CHAPTER 7
DIFFERENTIAL EQUATIONS
Tutorial Solutions

Additional Practice Questions

1. HCI/2003/II/2

Liquid is poured into a container at a constant rate of 30 cm^3 s^{-1} and leaks out at a rate which is proportional to the volume of liquid in the container. At time t seconds, the volume of the water in the container is V cm^3. When the volume reaches 450 cm^3, it decreases at a rate of 30 cm^3 s^{-1}.

Show that $-15\frac{\mathrm{d}V}{\mathrm{d}t} = 2V - 450.$ [2]

It is given that V = 1000 cm^3 when t = 0.

(i) Show that $V = Ae^{\alpha t} + B$, where α, A and B are constants to be determined. [4]

(ii) Sketch the graph of V against t. [1]

1

$$\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}$$

$$\frac{dV}{dt} = 30 - kV$$

At V = 450, $\frac{dV}{dt} = 30$: –30 = 30 – k(450)

=> k = 2/15

$$\therefore \frac{dV}{dt} = 30 - \frac{2}{15}V$$

=> $-15\frac{dV}{dt} = -450 + 2V$ (shown)

(i)

$$\int \frac{-15}{2V-450}\,dV = \int dt$$

$$-\frac{15}{2}\ln|2V-450| = t + C$$

When t =0, V = 1000: $C = -\frac{15}{2}\ln 1550$

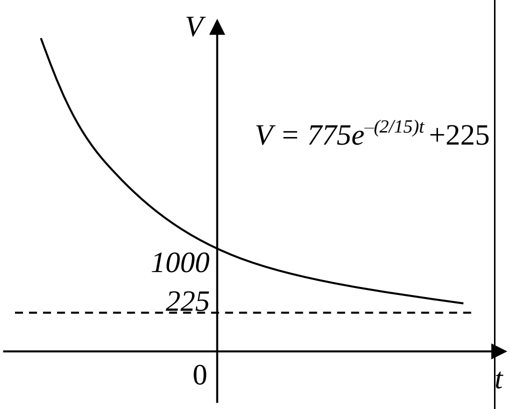
=> $-\frac{15}{2}\ln|2V-450| = t - \frac{15}{2}\ln 1550$

$$\ln|2V-450| = \frac{-2t}{15} + \ln 1550$$

$$2V - 450 = 1550e^{-(2/15)t}$$

$$\therefore V = 775e^{-(2/15)t} + 225$$

(ii)



2. RJC/2003/I/16(b)

At the instant when the radius of a sphere is r cm, its volume, V cm^3, is increasing at the rate of $\pi(\lambda - r^3)$ cm^{3}min^{-1}, where λ is a constant. Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ and show that $4r^2\frac{dr}{dt} = \lambda - r^3$. [3]

If the initial volume of the sphere is zero, find t when the volume is $\frac{\pi\lambda}{6}$ cm^3. [5]

[Volume of a sphere = $\frac{4}{3}\pi r^3$]

2	$\frac{dV}{dt} = \pi(\lambda - r^3)$ -----(1) $V = \frac{4}{3}\pi r^3$ => $\frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ Hence, from (1): $\pi(\lambda - r^3) = 4\pi r^2 \cdot \frac{dr}{dt}$ => $4r^2\frac{dr}{dt} = \lambda - r^3$ (shown) $\int \frac{4r^2}{\lambda - r^3}\,dr = \int dt$ $-\frac{4}{3}\int \frac{-3r^2}{-r^3+\lambda}\,dr = t + C$ $-\frac{4}{3}\ln\left\|\lambda - r^3\right\| = t + C$ When $t = 0$, $r = 0$: $-\frac{4}{3}\ln\lambda = C$ When $V = \frac{\pi\lambda}{6}$, $\frac{4}{3}\pi r^3 = \frac{\pi\lambda}{6}$ => $r^3 = \frac{\lambda}{8}$ At this instance, $-\frac{4}{3}\ln\left\|\lambda - \frac{\lambda}{8}\right\| = t - \frac{4}{3}\ln\lambda$ => $t = \frac{4}{3}\ln\left(\frac{8}{7}\right) = 0.178$ min

