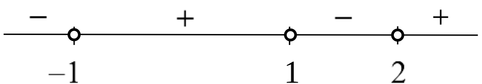
Solutions for 2024 JC1 H2 Mathematics Promotional Examination

1 **(a)**	Let x, y, z be the original price (\$) of a ticket for 'Senior Citizen', 'Adult' and 'Child' respectively. $5x+12y+6z=2440$ --- (1) $15(0.8)x+10y+5z=2660$ $12x+10y+5z=2660$ --- (2) $8x+10y+12\left(\frac{y}{2}\right)=2560$ $8x+16y=2560$ --- (3) Using GC, x = 80, y = 120 and z = 100. The original price of a ticket for 'Senior Citizen', 'Adult' and 'Children' is \$80, \$120 and \$100 respectively.
(b)	$2(80)+2(120)+3(100)=700$ Lee family spends \$700.
2	$\displaystyle\sum_{r=n+1}^{2n}\left(4r^3+n\right)$ $\displaystyle=4\sum_{r=n+1}^{2n}r^3+\sum_{r=n+1}^{2n}n$ $\displaystyle=4\left(\sum_{r=1}^{2n}r^3-\sum_{r=1}^{n}r^3\right)+(n)(n)$ $\displaystyle=4\left(\frac{(2n)^2}{4}(2n+1)^2-\frac{n^2}{4}(n+1)^2\right)+n^2$ $=4n^2(2n+1)^2-n^2(n+1)^2+n^2$ $=n^2\left[4\left(4n^2+4n+1\right)-\left(n^2+2n+1\right)+1\right]$ $=n^2\left(15n^2+14n+4\right)$

3 **(a)**	$x^2-2x+3=\left(x-1\right)^2+2$ Since $\left(x-1\right)^2 \geq 0$ for all $x \in \mathbb{R}$, $\left(x-1\right)^2+2 \geq 2 > 0$. $\therefore$ x^2-2x+3 is always positive for all values of x .
(b)	$\dfrac{\left(x^2-2x+3\right)\left(1-x\right)}{x^2-x-2}<0$ Since x^2-2x+3 is always positive, $\dfrac{\left(1-x\right)}{x^2-x-2}<0$. $\dfrac{\left(1-x\right)}{x^2-x-2}<0$ $\dfrac{\left(x-1\right)}{\left(x+1\right)\left(x-2\right)}>0$  $\therefore$ $-1<x<1$ or $x>2$
(c)	Replace x with $-x$ in $\dfrac{\left(x^2-2x+3\right)\left(1-x\right)}{x^2-x-2}<0$ $\therefore$ $-1<\sqrt{x}<1$ or $\sqrt{x}>2$ $0 \leq x<1$ or $x>4$

4 **(a)**	$\ln y=(\cos x)\ln x$ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=(-\sin x)\ln x+(\cos x)\frac{1}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=y\left[(-\sin x)\ln x+(\cos x)\frac{1}{x}\right]$ $=x^{\cos x}\left[\frac{\cos x}{x}-(\sin x)\ln x\right]$ OR $\ln y=(\cos x)\ln x$ $y=\mathrm{e}^{(\cos x)\ln x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=[(-\sin x)(\ln x)+(\cos x)(\frac{1}{x})]\ \mathrm{e}^{(\cos x)\ln x}$
(b) **(i)**	$\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy\right)-\left(x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y^2\right)=0$ $\left(x^2-2xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-2xy$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2xy}{x^2-2xy}$
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2xy}{x^2-2xy}=1$ $y^2-2xy=x^2-2xy$ $y^2=x^2$ $y=\pm x$ $y=x\Rightarrow x^3-x^3=6$ $0=6$ Thus, no solution. $y=-x\Rightarrow -x^3-x^3=6$ $x^3=-3$ $x=-\sqrt[3]{3}$ $y=\sqrt[3]{3}$ Thus $\left(-\sqrt[3]{3},\sqrt[3]{3}\right)$.

