2024 RI H2 Math Year 5 Promotion Examination: Solutions with Comments

1 A curve D has equation

$$y = ax + b\ln x + \frac{c}{x}, \quad x > 0,$$

where a, b and c are constants. It is given that D has a stationary point at $x = \frac{3}{2}$ and the tangent to D at the point where $x = 1$ is $y = x - 5$. Find the values of a, b and c. [4]

	Solution	**Comments**
[4]	**Method 1** $\frac{dy}{dx} = a + \frac{b}{x} - \frac{c}{x^2}$ At $x = \frac{3}{2}$, $\frac{dy}{dx} = 0 \quad \Rightarrow a + \frac{2}{3}b - \frac{4}{9}c = 0 \quad ----(1)$ The gradient of D at $x = 1$ is equal to the gradient of the line $y = x - 5$. At $x = 1$, $\frac{dy}{dx} = 1 \quad \Rightarrow a + b - c = 1 \quad ----(2)$ y-coordinate of D at $x = 1$ is $1 - 5 = -4$. Substituting $x = 1$ and $y = -4$ into equation of D, we get $a + c = -4 \quad ----(3)$ From GC, $a = 2, b = -7, c = -6$. **Method 2** $\frac{dy}{dx} = a + \frac{b}{x} - \frac{c}{x^2}$ At $x = \frac{3}{2}$, $\frac{dy}{dx} = 0 \quad \Rightarrow a + \frac{2}{3}b - \frac{4}{9}c = 0 \quad ----(1)$ Gradient of D at $x = 1$ is $\left.\frac{dy}{dx}\right\|_{x=1} = a + b - c$. y-coordinate of D at $x = 1$ is $a + c$. So, the equation of tangent to D at $x = 1$ is $y - (a + c) = (a + b - c)(x - 1) \Rightarrow y = (a + b - c)x - b + 2c$ Comparing this line with $y = x - 5$, we get $a + b - c = 1 \quad ----(2)$ $-b + 2c = -5 \quad ----(3)$ From GC, $a = 2, b = -7, c = -6$.	Most students did well and got full marks for this question. Those who did not get full marks, were able to get the equations (1) and (2). Need to observe that since the tangent line and the curve touches at a point, they should share the same y-coordinate at that point. So, use the equation of the given tangent line to find the y-coordinate and obtain equation (3) There are a few students who did not use the GC to solve the simultaneous equations and made careless mistakes in their workings and thus lost some marks. Common Mistakes: • $\frac{dy}{dx} = x + \frac{b}{x} + \frac{c}{x^2}$ • Substituting $x = 1$ and $y = -5$ into equation of D, instead of $y = -4$

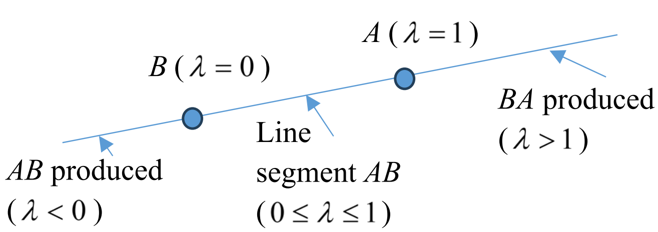
2 With respect to the origin O, the fixed points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel.

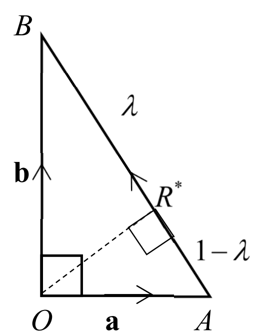
(a) The variable point R has position vector $\mathbf{r} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a real parameter. Describe geometrically the set of all possible positions of R. [1]

It is given that the angle AOB is $90°$.

(b) Explain why $\mathbf{a}.\mathbf{b} = 0$. [1]

(c) Among the set of all possible points R, the point R^* is the closest to the origin O. Find the position vector of R^*. Hence state the ratio $AR^* : BR^*$ in terms of magnitudes of $\mathbf{a}$ and $\mathbf{b}$. [4]

	Solution	**Comments**
(a) **[1]**	The set of all possible positions of R is the **line** that passes through points A and B, or The set of all possible positions of R is the **line** that passes through point B (or A) and parallel to the vector $\overrightarrow{AB}$. **Additional Notes:** When $\lambda = 0$, we get $\mathbf{r} = \mathbf{b}$, which corresponds to the point B. When $\lambda = 1$, we get $\mathbf{r} = \mathbf{a}$, which corresponds to the point A.  Note that a vector is NOT a line, and if we wish to talk about the position vector $\mathbf{b}$, we should say the line passes through the point with position vector $\mathbf{b}$ (or simply point B). A line cannot pass through a vector, and a point cannot be on a vector as well. Mathematical language has to be accurate.	Since R is a point, when describing its possible positions, it should be clear that we are describing possible points. When describing a line, you need 2 points or 1 point and a direction vector which the line is parallel to. The line segment AB refers to the part of the line AB which is between A and B inclusive. Since λ is real, R is any point on the line AB.
(b) **[1]**	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 90° = 0$ since $\cos 90° = 0$ **Additional Note:**	The definition of the dot product is $\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$.

	That the scalar product is 0 because 2 vectors are perpendicular (or the angle between them is $90°$) is a consequence of this definition.	
(c) **[4]**	$\overrightarrow{OR^*} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ for some $\lambda \in \mathbb{R}$. $$\overrightarrow{OR^*} \cdot (\mathbf{a}-\mathbf{b}) = 0$$ $$\left[\lambda\mathbf{a} + (1-\lambda)\mathbf{b}\right] \cdot (\mathbf{a}-\mathbf{b}) = 0$$ $$\lambda\|\mathbf{a}\|^2 - (1-\lambda)\|\mathbf{b}\|^2 + (1-2\lambda)\mathbf{a}\cdot\mathbf{b} = 0$$ Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}\cdot\mathbf{b} = 0$ $$\therefore \lambda = \frac{\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}$$ $$\overrightarrow{OR^*} = \frac{\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}\mathbf{a} + \frac{\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}\mathbf{b}$$ $AR^* : BR^* = 1-\lambda : \lambda$ $= \frac{\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2} : \frac{\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}$ $= \|\mathbf{a}\|^2 : \|\mathbf{b}\|^2$  	The point F on a line closest to a given point P is the foot of the perpendicular from P to the line. This point is not necessarily the midpoint of the line segment AB. When expanding the scalar product, note that $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|\|\mathbf{a}\|\cos 0 = \|\mathbf{a}\|^2$ Note that if $\mathbf{r} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, as shown in the diagram, $AR^* : BR^* = 1-\lambda : \lambda$, <u>not</u> $AR^* : BR^* = \lambda : 1-\lambda$

3 **Do not use a calculator in answering this question.**

Solve the inequality

$$\frac{6x^2+2x-3}{2x-1} \geq 2\left(x+1\right). \qquad [4]$$

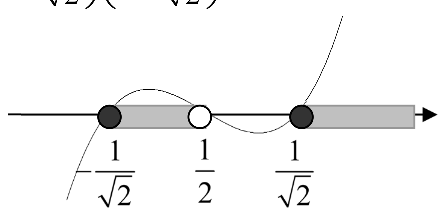
Hence solve $\dfrac{6x^2+2|x|-3}{2|x|-1} \geq 2\left(|x|+1\right)$. [3]

<table>
<tr><th></th><th>Solution</th><th>Comments</th></tr>
<tr><td>[4]</td><td>

$$\frac{6x^2+2x-3-2\left(x+1\right)\left(2x-1\right)}{2x-1} \geq 0, \qquad x \neq \frac{1}{2}$$

$$\frac{2x^2-1}{2x-1} \geq 0$$

$$2\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)\left(2x-1\right) \geq 0$$



$-\frac{1}{\sqrt{2}} \leq x < \frac{1}{2}$ or $x \geq \frac{1}{\sqrt{2}}$

Additional Notes:
You are strongly advised to use ( ) if you are making algebraic errors in arriving at $\frac{2x^2-1}{2x-1} \geq 0$.