3. TPJC/2007/CT/3

(a) Show that the differential equation $\frac{dy}{dx} - y^2 = -\frac{2y}{x}$ can be reduced to $\frac{du}{dx} = \frac{u^2}{x^2}$ using the substitution $y = \frac{u}{x^2}$. Hence, find y in terms of x, given that $y = 3$ when $x = 1$. [6]

(b) At time $t = 0$, there are 1000 students studying in Tenpin Junior College. At time t years, the admission rate is equal to one tenth of the number of students N studying in the college. There is a constant expulsion of 10 students per year. Assuming nobody transfers into or out of the college, show that $10\frac{dN}{dt} = N - 100$.
Solve the differential equation to find N in terms of t. Find the time taken for the number of students studying in Tenpin Junior College to increase to 1200. [6]

3. (a) $y = \frac{u}{x^2} \Rightarrow yx^2 = u \Rightarrow 2xy + \frac{dy}{dx}x^2 = \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} - y^2 = -\frac{2y}{x} \Rightarrow x^2\frac{dy}{dx} - x^2y^2 = -2yx$

$\Rightarrow \frac{du}{dx} - 2xy - x^2y^2 = -2yx$

$\Rightarrow \frac{du}{dx} - x^2\left(\frac{u}{x^2}\right)^2 = 0 \Rightarrow \frac{du}{dx} = \frac{u^2}{x^2}$ (shown)

$\int \frac{1}{u^2}\,du = \int \frac{1}{x^2}\,dx \Rightarrow -\frac{1}{u} = -\frac{1}{x} + C$

$\Rightarrow \frac{1}{u} = \frac{1}{x} - C = \frac{1 - Cx}{x}$

$\Rightarrow u = \frac{x}{1 - Cx}$

$\Rightarrow yx^2 = \frac{x}{1 - Cx} \Rightarrow y = \frac{1}{x(1 - Cx)}$

When x = 1, y = 3, $3 = \frac{1}{1 - C} \Rightarrow C = \frac{2}{3}$

$\therefore \quad y = \frac{1}{x\left(1 - \frac{2}{3}x\right)} = \frac{3}{x(3 - 2x)}$

(b) Since N is the number of people studying in Tenpin Junior College, at year t, the rate of change in the number of people studying in Tenpin Junior College = $\frac{dN}{dt}$

Also, $\frac{dN}{dt}$ = rate of number of people admitted – number of people expelled

$= \frac{1}{10}N - 10$

$\therefore \quad 10\frac{dN}{dt} = N - 100$ (shown)

$$\int \frac{1}{N-100}\,dN = \int \frac{1}{10}\,dt \quad \Rightarrow \quad \ln|N-100| = \frac{1}{10}t + C_1$$

$$\Rightarrow \quad N - 100 = C_2 e^{\frac{1}{10}t} \quad \Rightarrow \quad N = 100 + C_2 e^{\frac{1}{10}t}$$

When $t = 0$, N = 1000. So $C_2 = 900$.

Therefore $N = 900e^{\frac{1}{10}t} + 100$

When N = 1200, $1200 = 100 + 900e^{\frac{1}{10}t}$ $\Rightarrow$ t = 2.0067

Hence, no. of years = 2.01

4. NJC/2007/CT/10

In a chemical plant, a tank initially contains 5000 litres of water dissolved with a certain amount of substance X. Solution containing 0.05kg of substance X per litre of water flows into the tank at a constant rate of 10 litres per minute. The mixture is thoroughly stirred and the resulting solution flows out of the tank at the same rate such that the volume of liquid in the tank is kept constant.

