < 1$ $\frac{1}{2\sqrt{3}} \le \frac{1}{\sqrt{3}(1+k)} < \frac{1}{\sqrt{3}}$ i.e. $\frac{1}{2\sqrt{3}} \le \frac{y}{x} < \frac{1}{\sqrt{3}}$

Qn	Solution
4(i)	$a^2b = \frac{1}{2}(1+\sqrt{3}\mathrm{i})^2(1-\mathrm{i})$ $= \frac{1}{2}(1+2\sqrt{3}\mathrm{i}-3)(1-\mathrm{i})$ $= (-1+\sqrt{3}\mathrm{i})(1-\mathrm{i})$ $= (\sqrt{3}-1)+(\sqrt{3}+1)\mathrm{i}$
(ii)	$\lvert a^2b\rvert = \lvert a\rvert^2 \lvert b\rvert$ $= 2^2\left(\frac{1}{\sqrt{2}}\right)$ $= 2\sqrt{2}$ $\arg(a^2b) = 2\arg(a) + \arg(b)$ $= 2\left(-\frac{2\pi}{3}\right) - \frac{\pi}{4}$ $= -\frac{19\pi}{12}$ $\therefore\ \arg(a^2b) = -\frac{19\pi}{12} + 2\pi = \frac{5\pi}{12}$.
(iii)	Considering the imaginary part of a^2b, we have $2\sqrt{2}\sin\frac{5\pi}{12} = \sqrt{3}+1$ $\Rightarrow \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iv)	$\overline{BA}$ can be obtained by rotating $\overline{BC}$ through 90° in the anticlockwise direction about B. $i(c-b)=a-b$ $\Rightarrow c=-i(a-b)+b$ $=-ia+b(1+i)$ $=i(1+\sqrt{3}i)+\frac{1}{2}(2)$ $=(1-\sqrt{3})+i$
5 **(i)**	${}^6C_2\times{}^5C_4=75$ ways
(ii)	Number of ways if at least one of the sisters are included = number of ways without restriction – number of ways if none of the sisters is included $={}^{11}C_6-{}^8C_6$ $=434$ **Alternative Method** ${}^3C_1\times{}^8C_5+{}^3C_2\times{}^8C_4+{}^3C_3\times{}^8C_3=434$
(iii)	Select a man to be between the 2 sisters and group the 3 of them as one unit and arrange 4 units round a table Number of ways $={}^3C_1\times 3!\times 2$ $=36$
(iv)	First arrange the other 4 persons round the table. There are 4 ways to insert the sisters. Number of ways $=3!\times 4$ $=24$

Qn	Solution
6 (i)	$P(C\mid M)=\dfrac{P(C\cap M)}{P(M)}$ $=\dfrac{200}{400}=\dfrac{1}{2}$
(ii)	$P(M\cup C)=P(M)+P(C)-P(M\cap C)$ $=\dfrac{400}{1000}+\dfrac{450}{1000}-\dfrac{200}{1000}$ $=\dfrac{650}{1000}=\dfrac{13}{20}$
(iii)	$P(M'\cap B')=\dfrac{250+300}{1000}=\dfrac{11}{20}$ $P(C)=\dfrac{9}{20}$ $P(C\mid M)=\dfrac{1}{2}\neq P(C)$ C and M are not independent.
(iv)	No. of international studens in the sample $=0.2(200+250)+0.3(130+300)+0.05(120)=225$ $P(C\mid\text{international student})=\dfrac{P(C\cap\text{international student})}{P(\text{international student})}$ $=\dfrac{\frac{(200+250)0.2}{1000}}{\frac{225}{1000}}$ $=0.4$
(v)	Number of international students studying Physics $=0.3(430)=129$ P(exactly one international student studying Physics) $=\dfrac{{}^{129}C_1\,{}^{871}C_2}{{}^{1000}C_3}$ $=0.294$
	<u>Alternative method</u> Required Probability $=\dfrac{129}{1000}\,\dfrac{871}{999}\,\dfrac{870}{998}\times 3$ $=0.294$

7 **(i)**	
(ii)	$r = 0.914099 \approx 0.914$ (to 3 s.f) Though the value of r shows a strong positive linear correlation, from the scatter diagram, it is possible that x and y may have a curvilinear relationship.
(iii)	For $y = c + dx^2$, $r = 0.93986 \approx 0.940$ Since the value of r for $y = c + dx^2$ is closer to the value of 1, $y = c + dx^2$ is a better model.
(iv)	$y = -0.88934 + 0.0015441x^2$ When $x = 55$, $y = -0.88934 + 0.0015441(55)^2$ $y = 3.7816 \approx 3.8$ (to 1 d.p)
	Since $x = 55$ is within the range of data given and $r \approx 0.940$ is close to 1, the estimation is reliable.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8
(i)</td><td>P(first person that uses Voyager is the third person selected)

$= 0.92 \times 0.92 \times 0.08$

$= 0.067712$</td></tr>
<tr><td>(ii)</td><td>1. Whether a person uses Voyager is independent of another person.
2. The probability that a person uses Voyager is constant for every person in the sample.</td></tr>
<tr><td>(iii)</td><td>Let Y be the number of people who use Voyager out of 80 people.

$Y \sim \mathrm{B}(80, 0.08)$

Since $n = 80 > 50$, $np = 6.4 > 5$, $nq = 73.6 > 5$,

$Y \sim \mathrm{N}(6.4, 5.888)$ *approx*

$\mathrm{P}(Y < 10) \xrightarrow{c.c} \mathrm{P}(Y \le 9.5)$

$= 0.899295$

$= 0.899$ (to 3 s.f.)</td></tr>
<tr><td>(iv)</td><td>Let V be the number of people who use Voyager out of n people.