For students who multiply by $\left(2x-1\right)^2$ in the first step, you should always factorize first before any expansion, as shown below:

$\left(6x^2+2x-3\right)\left(2x-1\right)-2\left(x+1\right)\left(2x-1\right)^2 \geq 0$

$\Leftrightarrow \left(2x-1\right)\left[\left(6x^2+2x-3\right)-2\left(x+1\right)\left(2x-1\right)\right] \geq 0$

This will avoid unnecessary algebraic manipulation.
</td><td>
As the instruction of not using a calculator is given in the question, you should
1) Give all answers in exact form (no 3 s.f)
2) Show all working clearly

This also means that graphical approaches need to be clearly justified to obtain full credit.

The use of the OR is required when having 2 solution ranges that are distinct. Using ‘,’ or AND is incorrect.

Once you have fully factorized the expression, do feel free to use the GC (you do not need to say that you used it) to verify the shape of the cubic graph to avoid unnecessary mistakes.
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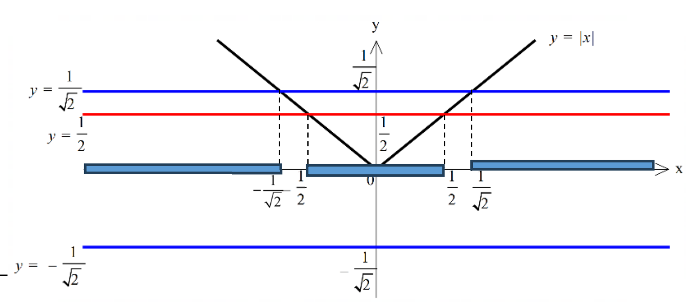
<tr><td>[3]</td><td>



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	By replacing x with $\lvert x\rvert$, $-\frac{1}{\sqrt{2}} \le \lvert x\rvert < \frac{1}{2}$ or $\lvert x\rvert \ge \frac{1}{\sqrt{2}}$ $x \le -\frac{1}{\sqrt{2}}$ or $-\frac{1}{2} < x < \frac{1}{2}$ or $x \ge \frac{1}{\sqrt{2}}$ **Additional Note:** Perhaps the simplest way to think of solving modulus inequalities like $\lvert x\rvert < \frac{1}{2}$ would be to tell yourself if the **magnitude*** of x is smaller than $\frac{1}{2}$, then x itself should not be too far from the origin, that is $-\frac{1}{2} < x < \frac{1}{2}$. Similarly, if $\lvert x\rvert \ge \frac{1}{\sqrt{2}}$, then x has to be at least $\frac{1}{\sqrt{2}}$ from the origin, and thus $x \le -\frac{1}{\sqrt{2}}$ or $x \ge \frac{1}{\sqrt{2}}$.	Note that $-\frac{1}{\sqrt{2}} \le \lvert x\rvert$ holds for all real x. Thus $-\frac{1}{\sqrt{2}} \le \lvert x\rvert < \frac{1}{2}$ is equivalent to $\lvert x\rvert < \frac{1}{2}$. *Note how "$\lvert\ \rvert$" is consistently used in various topics. $\lvert \mathbf{a}\rvert$ denotes the magnitude of a vector **a**, or equivalently the distance of the point A from the origin. $\lvert z\rvert$ denotes the magnitude of a complex number z, or equivalently the length OP, if P represents the complex number z on the Argand diagram.

4 A curve C has parametric equations

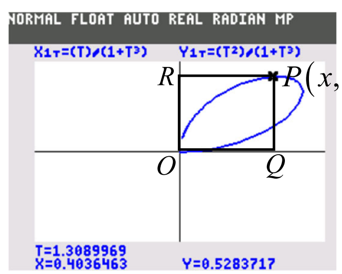
$$x = \frac{\lambda}{1+\lambda^3} \text{ and } y = \frac{\lambda^2}{1+\lambda^3}, \qquad \text{for } \lambda > 0.$$

The point P is a variable point on C.

(a) With reference to the origin O, OP forms the diagonal of the rectangle $OQPR$, where vertices Q and R lie on the x- and y-axis respectively. Using differentiation, find the value of λ which maximises the area of rectangle $OQPR$. You need to show that your answer gives a maximum. [5]

When the area of rectangle $OQPR$ is at its maximum, the rate of change of the x-coordinate of the point P is 1 unit per second.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence determine the rate of change of the y-coordinate of the point P at this instant. [4]

	Solution	**Comments**
(a) **[5]**	Area of rectangle $OQPR$, $A = xy = \frac{\lambda^3}{\left(1+\lambda^3\right)^2}$  $\frac{\mathrm{d}A}{\mathrm{d}\lambda} = \frac{\left(1+\lambda^3\right)^2\left(3\lambda^2\right)-\lambda^3(2)\left(1+\lambda^3\right)\left(3\lambda^2\right)}{\left(1+\lambda^3\right)^4}$ $= \frac{\left(1+\lambda^3\right)^2\left(3\lambda^2\right)-\lambda^3(2)\left(1+\lambda^3\right)\left(3\lambda^2\right)}{\left(1+\lambda^3\right)^4}$ $= \frac{3\lambda^2\left(1-\lambda^3\right)}{\left(1+\lambda^3\right)^3}$	It is important to READ the question properly. Here, the question aims to maximise the AREA of the rectangle with respect to λ, and so we should solve for $\frac{\mathrm{d}A}{\mathrm{d}\lambda}=0$. Attempts to solve other equations are not equivalent (see additional notes).

	Additional Notes for Comments **Case 1 :** $A = xy \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\lambda} = y\frac{\mathrm{d}x}{\mathrm{d}\lambda} + x\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$ Hence none of the solutions for $\frac{\mathrm{d}x}{\mathrm{d}\lambda} = 0$ or $\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ gives desired result. In fact, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\lambda - \lambda^4}{1-2\lambda^3}$(from (b)) $\Rightarrow \frac{2\lambda - \lambda^4}{1-2\lambda^3} = -\lambda \Rightarrow 3\lambda(1-\lambda^3) = 0$ which then gives the correct value of λ. **Case 2 :** $A = xy \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\lambda} = y\frac{\mathrm{d}x}{\mathrm{d}\lambda} + x\frac{\mathrm{d}y}{\mathrm{d}\lambda} \neq \frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{\mathrm{d}y}{\mathrm{d}\lambda}$ Hence, in general, maximizing Area with respect to λ is not the same as maximizing $x + y$ with respect to λ. **Case 3 :** Consider $D = OP$. $A = xy \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}\lambda} = y\frac{\mathrm{d}x}{\mathrm{d}\lambda} + x\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$ $D^2 = x^2 + y^2 \Rightarrow 2D\frac{\mathrm{d}D}{\mathrm{d}\lambda} = 2x\frac{\mathrm{d}x}{\mathrm{d}\lambda} + 2y\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ Note that solving for $\frac{\mathrm{d}A}{\mathrm{d}\lambda} = 0$ may not give the same solution as solving $\frac{\mathrm{d}D}{\mathrm{d}\lambda} = 0$. When the condition $\lambda = 1$ is not given, the maximization is thus not equivalent.	
	For stationary A, $\frac{\mathrm{d}A}{\mathrm{d}\lambda} = 0$, giving $\lambda = 1$ (since λ>0) **MTD 1 : First Derivative Test**	Using the first derivative test to show that Area is maximized, it is necessary to factorize the expression

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$\dfrac{dA}{d\lambda}=\dfrac{3\lambda^2\left(1-\lambda^3\right)}{\left(1+\lambda^3\right)^3}$

Since $\dfrac{3\lambda^2}{\left(1+\lambda^3\right)^3}>0$ for all $\lambda>0$,

λ	1^-	1	1^+
$\left(1-\lambda^3\right)$	+ve	0	− ve
$\dfrac{dA}{d\lambda}$	+ve	0	− ve

Maximum A when $\lambda=1$.

MTD 2 : Second Derivative Test

$\dfrac{dA}{d\lambda}=\dfrac{3\left(\lambda^2-\lambda^5\right)}{\left(1+\lambda^3\right)^3}$

$\dfrac{d^2A}{d\lambda^2}=3\left[\dfrac{\left(1+\lambda^3\right)^3\left(2\lambda-5\lambda^4\right)-\left(\lambda^2-\lambda^5\right)(3)\left(1+\lambda^3\right)^2\left(3\lambda^2\right)}{\left(1+\lambda^3\right)^6}\right]$

$=3\left[\dfrac{4\lambda^7-12\lambda^4+2\lambda}{\left(1+\lambda^3\right)^4}\right]$

When $\lambda=1$, $\dfrac{d^2A}{d\lambda^2}=-\dfrac{9}{8}<0$

Maximum A when $\lambda=1$.