If x (kg) is the amount of substance X in the tank at time, t (min), show that the rate of change of the amount of substance X in the tank can be modelled by the differential equation $\frac{dx}{dt} = \frac{250 - x}{500}$. [2]

(i) Solve the differential equation, leaving your answer in the form $x = B - Ae^{-Ct}$ where A is an arbitrary constant, and B and C are positive constants to be determined. [3]

(ii) Deduce the long-term steady state amount of substance X in the tank? [1]

4.	Amount of X entering per minute = $10 \times 0.05 = \frac{1}{2}$ kg Amount of X leaving per minute = $\frac{x}{500}$ kg Thus the rate of change of the amount of substance X in the tank = $\frac{dx}{dt} = \frac{1}{2} - \frac{x}{500} = \frac{250 - x}{500}$ (Shown)
4i.	Integrating the above differential equation on both sides with respect to t, we have

	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{250-x}{500} \Rightarrow \int \frac{1}{250-x}\,\mathrm{d}x=\int \frac{1}{500}\,\mathrm{d}t$ $\Rightarrow -\ln\lvert 250-x\rvert=\frac{1}{500}t+C$ $\Rightarrow \lvert 250-x\rvert=\mathrm{e}^{-\frac{1}{500}t+B}$ $\Rightarrow 250-x=A\mathrm{e}^{-\frac{1}{500}t}$ $\Rightarrow x=250-A\mathrm{e}^{-\frac{1}{500}t}$ (Ans)
4ii.	As $t\to\infty$, $\mathrm{e}^{-\frac{1}{500}t}\to 0$. Hence $x\to 250$. (Ans)

5. AJC/2008/I/14

By means of the substitution $z=\frac{1}{y^2}$, show that $e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2xy^2\sqrt{y^2-1}$, where $y>1$ can be reduced to the form $\frac{1}{\sqrt{1-z}}\frac{\mathrm{d}z}{\mathrm{d}x}=-4xe^{-x^2}$. Hence find the general solution of *y* in terms of *x*.

$$z=\frac{1}{y^2}\Rightarrow \frac{dz}{dx}=\frac{-2}{y^3}\frac{dy}{dx}\Rightarrow \frac{dy}{dx}=-\frac{y^3}{2}\frac{dz}{dx}$$

$$e^{x^2}\left[-\frac{y^3}{2}\frac{dz}{dx}\right]=2xy^2\sqrt{y^2-1}$$

$$e^{x^2}\frac{dz}{dx}=-4x\left(\frac{1}{y}\right)\sqrt{y^2-1}$$

$$e^{x^2}\frac{dz}{dx}=-4x\sqrt{1-\frac{1}{y^2}}$$

$$e^{x^2}\frac{dz}{dx}=-4x\sqrt{1-z}$$

$$\therefore \frac{1}{\sqrt{1-z}}\frac{dz}{dx}=-4xe^{-x^2} \quad \text{(Shown)}$$

$\int \frac{1}{\sqrt{1-z}} dz = \int -4xe^{-x^2} dx$ $-2\sqrt{1-z} = 2e^{-x^2} + C \Rightarrow \sqrt{1-z} = -e^{-x^2} + A$ $1-z = \left(A - e^{-x^2}\right)^2$ $z = 1 - \left(A - e^{-x^2}\right)^2$ $y^2 = \frac{1}{1-\left(A-e^{-x^2}\right)^2}$ $y = \sqrt{\frac{1}{1-\left(A-e^{-x^2}\right)^2}}$ since $y > 1$

6. DH/2008/II/3

The height x, of a certain plant, at time t is proportional to the amount of nutrients it absorbs. The plant loses nutrients at a rate proportional to the height of the plant and absorbs nutrients at a rate inversely proportional to the surface area of the stem of the plant. Taking the cross-section of the stem to be circular with radius $r = kx$, where k is a constant, and the surface area of the stem to be $2\pi rx$, show that x satisfies the differential equation $\frac{dx}{dt} = \frac{p}{x^2} - qx$, where p and q are positive constants. [2]

For the case $p = 2q$, it is known that $x = 0$ when $t = 0$.