5
(a)

$$3[xy]+2\left[\frac{1}{2}x^2\sin 60°\right]=300$$

$$3xy+\frac{\sqrt{3}}{2}x^2=300$$

$$3xy=300-\frac{\sqrt{3}}{2}x^2$$

$$y=\frac{100}{x}-\frac{x}{2\sqrt{3}}$$

$$V=\left(\frac{1}{2}x^2\sin 60°\right)y$$

$$=\left(\frac{\sqrt{3}}{4}x^2\right)\left(\frac{100}{x}-\frac{x}{2\sqrt{3}}\right)$$

$$=25\sqrt{3}x-\frac{1}{8}x^3 \text{ (Shown)}$$

(b)

$$\frac{\mathrm{d}V}{\mathrm{d}x}=25\sqrt{3}-\frac{3}{8}x^2=0$$

$$x^2=\frac{200}{\sqrt{3}}$$

$$x=\pm\frac{\sqrt{200}}{\sqrt[4]{3}}$$

Since $x>0$, $x=\frac{\sqrt{200}}{\sqrt[4]{3}}=10.746 \text{ (5 sf)}$.

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=-\frac{3}{4}x=-\frac{3\sqrt{200}}{4\sqrt[4]{3}}<0$$

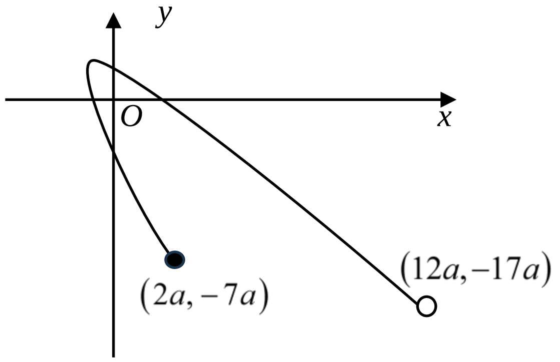
Hence V is maximum when $x=\frac{\sqrt{200}}{\sqrt[4]{3}}$.

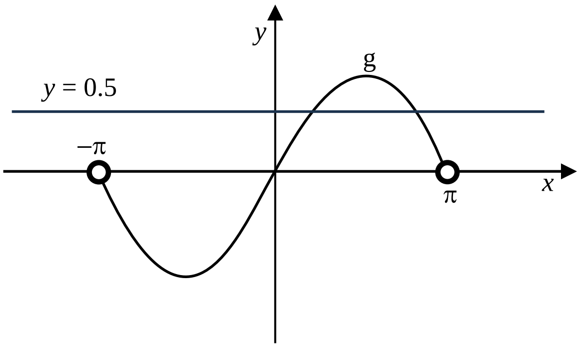
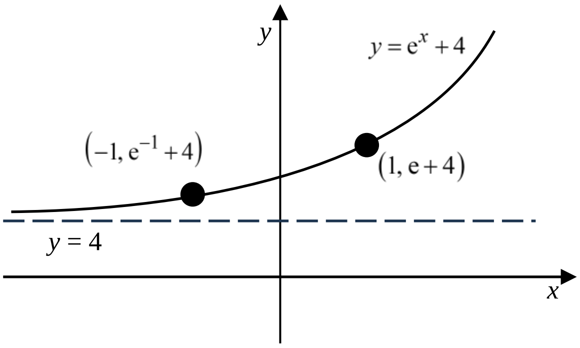
Alternatively:

x	10.7	$\frac{\sqrt{200}}{\sqrt[4]{3}}$	10.8
$\frac{\mathrm{d}V}{\mathrm{d}x}$	0.368	0	–0.439
Slope	/	–	\

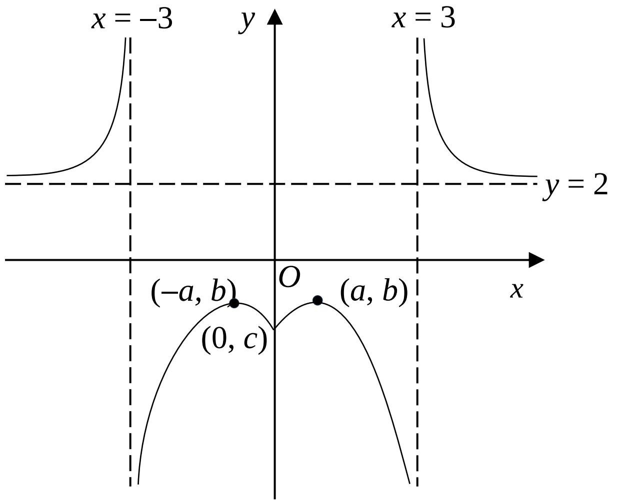
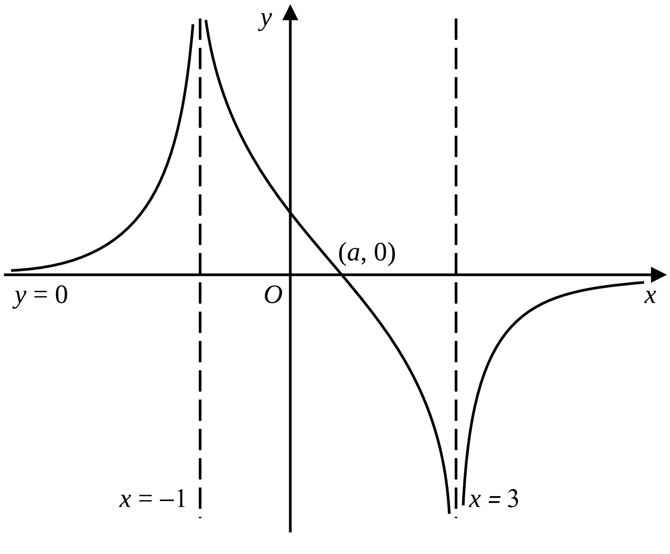
Maximum V