</td><td>completely so that it is easy to see and explain which are the terms that are clearly positive. Next, use the table to explain the behavior of $1-\lambda^3$ in the vicinity of $\lambda=1$.

In using the second derivative test, one can use the GC to evaluate the value at the stationary point.</td></tr>
<tr><td>**(b)**
[4]</td><td>

$\dfrac{dy}{dx}=\dfrac{dy}{d\lambda}\cdot\dfrac{d\lambda}{dx}=\dfrac{\left(1+\lambda^3\right)(2\lambda)-\lambda^2\left(3\lambda^2\right)}{\left(1+\lambda^3\right)^2}\cdot\dfrac{\left(1+\lambda^3\right)^2}{1-2\lambda^3}=\dfrac{2\lambda-\lambda^4}{1-2\lambda^3}$

$\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}=\dfrac{2\lambda-\lambda^4}{1-2\lambda^3}\cdot\dfrac{dx}{dt}$

When $\lambda=1$, $\dfrac{dy}{dt}=(-1)(1)=-1$ unit per sec.

Rate of change of the y-coordinate of the point P at this instant is -1 unit

</td><td>Clear working of how chain rule is applied is expected. Where the relation between $\dfrac{dy}{dt},\dfrac{dy}{dx},\dfrac{dx}{dt}$ is not stated</td></tr>
</table>

	per second or Rate of decrease of the y-coordinate of the point P at this instant is 1 unit per second.	explicitly, marks cannot be awarded.
	<u>General comments for both parts :</u> Poor algebraic manipulation skills led to loss of marks for many students. Poor handwriting makes it extremely difficult for the marker to follow the thought process. For example, it was difficult to make out if the written is x or λ. It is important to note that if the marker is not able to read the solution clearly, marks may not be awarded.	

5 **(a)** Find $\sum_{r=0}^{n}[(n+2)r+n]$, giving your answer in terms of n. [3]

[You may use the result $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for the rest of this question.]

(b) By writing down the first two and the last two terms in the series, find $\sum_{r=1}^{n}(r+2)^3$, giving your answer in terms of n. [3]

(c) Find $1^3-2^3+3^3-4^3+5^3-6^3+\ldots+(2n-1)^3-(2n)^3$, giving your answer in terms of n. [3]

	Solution	Comments
(a) **[3]**	$\sum_{r=0}^{n}[(n+2)r+n]$ $=\frac{n+1}{2}[n+n(n+2)+n]$ $=\frac{n+1}{2}(n^2+4n)$ $=\frac{n}{2}(n+1)(n+4)$ **Alternative** $\sum_{r=0}^{n}[(n+2)r+n]=\sum_{r=0}^{n}(n+2)r+\sum_{r=0}^{n}n$ $=(n+2)\sum_{r=1}^{n}r+(n+1)n$ $=(n+2)\cdot\frac{n}{2}(n+1)+(n+1)n$ $=\frac{n}{2}(n+1)(n+4)$	Many do not recognize that the entire series can be seen as an Arithmetic series with $n+1$ terms, first term n and common difference of $n+2$. Common mistakes: • Lack of basic knowledge of arithmetic or geometric series. • Lack of understanding of number of terms represented by the summation notation. • Carelessness in applying sum of terms of arithmetic series correctly, such as ○ $\sum_{r=0}^{n}n=n^2$ ○ $\sum_{r=0}^{n}[(n+2)r+n]=\frac{n}{2}[n+n(n+2)+n]$ ○ Factorising r "out of Σ notation" Eg $\sum_{r=0}^{n}(n+2)r=r\sum_{r=0}^{n}(n+2)$
(b) **[3]**	$\sum_{r=1}^{n}(r+2)^3=3^3+4^3+\ldots+(n+1)^3+(n+2)^3$ $\sum_{r=1}^{n}(r+2)^3=\sum_{r=3}^{n+2}r^3$ $=\sum_{r=1}^{n+2}r^3-1^3-2^3$ $=\frac{1}{4}(n+2)^2(n+3)^2-9$	To demonstrate understanding of representation of series in r^3 form, students should present the first and last 2 terms in a series instead of simply listing out isolated individual terms .

<table>
<tr><td></td><td></td><td>Students should learn to interpret the series as a difference of 2 sums.
Some common mistakes and misconceptions are:
• $\sum_{r=1}^{n}(r+2)^3=\frac{1}{4}(n+2)^2(n+3)^2$
• $\sum_{r=1}^{n}(r+2)^3=\sum_{r=-1}^{n-2}(r+2)^3$
• $\sum_{r=1}^{n}(r+2)^3=\sum_{r=3}^{n}r^3-1-8$
• $\sum_{r=1}^{n}(r+2)^3=\sum_{r=1}^{n}r^3-27-64$
• Wrongly identifying the series as geometric or arithmetic series and then applying sum of terms formula.
• Not following instructions to write down the first two and last two terms in the series but instead attempt to expand $(r+2)^3$ to work on the summation of terms, which is more tedious. Several students expanded $(r+2)^3$ wrongly. Some examples:
$(r+2)^2=r^2+2r+4$
$(r+2)^3=r^3+2^3$</td></tr>
<tr><td>(c)
[3]</td><td>$1^3-2^3+3^3-4^3+5^3-6^3+\ldots+(2n-1)^3-(2n)^3$
$=\sum_{r=1}^{2n}r^3-2\sum_{r=1}^{n}(2r)^3$
$=\sum_{r=1}^{2n}r^3-16\sum_{r=1}^{n}r^3$
$=\frac{1}{4}(2n)^2(2n+1)^2-16\left[\frac{1}{4}n^2(n+1)^2\right]$
$=n^2(2n+1)^2-4n^2(n+1)^2$
$=n^2\left[(2n+1)^2-(2n+2)^2\right]$
$=-n^2(4n+3)$</td><td>Mistakes in series representation include different variants of the following:
$\sum_{r=1}^{2n}r^3-\sum_{r=1}^{n}(2r)^3$
$\sum_{r=1}^{2n}r^3-2\sum_{r=1}^{n}2r^3$
$\sum_{r=1}^{2n}n^3-2\sum_{r=1}^{n}2n^3$
$\sum_{r=1}^{n}(2n-1)^3-\sum_{r=1}^{n}(2n)^3$
Examples of misuse of formulae during simplification:
$\sum_{r=1}^{n}(2r-1)^3\neq\frac{1}{4}(2n-1)^2(2n)^2$</td></tr>
</table>

		$\sum_{r=1}^{n}(2r)^3 \neq \frac{1}{4}(2n)^2(2n+1)^2$. Some other mistakes involved in simplification are shown below: $\sum_{r=1}^{2n} r^3(-1)^{r+1} = \sum_{r=1}^{2n}(-1)^{r+1}\sum_{r=1}^{2n} r^3$ or $\sum_{r=1}^{2n} r^3(-1)^{r+1} = (-1)^{r+1}\sum_{r=1}^{2n} r^3$

6 It is given that $e^y = 1 + \sin 3x$.

(a) Show that

$$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 9 = 9e^{-y}.$$

Hence find the series expansion of y in ascending powers of x, up to and including the term in x^3, simplifying your answer. [5]

(b) Using standard series from the List of Formulae, verify that the series expansion obtained in **part (a)** is correct. [3]