(i) Solve this differential equation, obtaining an expression for x in terms of t and q. [5]

(ii) Deduce the value of x as t becomes very large. [1]

6 (i)	Surface area of stem = $2\pi rx = 2\pi(kx)x = 2k\pi x^2$ $\frac{dx}{dt} = -qx + \frac{c}{2k\pi x^2}$ $= \frac{p}{x^2} - qx$ where $p = \frac{c}{2k\pi}$ (shown) $p = 2q$: $\frac{dx}{dt} = \frac{2q}{x^2} - qx$ $\frac{dx}{dt} = q\left(\frac{2-x^3}{x^2}\right)$ $\int \frac{x^2}{2-x^3}\, dx = \int q\, dt \Rightarrow -\frac{1}{3}\ln\lvert 2-x^3\rvert = qt + C$ $\Rightarrow \ln\lvert 2-x^3\rvert = -3qt + D,$ where $D = -3C$ $2 - x^3 = Ae^{-3qt} \Rightarrow x^3 = 2 - Ae^{-3qt}$

(ii)	$t = 0, x = 0$: $0 = 2 - Ae^0 \Rightarrow A = 2$ $x^3 = 2 - 2e^{-3qt}$ $x = \sqrt[3]{2 - 2e^{-3qt}}$ As $t \to \infty$, $e^{-3qt} \to 0$ since $q > 0$ $x \to \sqrt[3]{2 - 0} = \sqrt[3]{2}$.

7. SRJC/2013/I/9

In a model of loan repayment offered by ABC Bank, the amount of money owed by the borrower, x (dollars) at time t (months), increases, due to interest, at a rate proportional to the amount of money owed. The borrower repays the loan at a constant rate r. Theoretically, the model suggests that when $x = a$, the interest and repayment balance, i.e. the amount of money owed by the borrower remains constant.

It is assumed that both x and t are continuous variables.

(i) Show that $\frac{dx}{dt} = \frac{r}{a}(x - a)$. [3]

A man borrows \$$A$ from the bank.

(ii) Find the amount owed by the man at time t, in terms of t, r, a and A. [3]
Determine with a suitable diagram if the loan can be repaid in a finite time if $A < a$. [2]

(i) $\frac{dx}{dt} = kx - r$

When $x = a$, $\frac{dx}{dt} = ka - r = 0 \Rightarrow k = \frac{r}{a}$

$\therefore \frac{dx}{dt} = \frac{r}{a}(x - a)$ (Shown)

(ii) $\int \frac{1}{x-a}\,dx = \int \frac{r}{a}\,dt \Rightarrow \ln|x - a| = \frac{r}{a}t + C$

$\therefore x = Be^{\frac{r}{a}t} + a$

Given that, when $t = 0, x = A$, we get $B = A - a$

Therefore, $x = (A - a)e^{\frac{r}{a}t} + a$

When the loan is repaid, $x = 0, t = T$

$0 = (A - a)e^{\frac{r}{a}T} + a$

$(a - A)e^{\frac{r}{a}T} = a$

$T = \frac{a}{r} ln\left(\frac{a}{a-A}\right)$

8. HCI/2016/II/4

In harvesting of renewable natural resources, it is desirable that policies are formulated to allow maximal harvest of the natural resources, and yet not deplete the resources below a sustainable level. A simple harvesting model devised for the rate of change of the population of wild salmon in a particular region in the Pacific Ocean is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P)-h,$$

where P is the population of wild salmon in millions at time t years and h is the constant harvest rate in millions.