$$=25\sqrt{3}\left(\frac{\sqrt{200}}{\sqrt[4]{3}}\right)-\frac{1}{8}\left(\frac{\sqrt{200}}{\sqrt[4]{3}}\right)^3=310 \text{ cm}^3 \text{ (3 sf)}$$

6 **(a)**	
(b)(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=a(2t-1)$ $\frac{\mathrm{d}y}{\mathrm{d}t}=a(-4t)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4t}{2t-1}$ At $t=-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4}{-3}$ Gradient of normal is $\frac{3}{4}$ $y-(-a)=\frac{3}{4}(x-2a)$ $y=\frac{3}{4}x-\frac{5}{2}a$
(b)(ii)	$a(1-2t^2)=\frac{3}{4}a(t^2-t)-\frac{5}{2}a$ $4-8t^2=3t^2-3t-10$ $11t^2-3t-14=0$ $(11t-14)(t+1)=0$ $t=\frac{14}{11}$ or $t=-1$ (rej. original point) $x=a\left[\left(\frac{14}{11}\right)^2-\frac{14}{11}\right]=\frac{42}{121}a$, $y=a\left[1-2\left(\frac{14}{11}\right)^2\right]=-\frac{271}{121}a$ $\left(\frac{42}{121}a,-\frac{271}{121}a\right)$

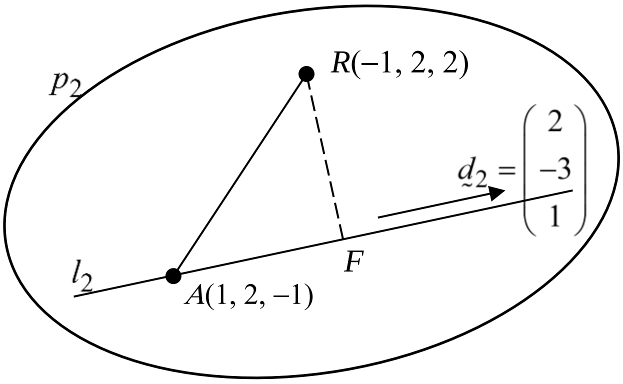
7 **(a)**	Let $y=\mathrm{f}(x)$ $y=\mathrm{e}^{x}+4$ $\mathrm{e}^{x}=y-4$ $x=\ln(y-4)$ $\mathrm{f}^{-1}(x)=\ln(x-4)$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=(4, 4+\mathrm{e}]$ $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=(-\infty, 1]$
(b)	 $y=0.5$ cuts the graph of g twice (more than once), g is not one-one, g^{-1} does not exist.
(c)	$\mathrm{R}_{\mathrm{g}}=[-1,1]$, $\mathrm{D}_{\mathrm{f}}=(-\infty,1]$ $\mathrm{R}_{\mathrm{g}}\subseteq\mathrm{D}_{\mathrm{f}}$, hence the composite function fg exists.
(d)	$\mathrm{fg}(x)=\mathrm{f}(\sin x)$ $\quad=\mathrm{e}^{\sin x}+4$ $\mathrm{D}_{\mathrm{fg}}=\mathrm{D}_{\mathrm{g}}=(-\pi, \pi)$
(e)	To find R_{fg} : Let $\mathrm{R}_{\mathrm{g}}=[-1,1]$ be new domain of f :  $\mathrm{R}_{\mathrm{fg}}=\left[\mathrm{e}^{-1}+4, \mathrm{e}+4\right]$

8 **(a)**	**Translation** by 2 units in the **negative *x*-direction**.