	Solution	**Comments**
(a) **[5]**	$e^y = 1 + \sin 3x$ ------ (1) Differentiating w.r.t. x, $e^y \frac{dy}{dx} = 3\cos 3x$ Differentiating w.r.t. x, $e^y \frac{d^2 y}{dx^2} + e^y \frac{dy}{dx}\frac{dy}{dx} = -9\sin 3x = -9\left(e^y - 1\right)$ (from (1)) $e^y \frac{d^2 y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 + 9e^y = 9$ Dividing throughout by e^y, $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 9 = 9e^{-y}$ (shown) Differentiating w.r.t. x, $\frac{d^3 y}{dx^3} + 2\frac{dy}{dx}\frac{d^2 y}{dx^2} = 9e^{-y}\left(-\frac{dy}{dx}\right)$ When $x = 0$, $y = 0$; $\frac{dy}{dx} = 3$; $\frac{d^2 y}{dx^2} = -9$; $\frac{d^3 y}{dx^3} = 27$ $\Rightarrow y = 0 + 3x + \frac{-9}{2!}x^2 + \frac{27}{3!}x^3 + \ldots$ $\therefore y = 3x - \frac{9}{2}x^2 + \frac{9}{2}x^3 + \ldots$	The most efficient option was to use the given equation as the starting point to find the first derivative, which many candidates did. Some candidates chose to make y the subject before differentiating, a method that proved to be inefficient and prone to errors. It should be emphasized that even if it is successful in proving the given statement this instance, candidates should learn to utilize the efficient method for future questions. A number of candidates made errors in differentiation e.g. $\frac{d}{dx}9e^{-y} = -9ye^{-y}\frac{dy}{dx}$, $\frac{d}{dx}9e^{-y} = -9e^{-y}$, $\frac{d}{dx}9e^{-y} = 9e^{-y}\frac{dy}{dx}$.

(b) **[3]**	$e^{y}=1+\sin 3x$ $\Rightarrow y=\ln\left(1+\sin 3x\right)$ $=\ln\left(1+\left(3x-\frac{(3x)^3}{3!}\right)+\ldots\right)$ $=\left(3x-\frac{(3x)^3}{3!}\right)-\frac{\left(3x-\frac{(3x)^3}{3!}\right)^2}{2}+\frac{\left(3x-\frac{(3x)^3}{3!}\right)^3}{3}+\ldots$ $=3x-\frac{27x^3}{6}-\frac{9x^2}{2}+\frac{27x^3}{3}+\ldots$ $=3x-\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$ which is same as the expansion for y found in part **(a)**, up to and including the term in x^3.	In this question, candidates were pointed to the expansions in the List of Formulae and most candidates made use of these formulae. The most common approach was to write down the Maclaurin expansion for sin $(3x)$ and then substitute it into the Maclaurin expansion for ln $(1+x)$. A number of candidates did not go far enough in their expansion; the answer was required up to and including the term in x^3, so it is essential to use $\sin 3x=3x-\frac{(3x)^3}{3!}+\ldots$

7 **Do not use a calculator in answering this question.**

(a) Given that $x=1+3\text{i}$ is a root of the equation

$$x^3+ax^2+18x+b=0,$$

find the values of the real numbers a and b, and the other roots. [5]

(b) The complex numbers z and w satisfy the following equations.

$$w^*+z=4-6\text{i}$$
$$w-2z=1+10\text{i}$$

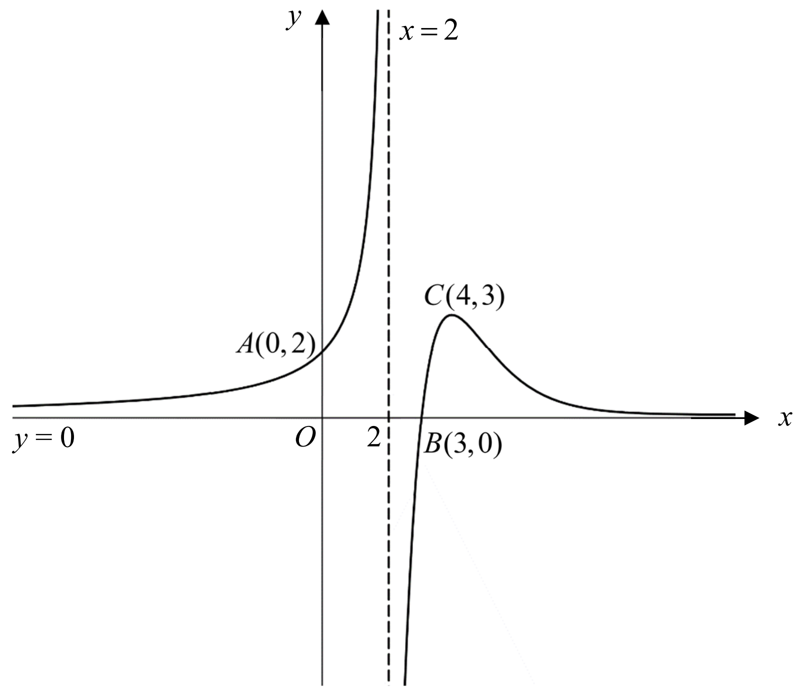
Find z and w, giving your answers in the form $c+\text{i}d$, where c and d are real numbers. [5]

	Solution	Comments
(a) **[5]**	Since $a,\ b\in\mathbb{R}$, then all the coefficients of the polynomial are real, complex roots will occur in conjugate pairs. Given $1+3\text{i}$ is a root, then $1-3\text{i}$ is also a root. $\left[x-(1+3\text{i})\right]\left[x-(1-3\text{i})\right]$ $=\left[(x-1)-3\text{i}\right]\left[(x-1)+3\text{i}\right]$ $=(x-1)^2-(3\text{i})^2$ $=x^2-2x+10$ $x^3+ax^2+18x+b=\left(x^2-2x+10\right)(x+p)$ Compare coefficients of $x\ :18=-2p+10\ \Rightarrow p=-4$ $x^2:a=p-2\ \Rightarrow a=-6$ $x^0:b=-40$ The other roots are 4 and $1-3\text{i}$.	This way of solving was chosen by the majority of students. However, a good number of them did not state the condition necessary for the existence of conjugate roots. A small number of students were also careless with their working although they know the correct method to apply. Students should also remember to answer the question explicitly, like stating what the other 2 roots are.
	Alternative: Since $x=1+3\text{i}$ is a root of the given equation,	This alternative was chosen by a small portion of students of which only about

	$(1+3i)^3+a(1+3i)^2+18x+b=0$ $(1+3i)(-8+6i)+a(-8+6i)+18(1+3i)+b=0$ $-8-24i+6i-18-8a+6ai+18+24i+b=0$ $(-8-8a+b)+(6a+36)i=0$ Equating real and imaginary parts: $6a+36=0 \Rightarrow a=-6$ $\therefore b=8a+8=-40$ Since the coefficients of the equation $x^3+ax^2+18x+b=0$ are all real numbers, the complex roots occur in conjugate pairs and so $1-3\mathrm{i}$ is also a root. Let the third root be a real number k. Note that $\left[x-(1+3\mathrm{i})\right]\left[x-(1-3\mathrm{i})\right]$ $=\left[(x-1)-3\mathrm{i}\right]\left[(x-1)+3\mathrm{i}\right]$ $=(x-1)^2-(3\mathrm{i})^2$ $=x^2-2x+10$ Therefore, we have $x^3-6x^2+18x-40=(x^2-2x+10)(x-k)$. Comparing constants, we have $k=4$. So $a=-6,\ b=-40$ and the other roots are $1-3\mathrm{i}$ and 4.	half of them were able to expand out the expression correctly without the aid of calculators. Students will lose credit if they did not show explicit working for the expansion of the term $(1+3i)^3$ as it meant a calculator was used. Some students wasted precious time when they also substituted $1-3\mathrm{i}$ into the equation and expanded it out too before equating real and imaginary parts.