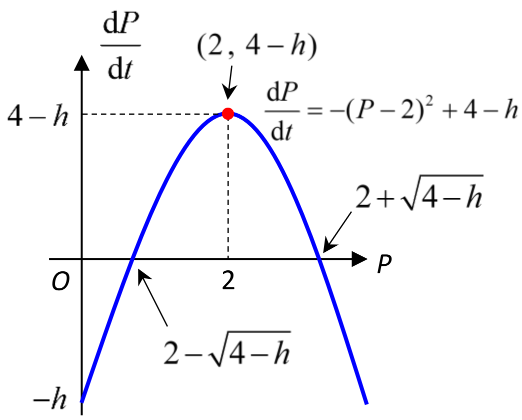
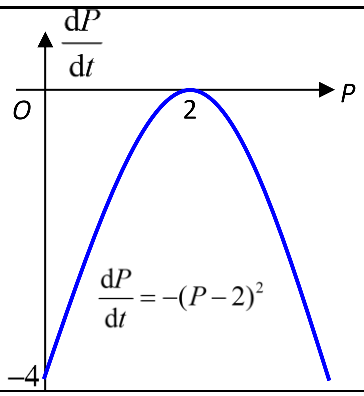
(i) Sketch a graph of $\frac{\mathrm{d}P}{\mathrm{d}t}$ against P, expressing the turning point in terms of h. [2]

(ii) The *Maximum Sustainable Yield (MSY)* is the largest harvest rate h that allows for a sustainable harvest of wild salmon without long-term depletion. State the *MSY* for the wild salmon. [1]

(iii) It is given that the population of wild salmon in that region was 3.2 million in 2015 and the constant harvest rate is 3 million. Find an expression for P at any time t. [5]

(iv) Hence find the population of the wild salmon in that region in 2016. [2]

(v) State one assumption you made in your calculation. [1]

8(i)	$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P)-h$ $= -P^2+4P-h$ $= -(P^2-4P)-h$ $= -(P-2)^2+4-h$ 
(ii)	From graph in (i), when $h=4$, $\frac{\mathrm{d}P}{\mathrm{d}t} = -(P-2)^2 = 0$ at $P=2$. $\therefore$ largest $h=4$ where the population P remains constant at $P=2$. Hence $MSY = 4$ million 
(iii)	$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P)-3 = -\left[(P^2-4P)+3\right] = -\left[(P-2)^2-1\right]$ <u>Method 1</u> $\int \frac{1}{(P-2)^2-1}\,\mathrm{d}P = -\int 1\,\mathrm{d}t$

	$\frac{1}{2}\ln\left\|\frac{(P-2)-1}{(P-2)+1}\right\|=-t+C$ $\frac{P-3}{P-1}=\pm e^{-2t+2C}=Ae^{-2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=\frac{1}{11}$ $\therefore\ P-3=\frac{1}{11}e^{-2t}(P-1)$ $11P-33=Pe^{-2t}-e^{-2t}$ Hence $P=\frac{33-e^{-2t}}{11-e^{-2t}}=\frac{33e^{2t}-1}{11e^{2t}-1}$ Method 2 $\frac{dP}{dt}=1-(P-2)^2$ $\int\frac{1}{1-(P-2)^2}\ dP=\int 1\ dt$ $\frac{1}{2}\ln\left\|\frac{1+(P-2)}{1-(P-2)}\right\|=t+C$ $\frac{P-1}{3-P}=\pm e^{2t+2C}=Ae^{2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=-11$ $P-1=-11e^{2t}(3-P)$ $P-1=-33e^{2t}+11Pe^{2t}$ $P=\frac{1-33e^{2t}}{1-11e^{2t}}$ or $P=\frac{33e^{2t}-1}{11e^{2t}-1}$ Method 3 $-\int\frac{1}{P^2-4P+3}\ dP=\int 1\ dt$ $-\int\frac{1}{(P-3)(P-1)}\ dP=\int 1\ dt$ $-\int\frac{1}{2(P-3)}\ dP+\int\frac{1}{2(P-1)}dP=\int 1\ dt$ (using partial fractions) $\frac{1}{2}\ln\left\|\frac{P-1}{P-3}\right\|=t+C$ $\frac{P-1}{P-3}=\pm e^{2t+2C}=Ae^{2t}$ In 2015, let $t=0$, $P=3.2$; hence $A=11$ $P-1=11e^{2t}(P-3)$ $P-1=11Pe^{2t}-33e^{2t}$

(iv)	$P=\frac{1-33e^{2t}}{1-11e^{2t}}$ or $P=\frac{33e^{2t}-1}{11e^{2t}-1}$ In 2016, $t=1$ Hence $P=\frac{33e^{2}-1}{11e^{2}-1}=3.02$ $\therefore$ the population of wild salmon is 3.02 million in 2016.
(v)	There are no external factors such as marine pollution or climate change that drastically affect the population of wild salmon in that region.