$V \sim \mathrm{B}(n, 0.08)$

$\mathrm{P}(V \ge 10) > 0.2$

$1 - \mathrm{P}(V \le 9) > 0.2$
$\mathrm{P}(V \le 9) < 0.8$

Using GC,

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl<table><tr><th>X</th><th>Y1</th></tr><tr><td>90</td><td>.81786</td></tr><tr><td>91</td><td>.80902</td></tr><tr><td>92</td><td>.79999</td></tr><tr><td>93</td><td>.79078</td></tr><tr><td>94</td><td>.7814</td></tr></table>
Least value of $n = 92$</td></tr>
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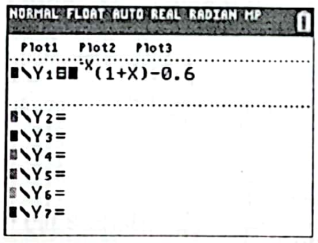
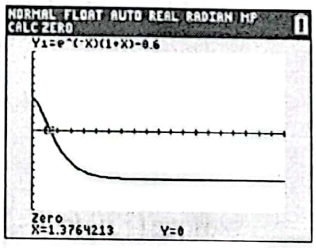
9 (i)	Let C be the number of boxes of chocolate biscuits sold in a day. $C \sim \text{Po}(2.2)$ $\text{P}(C=0)=0.11080$ $=0.111$ (to 3 s.f.)
(ii)	Let D be the number of days that no boxes of chocolate biscuits were sold out of 7 days. $D \sim \text{B}(7, 0.11080)$ $\text{E}(D)=7\times 0.11080$ $=0.77562$ $=0.776$
(iii)	Let S be the number of boxes of strawberry biscuits sold in a day. $S \sim \text{Po}(\lambda)$ $\text{P}(S<2)=0.6$ $\text{P}(S=0)+\text{P}(S=1)=0.6$ $e^{-\lambda}\left(\frac{\lambda^0}{0!}\right)+e^{-\lambda}\left(\frac{\lambda^1}{1!}\right)=0.6$ $e^{-\lambda}(1+\lambda)=0.6$
	Using GC,   $\lambda=1.376$ $=1.4$ (to 1 d.p)

(iv)	Let T be the total number of boxes of chocolate and strawberry biscuits sold in 7 days. $T \sim \text{Po}\left(7\times2.2+7\times1.376\right)=\text{Po}(25.032)$ $\text{P}(T>25)=1-\text{P}(T\le25)$ $=0.44962$ $=0.450$ (to 3 s.f)
(v)	Let X be number of boxes of chocolate biscuits sold in 30 days. $X \sim \text{Po}\left(30\times2.2\right)=\text{Po}(66)$ Since $\lambda=66>10$, $X \sim \text{N}(66,66)$approx Let Y be number of boxes of strawberry biscuits sold in 30 days. $Y \sim \text{Po}\left(30\times1.376\right)=\text{Po}(41.28)$ Since $\lambda=41.28>10$, $Y \sim \text{N}(41.28,41.28)$approx $X-Y \sim \text{N}(24.72,107.28)$approx $\text{P}\left(X-Y>0\right)\xrightarrow{c.c}\text{P}\left(X-Y>0.5\right)$ $=0.99032$ $=0.990$ (to 3 s.f.)

10 (i)	Choose a plant randomly from the first 10 plants, say the 5^{th} plant. Choose every 10^{th} plant thereafter until 8 plants are selected i.e. 5^{th}, 15^{th}, 25^{th}, … The 8 plants selected will be *evenly spread out* across the row of 80 plants.
(ii)	Unbiased estimate of the population mean, $\hat{\mu}$ $= \frac{\sum(x-150)}{80} + 150$ $= -\frac{160}{80} + 150$ $= 148$ Unbiased estimate of the population variance, s^2 $= \frac{1}{80-1}\left[\sum(x-150)^2 - \frac{\left(\sum(x-150)\right)^2}{80}\right]$ $= \frac{1}{79}\left[5520 - \frac{(-160)^2}{80}\right]$ $= \frac{5200}{79}$
(iii)	$H_0 : \mu = 150$ $H_1 : \mu < 150$ Under H_0, since n =80 >50, by Central Limit Theorem, $\overline{X} \sim N\left(150, \frac{5200}{79(80)}\right)$ approx. Test statistic $Z = \frac{\overline{X} - 150}{\sqrt{\frac{5200}{79(80)}}} \sim N(0,1)$ approx. From GC, p–value = 0.013731 = 0.0137 (to 3 s.f.) $\alpha = 0.01$ Since p–value = 0.0137 > $\alpha = 0.01$, we do not reject H_0 at 1% level of significance and conclude that there is insufficient evidence that the researcher's claim is invalid.

(iv)	It means that there is a probability of 0.01 of concluding that the population mean mass of a new variety of potato is less than 150g given that the population mean mass of a new variety of potato is in fact 150g.
	Unbiased estimate of the population variance $=\frac{8}{7}k^2$ $H_0 : \mu = 150$ $H_1 : \mu < 150$ Under H_0, test statistic $T = \frac{\overline{X} - 150}{\sqrt{\frac{S^2}{8}}} \sim t(7)$ $\alpha = 0.01$ Researcher's claim is invalid at 1% level of significance $\Rightarrow H_0$ is rejected at 1% level of significance $\Rightarrow t \le -2.9980$ $\Rightarrow \frac{148.5-150}{\sqrt{\frac{k^2}{7}}} \le -2.9980$ $\Rightarrow k \le 1.3238$ $\therefore k \le 1.32$ (to 3 s.f)