(b) **[5]**	$w^*+z=4-6\text{i}$ ------ (1) $w-2z=1+10\text{i}$ ------ (2) $2(1)+(2):\quad 2w^*+w=9-2\text{i}$ Sub $w=c+d\text{i}$: $3c-d\text{i}=9-2\text{i}$ Comparing real part: $3c=9\Rightarrow c=3$ Comparing imaginary part: $d=2$ $\therefore w=3+2\text{i}$ From (1) : $z=4-6\text{i}-(3-2\text{i})=1-4\text{i}$ $\therefore w=3+2\text{i},\ z=1-4\text{i}$ **Alternative 1** Substitute $z=4-6\text{i}-w^*$ into (2) : $w-2(4-6\text{i}-w^*)=1+10\text{i}$ $\Rightarrow 2w^*+w=9-2\text{i}$ Sub $w=c+d\text{i}$: $3c-d\text{i}=9-2\text{i}$ Comparing real part: $3c=9\Rightarrow c=3$ Comparing imaginary part: $d=2$ $\therefore w=3+2\text{i}$ From (1) : $z=4-6\text{i}-(3-2\text{i})=1-4\text{i}$ $\therefore w=3+2\text{i},\ z=1-4\text{i}$	This question only required students to solve the set of simultaneous equations by eliminating one variable from the equations, set the remaining complex number to the form $c+di$ and then equate real and imaginary parts to solve for the values. It was generally well done except for the few students who used the wrong definition of w^* as well as those who were careless with their working.
	Alternative 2 Let $w=a+bi$ and $z=g+hi$. Then we have $a-bi+g+hi=4-6i$ $\Rightarrow a+g=4$ ---------- (1) $h-b=-6$ ---------- (2) and $a+bi-2(g+hi)=1+10i$ $\Rightarrow a-2g=1$ ---------- (3) $b-2h=10$ ---------- (4) From (1) and (3), $3g=3\Rightarrow g=1$ and $a=3$. From (2) and (4), $-h=4\Rightarrow h=-4$ and $b=2$. $\therefore w=3+2i$ and $z=1-4i$.	This simple substitution and comparison method surprised many students with its simple calculations as most students only attempted this method after they were unable to solve by other methods. Students who declared their use of GC of course lose some marks.

8 **(a)** The diagram shows the curve $y = \mathrm{f}(x)$ with a maximum point at $C(4,3)$. The curve crosses the axes at the points $A(0,2)$ and $B(3,0)$. The lines $x = 2$ and $y = 0$ are the asymptotes of the curve.



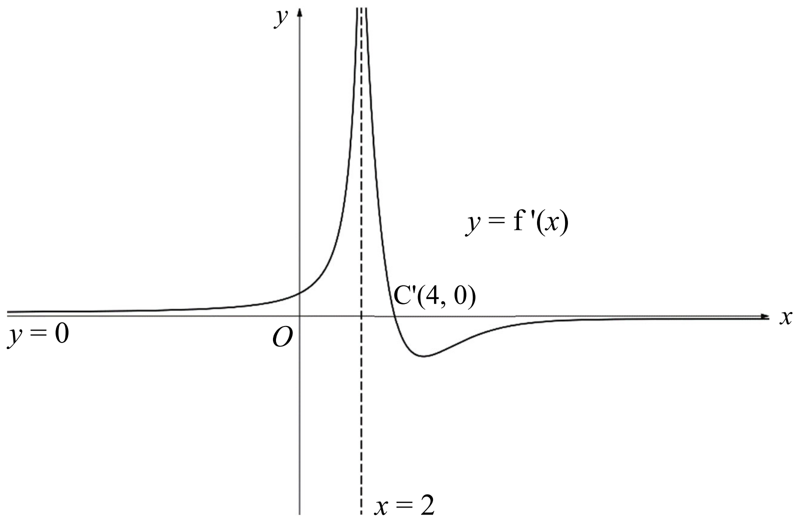
Sketch the graph of $y = \mathrm{f}'(x)$, clearly stating the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where appropriate. [3]

(b) The curve C_1 has equation $y = \dfrac{ax^2 + bx - 8}{x - 2}$, where a and b are constants. It is given that C_1 has an asymptote $y = 3 - 2x$.

(i) State the value of a and show that $b = 7$. [3]

(ii) Sketch C_1, clearly stating the equations of any asymptotes, the coordinates of any stationary points and of any points where C_1 crosses the axes. [3]

(iii) The curve C_1 is transformed by a translation of 2 units in the negative x-direction, followed by a stretch with scale factor $\dfrac{1}{2}$ parallel to the y-axis, to form the curve C_2. Find the equation of C_2. [2]

<table>
<tr><th></th><th>Solution</th><th>Comments</th></tr>
<tr><td>(a)
[3]</td><td>
</td><td>Students are reminded that the corresponding points of A and B are not (0,2) and (3,0) respectively as we do not know the gradient of the original graph at these points.</td></tr>
<tr><td>(b)(i)
[3]</td><td>$a = -2$

By long division, $y = (b-4) - 2x + \dfrac{2b-16}{x-2}$.

$b - 4 = 3 \Rightarrow b = 7$ (shown)

Alternative
$y = \dfrac{ax^2+bx-8}{x-2} = 3-2x+\dfrac{A}{x-2}$
$= \dfrac{(3-2x)(x-2)+A}{x-2}$
Comparing the numerators,
Coefficient of x^2 : $a=-2$
Coefficient of x: $b=3+4=7$

<u>Note</u> Coefficient of x^0 : $-8=-6+A \Rightarrow A=-2$</td><td>Many students erroneously wrote $(3-2x)(x-2)$ $= ax^2+bx-8$
Note that this is not true as the constant differs. However, students were awarded marks based on the correct method of comparing coefficients of the x^2 and x term.

There were many careless mistakes in this part while dealing with long division too.</td></tr>
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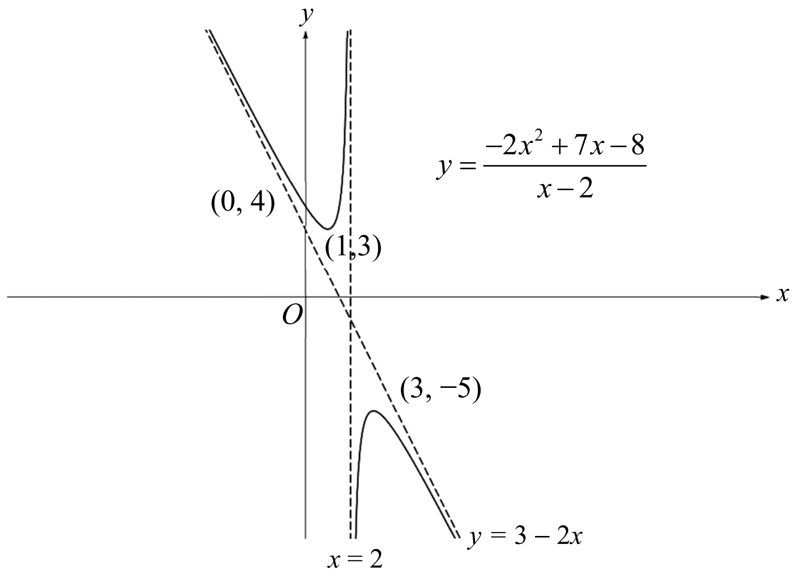
(b)(ii) [3]		Students are reminded to read the question carefully and denote all axial intercepts and turning points clearly on the graph.
(b)(iii) [2]	$y = 3 - 2x - \frac{2}{x-2}$ $\big\downarrow$ Replace x with $x+2$ $y = 3 - 2(x+2) - \frac{2}{(x+2)-2}$ $\Rightarrow y = -1 - 2x - \frac{2}{x}$ $\big\downarrow$ Replace y with $2y$ $2y = -1 - 2x - \frac{2}{x}$ $\Rightarrow y = -\frac{1}{2} - x - \frac{1}{x}$	Many careless mistakes were present in the working for this part, resulting in wrong final answers. Also, students should simplify their answers.

9 The planes p and q have equations

$$\mathbf{r} = (2+\lambda-2\mu)\mathbf{i} + (-3\lambda+\mu)\mathbf{j} + (3-\lambda+2\mu)\mathbf{k} \quad \text{and} \quad \mathbf{r}.\begin{pmatrix} a \\ -1 \\ b \end{pmatrix} = 1 \text{ respectively,}$$

where a and b are constants and λ and μ are parameters.

The line l passes through the point $(5,4,0)$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$. The planes p and q meet in the line l.