9. NYJC/2017/I/10

A Nanyang space probe P lands at a point O on the surface of Mars (assumed flat) and immediately begins to move in such a way that its coordinates (x, y) relative to O are determined by the pair of differential equations

$$\frac{d^2x}{dt^2}=-e^{-t} \qquad (A)$$

$$\frac{dy}{dt}=y+t \qquad (B)$$

where t denotes the time that elapses immediately after the space probe lands at O on the surface of Mars.

Through precise monitoring over a very long period of time, it is found that the x-coordinate of P approaches a finite value ℓ.

(i) Solve the differential equation (A) and show that $x=1-e^{-t}$, justifying your answer. State the exact value of ℓ. [5]

(ii) Use the substitution $z=y+t$ to solve the differential equation (B), expressing y in terms of t. [4]

(iii) Show that the cartesian equation of the path of P is given by

$$y=\ln(1-x)+\frac{x}{1-x}$$

and sketch the graph of the above cartesian equation which is relevant to the context of the question, showing clearly any asymptotic behaviour. [4]

9(i) $\frac{d^2x}{dt^2}=-e^{-t} \Rightarrow \frac{dx}{dt}=e^{-t}+A$ on integrating w.r.t x

$\Rightarrow x=-e^{-t}+At+B$ on integrating w.r.t x

Since when t = 0, x = 0, $0=-1+B \Rightarrow B=1$.

Thus $x=-e^{-t}+At+1$.

Since as $t\to\infty, x\to\ell$ where ℓ is finite, A = 0,

for otherwise, $x\to\pm\infty$.

Therefore $x=1-e^{-t}$ ------ (1)

and ℓ = 1.

9(ii) $z=y+t \Rightarrow \frac{dy}{dt}=\frac{dz}{dt}-1$. Substitute into DE gives

$$\frac{dz}{dt}-1=z \Rightarrow \int \frac{1}{1+z}\,dz=\int dt$$
$$\Rightarrow \ln|1+z|=t+C$$
$$\Rightarrow 1+z=\underbrace{\pm e^{C}}_{K}\cdot e^{t}$$
$$\Rightarrow y+t=Ke^{t}-1$$
$$\Rightarrow y=Ke^{t}-t-1$$

Since when $t = 0$, $y = 0$, $0=K-1 \Rightarrow K=1$.

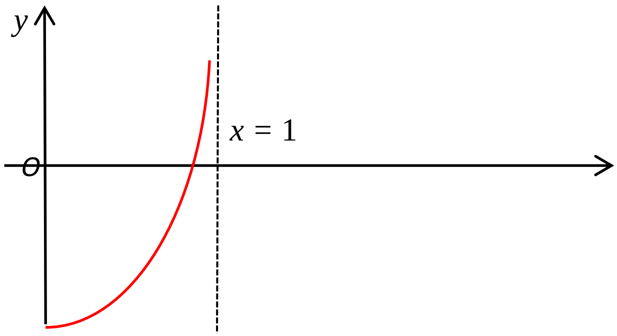
Hence $y=e^{t}-t-1$ ------ (2)

9(iii) From (1),

$$x=1-e^{-t} \Rightarrow e^{t}=\frac{1}{1-x}$$
$$\Rightarrow t=-\ln(1-x)$$

Substitute into (2) gives

$$y=\frac{1}{1-x}+\ln(1-x)-1$$
$$=\ln(1-x)+\frac{x}{1-x}$$



10. SAJC/2018/MYE/II/4 (modified)

In a chemical reaction, the mass, x grams, of a certain salt present at time t minutes satisfies the differential equation

$$\frac{dx}{dt}=k(2+x-x^{2}),$$

where $0 \le x \le 1$ and k is a constant. Initially, the mass of salt present is 1 gram and $\frac{dx}{dt}=-\frac{1}{5}$.