	Scaling by a scale **factor** of $\frac{1}{3}$, **parallel to the *x*-axis**. **Translation** by 1 unit in the **positive *y*-direction**. OR **Scaling** by a scale **factor** of $\frac{1}{3}$, **parallel to the *x*-axis**. **Translation** by $\frac{2}{3}$ units in the **negative *x*-direction**. **Translation** by 1 unit in the **positive *y*-direction**.
(b) **(i)**	
(b) **(ii)**	

9 (a)	$\int \sin^2 3\theta \, \mathrm{d}\theta = \int \frac{1-\cos 6\theta}{2} \, \mathrm{d}\theta$ $= \frac{1}{2}\theta - \frac{1}{12}\sin 6\theta + C$
9 (b)	$\int x(\ln x)^2 \, \mathrm{d}x = \frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2} \cdot 2\ln x \frac{1}{x} \, \mathrm{d}x$ $= \frac{1}{2}(x\ln x)^2 - \int x\ln x \, \mathrm{d}x$ $= \frac{1}{2}(x\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, \mathrm{d}x\right)$ $= \frac{1}{2}(x\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{x^2}{4} + C$
9 (c)	$\int_1^3 \frac{x^2}{\sqrt{2x^3-1}} \, \mathrm{d}x = \frac{1}{6}\int_1^3 6x^2\left(2x^3-1\right)^{-\frac{1}{2}} \, \mathrm{d}x$ $= \frac{1}{6}\left[\frac{\left(2x^3-1\right)^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^3$ $= \frac{1}{3}\left[\left(2(3)^3-1\right)^{\frac{1}{2}} - \left(2(1)^3-1\right)^{\frac{1}{2}}\right]$ $= \frac{1}{3}\left(\sqrt{53}-1\right)$
9 (d)	$\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u$ When $x = 0$, $u = 1$ When $x = \ln\sqrt{3}$, $u = \sqrt{3}$ $\int_0^{\ln\sqrt{3}} \frac{\mathrm{e}^{3x}}{\mathrm{e}^{2x}+1} \, \mathrm{d}x = \int_1^{\sqrt{3}} \frac{u^3}{u^2+1} \cdot \frac{1}{u} \, \mathrm{d}u$ $= \int_1^{\sqrt{3}} 1 - \frac{1}{u^2+1} \, \mathrm{d}u$ $= \left[u - \tan^{-1}u\right]_1^{\sqrt{3}}$ $= \left(\sqrt{3} - \tan^{-1}\sqrt{3}\right) - \left(1 - \tan^{-1}1\right)$ $= \sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}$ $= \sqrt{3} - 1 - \frac{\pi}{12}$

10 (a) (i)	$\lambda = \frac{x+1}{2} = y-4 = \frac{2-z}{3}$

	$x=-1+2\lambda$ $y=4+\lambda$ $z=2-3\lambda$ $\therefore \underset{\sim}{r}=\begin{pmatrix}-1\\4\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	Plane p_1: $x-4z=5\Rightarrow\underset{\sim}{r}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}=5$ Let θ be the acute angle between l_1 and p_1. $\sin\theta=\frac{\left\lvert\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}2\\1\\-3\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}1\\0\\-4\end{pmatrix}\right\rvert}=\frac{\lvert 2+12\rvert}{\sqrt{14}\sqrt{17}}=\frac{14}{\sqrt{14}\sqrt{17}}$ $\theta=65.2^\circ$
(iii)	Substituting line l_1 equation into p_1 equation: $\left(\begin{pmatrix}-1\\4\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}=5$ $\begin{pmatrix}-1+2\lambda\\4+\lambda\\2-3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}=5$ $-1+2\lambda-8+12\lambda=5$ $\lambda=1$ Required position vector $=\begin{pmatrix}-1\\4\\2\end{pmatrix}+(1)\begin{pmatrix}2\\1\\-3\end{pmatrix}=\begin{pmatrix}1\\5\\-1\end{pmatrix}$
(b) **(i)**	 Let *F* be foot of perpendicular from *R* to l_2.

$$\overrightarrow{AR} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

<u>Method 1: Direct formula</u>

$$RF = \frac{\left| \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|} = \frac{\left| \begin{pmatrix} 9 \\ 8 \\ 6 \end{pmatrix} \right|}{\sqrt{14}} = \frac{\sqrt{181}}{\sqrt{14}} = 3.60 \text{ units (3 s.f.)}$$