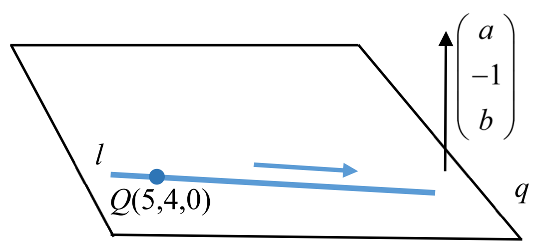
(a) Show that $a=1$ and $b=\dfrac{1}{2}$. [2]

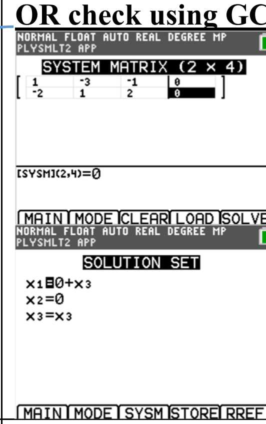
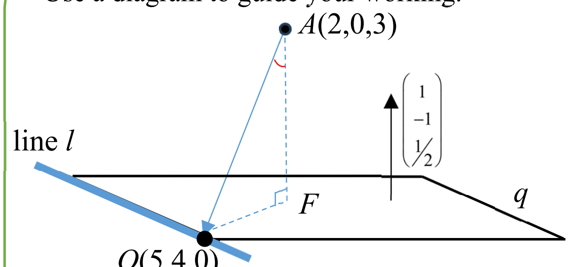
(b) Find the exact acute angle between the planes p and q. [3]

(c) Find the distance from the point $A(2,0,3)$ to the plane q. Hence deduce the shortest distance from A to l. [4]

The plane q is reflected about the plane p to obtain the plane q'.

(d) Find a cartesian equation of the plane q'. [3]

	Solution	**Comments**
(a) **[2]**	Since l lies in plane q, $\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ -1 \\ b \end{pmatrix} = 1 \Rightarrow 5a-4=1 \quad \therefore a=1$ $\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ -1 \\ b \end{pmatrix} = 0 \Rightarrow -2a+1+2b=0$ ----- (2) Substitute $a=1$, $-2+1+2b=0 \Rightarrow b=\dfrac{1}{2}$ Working required to "show" how you obtain the values of a and b. **Alternative** Equation of l : $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \alpha \in \mathbb{R}$.	A simple diagram will help you visualise. 

	Generate 2 points on l, say $(5,4,0)$ and $(3,3,2)$. $\begin{pmatrix}5\\4\\0\end{pmatrix}\cdot\begin{pmatrix}a\\-1\\b\end{pmatrix}=1\Rightarrow 5a-4=1 \quad \therefore a=1$ $\begin{pmatrix}3\\3\\2\end{pmatrix}\cdot\begin{pmatrix}a\\-1\\b\end{pmatrix}=1\Rightarrow 3a-3+2b=1$ Substitute $a=1$, $2b=1 \Rightarrow b=\frac{1}{2}$ With both points lying on q, each position vector must satisfy the equation of q.	
(b) **[3]**	$p:\ \mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\-1\end{pmatrix}+\mu\begin{pmatrix}-2\\1\\2\end{pmatrix},\quad \lambda,\mu\in\mathbb{R}$ A vector perpendicular to $p=\begin{pmatrix}1\\-3\\-1\end{pmatrix}\times\begin{pmatrix}-2\\1\\2\end{pmatrix}=\begin{pmatrix}-5\\0\\-5\end{pmatrix}=-5\begin{pmatrix}1\\0\\1\end{pmatrix}$ Let θ be the acute angle between the planes p and q. $\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}=\sqrt{2}\sqrt{\frac{9}{4}}\cos\theta$ $\frac{3}{2}=\frac{3}{2}\sqrt{2}\cos\theta$ $\cos\theta=\frac{1}{\sqrt{2}}$ $\theta=45°$ The 'exact' angle should be one of those special angles. We see that any vector parallel to $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is perpendicular to both $\begin{pmatrix}1\\-3\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\2\end{pmatrix}$.	<u>**Always check your cross product:**</u> $\begin{pmatrix}1\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=0$ and $\begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=0$, since the cross product MUST be <u>perpendicular</u> to the original 2 vectors used. <u>**OR check using GC**</u> 
(c) **[4]**	Distance from A to q = AF $=\left\lvert\overrightarrow{AQ}\cdot\frac{1}{\sqrt{\frac{9}{4}}}\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}\right\rvert$ $=\frac{2}{3}\left\lvert\left[\begin{pmatrix}5\\4\\0\end{pmatrix}-\begin{pmatrix}2\\0\\3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}\right\rvert$ Use a diagram to guide your working.  	

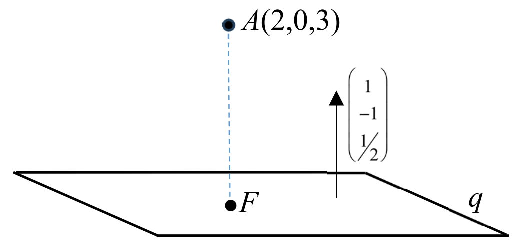
$$= \frac{2}{3}\left|3-4-\frac{3}{2}\right| = \frac{5}{3}$$

Alternative
Solving simultaneously

$l_{AF}: \quad \mathbf{r} = \begin{pmatrix}2\\0\\3\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}$ and

$q: \quad \mathbf{r}\cdot\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix} = 1$,

Always use a diagram to guide your working.



$$\left[\begin{pmatrix}2\\0\\3\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix} = 1$$

$$2+\frac{3}{2}+\alpha\left(1+1+\frac{1}{4}\right)=1$$

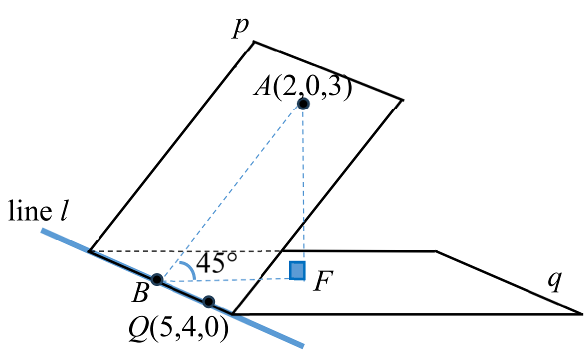
$$\alpha = -\frac{10}{9}$$

In simplifying equation of q, $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix} = 1$ is the same equation as $\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = 2$

Substitute $\alpha = -\frac{10}{9}$ into equation of l_{AF},

$\overrightarrow{OF} = \begin{pmatrix}2\\0\\3\end{pmatrix} + \left(-\frac{10}{9}\right)\begin{pmatrix}1\\-1\\1/2\end{pmatrix}$ and $\overrightarrow{AF} = \left[\begin{pmatrix}2\\0\\3\end{pmatrix} + \left(-\frac{10}{9}\right)\begin{pmatrix}1\\-1\\1/2\end{pmatrix}\right] - \begin{pmatrix}2\\0\\3\end{pmatrix}$

Distance from A to q = $AF = \left|-\frac{10}{9}\begin{pmatrix}1\\-1\\\frac{1}{2}\end{pmatrix}\right| = \left(\frac{10}{9}\right)\left(\frac{3}{2}\right) = \frac{5}{3}$

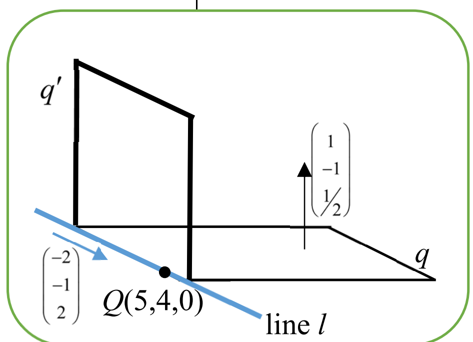
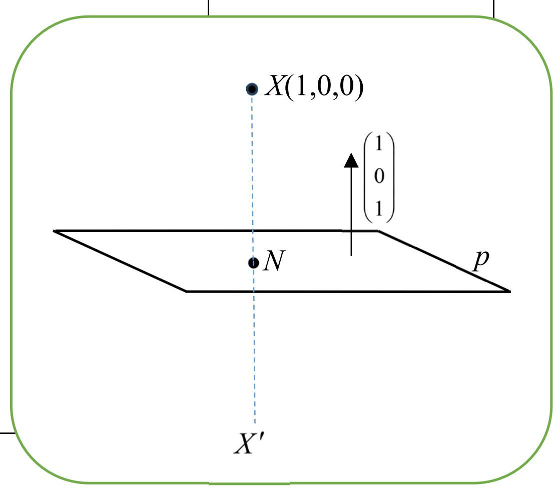


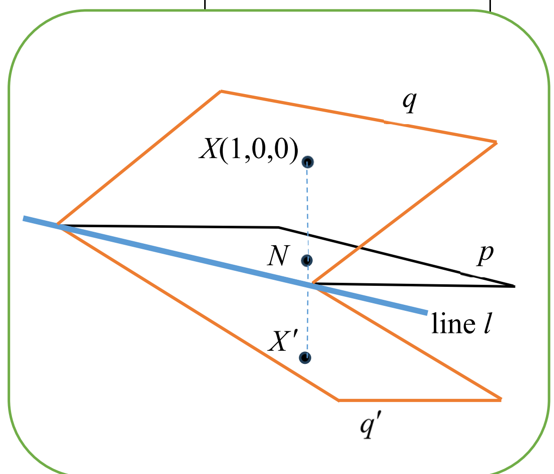
From **part (b)**, the angle between the 2 planes is $45°$.