(i) Show that $k=-\frac{1}{10}$. [1]

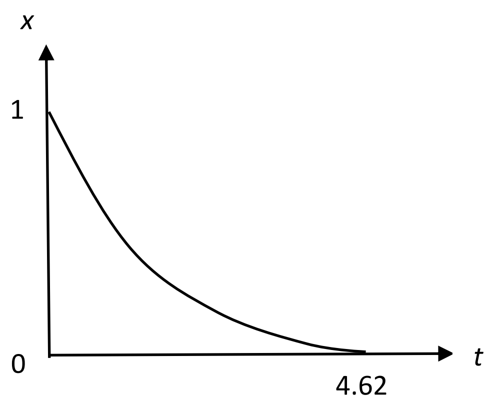
(ii) By first expressing $2+x-x^{2}$ in completed square form, find t in terms of x. [6]

(iii) Find the time taken for there to be no salt present in the chemical reaction. [1]

(iv) Express the solution of the differential equation in the form $x = \mathrm{f}(t)$. Find the mass of salt present in the chemical reaction after 3 minutes. [4]

(v) Sketch the curve $x = \mathrm{f}(t)$. [2]

(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(2+x-x^2)$ when $t=0$, $x=1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{5}$, $-\frac{1}{5} = k(2+1-1) \Rightarrow k = -\frac{1}{10}$ (shown)
(ii)	$2+x-x^2 = 2-(x^2-x)$ $= 2-\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}\right]$ $= \frac{9}{4} - \left(x-\frac{1}{2}\right)^2$ $\frac{\mathrm{d}x}{\mathrm{d}t} = k(2+x-x^2)$ $\int \frac{1}{2+x-x^2}\,\mathrm{d}x = \int k\,\mathrm{d}t$ $\int \frac{1}{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}\,\mathrm{d}x = \int -\frac{1}{10}\,\mathrm{d}t$ $\frac{1}{2(\frac{3}{2})} \ln\left\lvert \frac{\frac{3}{2}+\left(x-\frac{1}{2}\right)}{\frac{3}{2}-\left(x-\frac{1}{2}\right)} \right\rvert = -\frac{1}{10}t + c$ $\frac{1}{3}\ln\left\lvert\frac{x+1}{2-x}\right\rvert = -\frac{1}{10}t + c$ When $t=0$, $x=1$ $c = \frac{1}{3}\ln 2$ $\frac{1}{3}\ln\left\lvert\frac{x+1}{2-x}\right\rvert = -\frac{1}{10}t + \frac{1}{3}\ln 2$ $t = \frac{10}{3}\ln\left\lvert\frac{x+1}{4-2x}\right\rvert^{-1}$ $t = \frac{10}{3}\ln\left(\frac{4-2x}{x+1}\right)$, $0 \le x \le 1$
(iii)	When $x=0$, $t = \frac{10}{3}\ln 4$ Time taken is 4.62 minutes

(iv)	$t=\frac{10}{3}\ln\left(\frac{4-2x}{x+1}\right)$ $\ln\left(\frac{4-2x}{x+1}\right)=\frac{3t}{10}$ $\frac{4-2x}{x+1}=e^{\frac{3t}{10}}$ $e^{-\frac{3t}{10}}(4-2x)=x+1$ $x(1+2e^{-\frac{3t}{10}})=4e^{-\frac{3t}{10}}-1$ $x=\frac{4e^{-\frac{3t}{10}}-1}{1+2e^{-\frac{3t}{10}}}$ When $t=3$, $x=\frac{4e^{-\frac{9}{10}}-1}{1+2e^{-\frac{9}{10}}}$ = 0.345 Amount of salt present after 3 minutes is 0.345 gram.
(v)	

11. TMJC/2020/JC2/MYE/4

In a farm, the rate of growth of population of cows, P, with respect to time, t years, can be modelled by the following two mathematical models.