<u>Method 2: Use length of projection</u>

$$AF = \frac{\left| \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|} = \frac{1}{\sqrt{14}}$$

$$AR = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$RF = \sqrt{\sqrt{13}^2 - \left(\frac{1}{\sqrt{14}} \right)^2} = \sqrt{\frac{181}{14}} = 3.60 \text{ units (3 s.f.)}$$

<u>Method 3 (Foot of perpendicular): (Not recommended)</u>

$$\overrightarrow{OF} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\overrightarrow{RF} = \overrightarrow{OF} - \overrightarrow{OR} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{RF} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \left(\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$4 + 4\lambda + 9\lambda - 3 + \lambda = 0$$

$$\lambda = \frac{-1}{14}$$

$$RF = \left| \frac{1}{14} \begin{pmatrix} 26 \\ 3 \\ -43 \end{pmatrix} \right| = \frac{1}{14} \sqrt{26^2 + 3^2 + 43^2} = 3.60 \text{ units (3 s.f.)}$$

(b) **(ii)**	Normal of plane $p_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 6 \end{pmatrix}$ Eqn of plane p_2: $\underset{\sim}{r} \cdot \begin{pmatrix} 9 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 8 \\ 6 \end{pmatrix} = 19$ $9x + 8y + 6z = 19$
(b) **(iii)**	$\begin{cases} x - 4z = 5 \\ 9x + 8y + 6z = 19 \end{cases}$ From GC: $\underset{\sim}{r} = \begin{pmatrix} 5 \\ -\frac{13}{4} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -\frac{21}{4} \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$

11 (a)

From 1 Jan 2024 to 1 Dec 2026, there are 36 months.

$$\frac{36}{2}\left(2x+(36-1)(10)\right)=25000$$

$$2x+350=1388.88889$$

$$x=519.44 \text{ (2 d.p.)}$$

11 (b) (i)

At the end of 31 Dec 2024, the first \$200 will be worth

$$\$200\left(1.002\right)^{12}\approx\$204.85 \text{ (2 d.p.)}$$

(ii)

Method 1:

Mth	Start	End
1	200	$200(1.002)$
2	$200(1.002)+200$	$200\left(1.002^2\right)+200(1.002)$
3	$200\left(1.002^2\right)+200(1.002)$ $+200$	$200\left(1.002^3\right)+200\left(1.002^2\right)$ $+200(1.002)$
$\vdots$	$\vdots$	$\vdots$
n		$200\left(1.002^n\right)+200\left(1.002^{n-1}\right)$ $+\cdots+200(1.002)$

Total amount in the account at end of last day of n^{th} month

$$=200\left(1.002^n\right)+200\left(1.002^{n-1}\right)+\cdots+200(1.002)$$

$$=200\left[\underbrace{1.002+1.002^2+\cdots+1.002^n}_{\text{GP}, a=1.002,\ r=1.002,\ n \text{ terms}}\right]$$

$$=200\left[\frac{1.002\left(1.002^n-1\right)}{1.002-1}\right]$$

$$=100200\left(1.002^n-1\right) \text{ (Shown)}$$

Method 2:

At the end of the nth month,

The 1st \$200 deposited is worth $\$200\left(1.002\right)^n$

The 2nd \$200 deposited is worth $\$200\left(1.002\right)^{n-1}$

The 3rd \$200 deposited is worth $\$200\left(1.002\right)^{n-2}$

$\vdots$

The n^{th} \$200 deposited is worth $\$200\left(1.002\right)$

Total amount in the account

	$= 200\underbrace{\left[1.002+1.002^2+\cdots+1.002^n\right]}_{\text{GP}, a=1.002,\ r=1.002,\ n \text{ terms}}$ $= 200\left[\dfrac{1.002\left(1.002^n-1\right)}{1.002-1}\right]$ $=100200\left(1.002^n-1\right)$ (Shown)
(iii)	total amount at end of *n*th month = $100200\left(1.002^n-1\right)>4500$ Using GC, When $n=21$, total amount = 4293.64 When $n=22$, total amount = 4502.63 Dan’s account will first exceed \$4500 in Oct 2025. Amount at start of Oct 2025 $=\$4293.64+\$200$ $=\$4493.64<\$4500$ It occurs at the end of the month.