"**Hence deduce**" indicates that the answer must follow from the distance from A to q.

Note that $A(2,0,3)$ lies on p. This prompts us to bring in plane p and consider how it relates with q.

Common mistakes

	Distance from A to $l = AB = \dfrac{AF}{\sin 45^\circ} = \dfrac{\frac{5}{3}}{\frac{1}{\sqrt{2}}} = \dfrac{5}{3}\sqrt{2}$ **Alternative** Since the angle between the 2 planes is 45°, Distance from A to q = Distance from F to l Hence distance from A to l $= AB = \sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{5}{3}\sqrt{1^2+1^2} = \frac{5}{3}\sqrt{2}$	• Students thought that distance from A to l is AF. Note this can only be true if AF // the normal of q. • Students thought that distance A to l is AQ. This can only be true if $\overrightarrow{AQ} \perp$ direction vector of l.
(d) **[3]**	Since $\theta = 45^\circ$, q and q' are perpendicular. Since l lies on q and q',  A normal to q' = $\begin{pmatrix}-2\\-1\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\ \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\3\\3\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\2\\2\end{pmatrix}$ Equation of q': $\mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}5\\4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = 13$ Cartesian equation of q': $x+2y+2z=13$ **Alternative (If you can't see that $q \perp q'$)** $X(1,0,0)$ is a point on plane q. Let X' be the point of reflection of X about plane p. <u>**To find** X'</u> Solving simultaneously p: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = 5$ and $L_{XX'}$: $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = 5 \Rightarrow 1+2\beta = 5 \Rightarrow \beta = 2$ $\overrightarrow{ON} = \begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix}$  	

	$\overrightarrow{ON}=\dfrac{\overrightarrow{OX}+\overrightarrow{OX'}}{2}\Rightarrow\overrightarrow{OX'}=\begin{pmatrix}5\\0\\4\end{pmatrix}$ **<u>To find equation of q'</u>** A vector parallel to $q'=\begin{pmatrix}5\\0\\4\end{pmatrix}-\begin{pmatrix}5\\4\\0\end{pmatrix}=4\begin{pmatrix}0\\-1\\1\end{pmatrix}$ Since line l lies on q', $q':\quad \mathbf{r}=\begin{pmatrix}5\\4\\0\end{pmatrix}+s\begin{pmatrix}-2\\-1\\2\end{pmatrix}+t\begin{pmatrix}0\\-1\\1\end{pmatrix},\quad s,t\in\mathbb{R}$ A normal to $q'=\begin{pmatrix}-2\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ Equation of q': $\mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}5\\4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=13$ Cartesian equation of q': $x+2y+2z=13$	

10 It is given that

$$f(x)=\begin{cases}\dfrac{1}{2}x^2, & \text{for } 1\le x<2,\\[2mm] \dfrac{1}{4}(-3x+14), & \text{for } 2\le x<4,\end{cases}$$

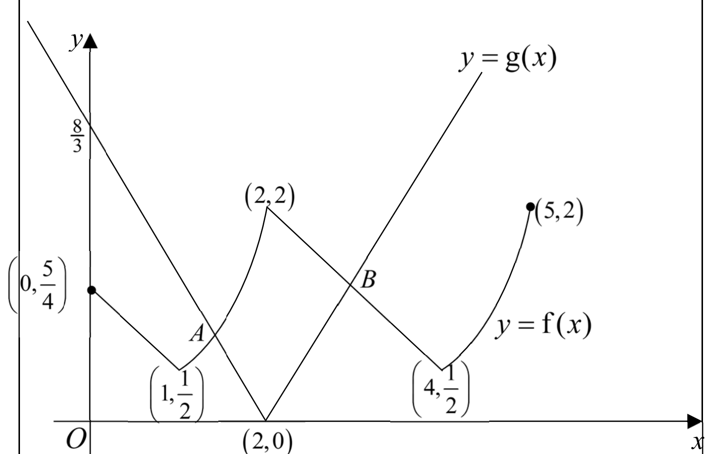
and that $f(x)=f(x+3)$ for all real values of x.

(a) State the value of $f(0)$. [1]

(b) Sketch the graph of $y=f(x)$ for $0\le x\le 5$. [3]

(c) The function g is given by $g(x)=\dfrac{4}{3}|x-2|$ for $x\in\mathbb{R}$. By sketching the graph of $y=g(x)$ on the same diagram as in **part (b)**, solve the inequality $f(x)>g(x)$. [4]

(d) The function h is given by $h(x)=\sqrt{3}\sin(\pi x)+\cos(\pi x)+1$ for $0\le x\le 2$. Explain why the composite function hf exists and find its range in exact form. [4]

	Solution	Comments
(a) **[1]**	By taking $x=0$ in $f(x)=f(x+3)$, $f(0)=f(3)=\dfrac{1}{4}[-3(3)+14]=\dfrac{5}{4}$.	Note that since 3 is in the interval [2,4), to find f(3), we should use $f(x)=\dfrac{1}{4}(-3x+14)$, NOT $f(x)=\dfrac{1}{2}x^2$.
(b) **[3]**		Note that the question asked to sketch the graph for $0\le x\le 5$. Many however sketch the graph from $1\le x\le 4$ only. Note that all extreme points and endpoints **should** be labelled as well.
(c) **[4]**	Finding the x-coordinates of the points of intersection A and B, Point A:	Question was generally well done even if the graphs from (b) were not drawn correctly

<table>
<tr>
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<td>

$$\frac{4}{3}(2-x)=\frac{1}{2}x^2$$
$$3x^2+8x-16=0$$
$$(3x-4)(x+4)=0$$
$$x=\frac{4}{3} \quad \text{or} \quad x=-4 \text{ (reject since } 1<x<2 \text{ at } A)$$

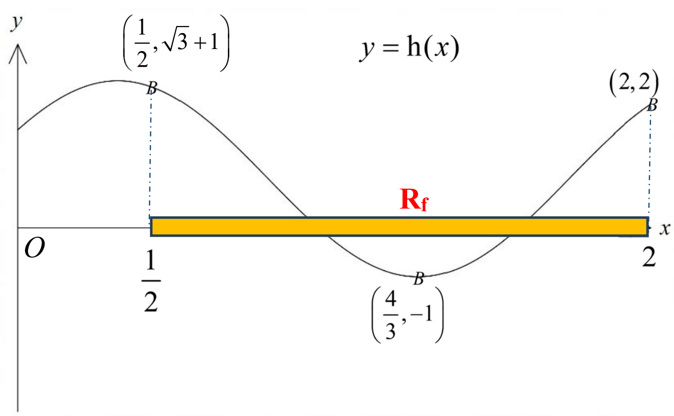
Point B:
$$\frac{4}{3}(x-2)=-\frac{3}{4}x+\frac{7}{2}$$
$$16(x-2)=-9x+42$$
$$x=\frac{74}{25} \text{ or } 2.96$$

Hence the solution to the inequality is $\left(\frac{4}{3},\frac{74}{25}\right)$.

</td>
<td>since the intersection points could be solved using GC.
However, many did not label the y-intercept for the graph of y=g(x).</td>
</tr>
<tr>
<td>(d)
[4]</td>
<td>

From the graph above, $R_f=\left[\frac{1}{2},2\right]$ and $D_h=[0,2]$.