Model I: $\frac{\mathrm{d}P}{\mathrm{d}t} = cP\ln\left(\frac{k}{P}\right)$, where c and k are positive constants,

Model II: $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = \mathrm{e}^{-\frac{t}{5}}$.

(i) By using the substitution $y = \ln\left(\frac{k}{P}\right)$, show that the differential equation in Model I reduces to $\frac{\mathrm{d}y}{\mathrm{d}t} = -cy$. [2]

(ii) Hence show that the general solution of the differential equation in Model I is $P = k\mathrm{e}^{-M\mathrm{e}^{-ct}}$, where M is an arbitrary constant. [4]

(iii) Given that $k = 80$ and $M > 0$ in Model I, describe what will happen to the population of the cows in the long run. [1]

(iv) Initially, the farm has 10 cows. Five years later, the population of cows is 30. Find the particular solution of the differential equation in Model II. [4]

(v) Give a reason, in context, which of the two models is more realistic in the long run. [1]

11(i)	$\frac{\mathrm{d}P}{\mathrm{d}t} = cP\ln\left(\frac{k}{P}\right)$ Given $y = \ln\left(\frac{k}{P}\right) = \ln k - \ln P$ differentiating with respect to t, $\frac{\mathrm{d}y}{\mathrm{d}t} = 0 - \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$ $\therefore \frac{\mathrm{d}P}{\mathrm{d}t} = -P\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)$ $\Rightarrow -P\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = cPy$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}t} = -cy$ Alternative: use chain rule $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}t}$ $\frac{\mathrm{d}y}{\mathrm{d}P} = -\frac{1}{P}$

	Alternatively Given $y=\ln\left(\frac{k}{P}\right)$ differentiating with respect to t, $\frac{\mathrm{d}y}{\mathrm{d}t}=\left(\frac{P}{k}\right)\left(\frac{-k}{P^2}\right)\frac{\mathrm{d}P}{\mathrm{d}t}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$ $\therefore \frac{\mathrm{d}P}{\mathrm{d}t}=-P\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)$ $\Rightarrow -P\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)=cPy$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}t}=-cy$
(ii)	$\int \frac{1}{y}\,\mathrm{d}y=\int -c\ \mathrm{d}t$ $\ln\lvert y\rvert=-ct+D$ $\lvert y\rvert=\mathrm{e}^{-ct+D}$ $y=\pm\mathrm{e}^{-ct+D}$ $y=M\mathrm{e}^{-ct}$, where $M=\pm\mathrm{e}^{D}$ $\Rightarrow \ln\left(\frac{k}{P}\right)=M\mathrm{e}^{-ct}$ $\frac{k}{P}=\mathrm{e}^{M\mathrm{e}^{-ct}}$ $\frac{P}{k}=\mathrm{e}^{-M\mathrm{e}^{-ct}}$ $\therefore P=k\mathrm{e}^{-M\mathrm{e}^{-ct}}$
(iii)	As $t\to\infty$, $\mathrm{e}^{-M\mathrm{e}^{-ct}}\to 1$, $\therefore P\to 80$. Since $M>0$, the population of cows **increases and approaches 80**.
(iv)	$\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=\mathrm{e}^{-\frac{t}{5}}$ $\frac{\mathrm{d}P}{\mathrm{d}t}=-5\mathrm{e}^{-\frac{t}{5}}+F$ $P=25\mathrm{e}^{-\frac{t}{5}}+Ft+G$ When $t=0, P=10$, $10=25\mathrm{e}^{0}+G\Rightarrow G=-15$ When $t=5, P=30$,

	$30=25\mathrm{e}^{-\frac{5}{5}}+5F-15$ $F=9-5\mathrm{e}^{-1}$ $\therefore P=25\mathrm{e}^{-\frac{t}{5}}+\left(9-5\mathrm{e}^{-1}\right)t-15$
(v)	For model II, as $t\to\infty$, $P\to\infty$. However, it is not possible for population of cows to increase indefinitely due to limited resources (e.g. food, water, space constraint, etc). So, Model I is a more realistic model.