Since $R_f \subseteq D_h$, hf exists.



$$R_f=\left[\frac{1}{2},2\right]\xrightarrow{h}\left[-1,\sqrt{3}+1\right]$$

$$\therefore R_{hf}=\left[-1,\sqrt{3}+1\right]$$

</td>
<td>Range of f must be correctly stated when proving that hf exists.

Another common mistake when finding the range of composite function hf is to claim that it is $\left[h(2),h\left(\frac{1}{2}\right)\right]$. It would be advisable to sketch the graph to determine the correct range. Note that question also asked for range in EXACT form, so there should not be any rounded off values or 3s.f.</td>
</tr>
</table>

11 A financial institution, Future Investments Inc., has introduced a new investment scheme. The scheme pays a compound interest of 5% per annum at the end of the year, based on the amount in the account at the beginning of each year.

John and Sarah are both interested in this scheme.

(a) John invests \$$x$ at the start of the first year and a further \$$x$ on the first day of each subsequent year. He chooses to leave the money in his account for the interest to accumulate.

(i) Write down the amount in John's account, including the interest, at the end of the first year. [1]

(ii) Show that John will have a total of $\$\,21x\left(1.05^n - 1\right)$ in his account at the end of n years. [3]

(iii) If John invests $\$10\,000$ at the start of every year, find the number of years for the total in his account to first exceed $\$500\,000$. Determine if this happens at the start or at the end of that year. [4]

(b) Sarah invests $\$6\,000$ at the start of the first year. On the first day of each subsequent year, she invests $\$400$ more than the amount invested at the start of the previous year.

(i) Explain why the amount in Sarah's account at the end of n years can be given by $\sum_{r=1}^{n}\left[a+b(r-1)\right](1.05)^{n-(r-1)}$, and determine the values of the constants a and b. [2]

(ii) Hence determine the number of complete years for the total amount in her account to first exceed \$500 000. [2]

<table>
<tr><th></th><th>Solution</th><th>Comments</th></tr>
<tr><td colspan="3">For this question, it is especially handy to know the formula for computing total amount obtained after investing $\$P$ for n units of time in an account which awards compound interest at r% per unit time, given by $P\left(1+\frac{r}{100}\right)^n$ -- assumed knowledge</td></tr>
<tr><td>(a)(i)
[1]</td><td>Amount in John's account, including the interest, at the end of the first year is $\$1.05x$.</td><td>Final answer should be simplified. For eg, $x+0.05x$ is not acceptable as it should be simplified to $1.05x$</td></tr>
<tr><td>(ii)
[3]</td><td>
<table>
<tr><th>Year</th><th>Amt at Beginning</th><th>Amt at the End</th></tr>
<tr><td>1</td><td>x</td><td>$1.05x$</td></tr>
<tr><td>2</td><td>$1.05x+x$</td><td>$(1.05x+x)(1.05)$
$=1.05x+1.05^2x$</td></tr>
<tr><td>3</td><td>$1.05^2x+1.05x+x$</td><td>$(1.05^2x+1.05x+x)1.05$
$=1.05x+1.05^2x+1.05^3x$</td></tr>
</table>
Total amount in the account at the end of n years
$=1.05x+1.05^2x+1.05^3x+\ldots+1.05^nx \quad \cdots(*)$
$=\dfrac{1.05x(1.05^n-1)}{1.05-1}$
$=21x(1.05^n-1)$
</td><td>As this part is a "show" question with the final form $21x(1.05^n-1)$ provided, derivation of the series (*) needs to be included in the working to earn full credit.</td></tr>
<tr><td>(iii)
[4]</td><td>
$21(10000)(1.05^n-1)>500000$
$\Rightarrow 1.05^n \geq \dfrac{50}{21}+1$
$\Rightarrow n \geq \dfrac{\ln\left(\frac{71}{21}\right)}{\ln(1.05)} \approx 24.967$

Alternative : Use GC table
<table>
<tr><th>n</th><th>$21(10000)(1.05^n-1)$</th></tr>
<tr><td>24</td><td>$467271 \not> 500000$</td></tr>
<tr><td>25</td><td>$501135>500000$</td></tr>
</table>
At the start of 25th year, total = \$467271 + \$10000 <\$500000
Or total = $\dfrac{501135}{1.05} \approx 477271 < \$500\,000$

The total in his account first exceeds \$500 000 at the <u>end</u> of the <u>25th</u> year.
</td><td>Students should formulate the inequality and solve it. Solving $21(10000)(1.05^n-1)=500000$ and obtaining 24.967 does not necessarily ensure least n is 25.

Marks are awarded for providing the table of values if using the GC method

You are expected to justify your choice of beginning or end of the year</td></tr>
</table>

(b)(i) [2]

Year r	**Amount Sarah invested at the start of year r**	**No. of years interest accumulated**	**Amt accumulated due to this investment at the end of nth year**
1	6000	n	$\$6000(1.05)^n$
2	6000+400(1)	$n-1$	$\$(6000+400)(1.05)^{n-1}$
3	6000+400(2)	$n-2$	$\$[6000+2(400)](1.05)^{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	6000+400(n−1)	1	$\$[6000+400(n-1)](1.05)^{n-(n-1)}$

The $\$\left[6000+400(r-1)\right]$ invested at the start of the rth year will become $\$\left[6000+(400)(r-1)\right](1.05)^{n-(r-1)}$ after n complete years since it accumulated interest for $n-(r-1)$ years.

Alternative

Yr	**Amt at Beginning**	**Amt at the End**
1	6000	(6000)1.05
2	$6000(1.05)+6400$	$6000(1.05)^2+6400(1.05)$
3	$6000(1.05)^2$ $+6400(1.05)+6800$	$6000(1.05)^3$ $+6400(1.05)^2+6800(1.05)$

The total amount at the end of n years will be

$$\sum_{r=1}^{n}[6000+400(r-1)](1.05)^{n-(r-1)}, \text{ where } a=6000, b=400.$$

(b)(ii) [2]

$$\sum_{r=1}^{n}[6000+400(r-1)](1.05)^{n-(r-1)}>500000$$

From GC,

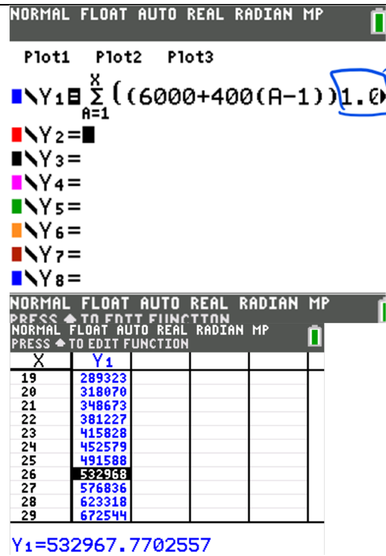
n	$\sum_{r=1}^{n}[6000+400(r-1)](1.05)^{n-(r-1)}$
25	$491588 \ngtr 500000$
26	$532968 > 500000$

It takes 26 complete years.

GC Method 1: Using Y=

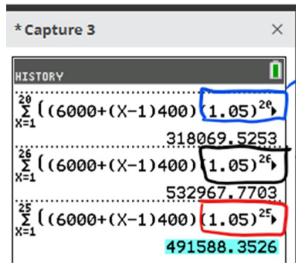
This is a GC exercise (refer to Method 1 and 2 for GC screenshots). Students are not expected to solve it algebraically.

These values obtained from GC are essential in your working and should be included in your solution, together with the right conclusion, to gain full credit.



$\rightarrow 1.05^{x-(A-1)}$

GC Method 2: Guess and Check (



$\rightarrow 1.05^{20-(x-1)}$

$\rightarrow 1.05^{26-(x-1)}$

$\rightarrow 1.05^{25-(x-1)}$

Note:

$$\sum_{r=1}^{n}\left[6000+400(r-1)\right](1.05)^{n-(r-1)}$$

$$\neq \sum_{r=1}^{n}\left[6000+400(r-1)\right]$$

$$\times \sum_{r=1}^{n}(1.05)^{n-(r-1)}$$