NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 2

NAME			
CLASS	2ma2	REGISTRATION NUMBER	

MATHEMATICS 9758/01

Preliminary Examination **09 September 2024**

Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class and registration number in the boxes above.
Please write clearly and use capital letters.

Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing and/or scientific calculator is expected, where appropriate.
All relevant working, statements and reasons must be shown in order to obtain full credit for your solution.

You are reminded of the need for clear presentation in your answers.
Up to 2 marks may be deducted for improper presentation.

The number of marks is given in the brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question Number	Marks Possible	Marks Obtained
1	4	
2	4	
3	5	
4	7	
5	8	
6	8	
7	8	
8	9	
9	10	
10	11	
11	12	
12	14	
Presentation Deduction		– 1 / – 2
TOTAL	100	

This document consists of **7** printed pages.

1 A circular sector has radius r cm and angle θ radians. This sector has area A cm^2 and fixed perimeter k cm.

(i) Show that $\dfrac{\mathrm{d}A}{\mathrm{d}r} = \dfrac{k}{2} - 2r$. [2]

(ii) Given that r is increasing at a constant rate of $\dfrac{k}{10}\,\mathrm{cm\,s^{-1}}$, find in terms of k, the rate at which A is changing when the arc length of the sector is equal to the radius. [2]

2 Two of the roots of the equation $z^3 + az^2 + bz + c = 0$ are $3\mathrm{e}^{\mathrm{i}\left(-\frac{2}{3}\pi\right)}$ and –2.
Given further that a, b and c are integer constants, find the values of a, b and c. [4]

3 **Do not use a calculator to solve this question.**

(i) Solve the inequality $\dfrac{x-6}{4x^2+x-5} \geq 1$. [3]

(ii) Hence solve the inequality $\dfrac{x-6x^2}{4+x-5x^2} \geq 1$. [2]

4 Relative to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors that are not parallel to one another. The points A, B and C are not collinear.

A point of trisection is a point that divides a line segment internally in the ratio $1:2$ or $2:1$. Suppose another two points D and E are points of trisection of line segments AB and AC respectively and both points are nearer to A than to B and C respectively. The lines BE and CD meet at point F.

(i) Show that the vector equations of the lines BE and CD can be expressed as $\mathbf{r} = \dfrac{2}{3}\lambda\mathbf{a} + (1-\lambda)\mathbf{b} + \dfrac{1}{3}\lambda\mathbf{c}$ and $\mathbf{r} = \dfrac{2}{3}\mu\mathbf{a} + \dfrac{1}{3}\mu\mathbf{b} + (1-\mu)\mathbf{c}$ respectively, where λ and μ are parameters. Hence, show that at point F, $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, where α, β and γ are constants, each to be expressed in terms of λ and μ. [4]

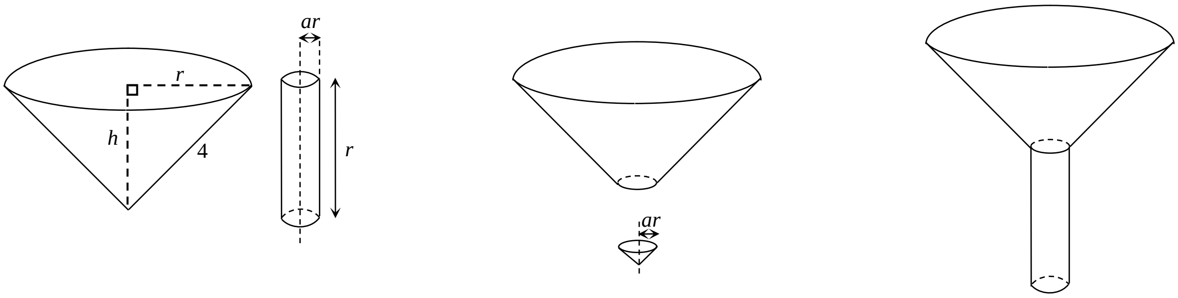
(ii) Given further that $OACB$ is a parallelogram, find the position vector of F in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

5 [The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

A manufacturer makes a funnel-shaped ornament from the same material which consists of two parts as shown in **Figure 1**.

- a right cone of radius r cm, height h cm and a slant height of 4 cm,
- a cylinder with radius ar cm and height r cm, where $0 < a < 1$.

Figure 1 **Figure 2** **Figure 3**



From the original cone, a similar cone with radius ar cm is removed from the vertex as shown in **Figure 2**. The remaining part of the cone is joined to the cylinder to form the funnel as shown in **Figure 3**. It may be assumed that the thickness of the funnel is negligible.

Given that the volume of the ornament is V cm^3, find V in terms of a and r. [3]

For the remainder of this question, assume that $a = 0.25$.

(a) The manufacturer wants V to be a maximum. If $r = r_1$ gives the maximum value of V, show that r_1 satisfies the equation $457r^4 - 9664r^2 + 50176 = 0$. [3]

(b) Show that one of the positive roots to the equation in part **(a)** does not give a stationary value of V. Hence find the value of h for which V is stationary. [2]

6 **(i)** Show that $\dfrac{e^{i\theta}}{1-e^{i\theta}}$ can be expressed as $k\left(i\cot\dfrac{\theta}{2} - 1\right)$, where k is a real constant to be determined exactly. [3]

(ii) Express the complex number i in three equivalent $re^{i\theta}$ forms, where $r > 0$ and $-3\pi < \theta \le 3\pi$. [2]

(iii) Hence find the roots of the equation $\left(\dfrac{w}{w+1}\right)^3 - i = 0$, leaving your answers in the form $k\left(i\cot\phi - 1\right)$, where $-\dfrac{\pi}{2} < \phi \le \dfrac{\pi}{2}$. [3]

7 **(i)** Given that $y = \operatorname{cosec}\left(2x + \frac{\pi}{4}\right)$, show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 8y^3 - 4y$. [3]

(ii) By further differentiation of the result in part **(i)**, find the first four terms of the Maclaurin series for $\operatorname{cosec}\left(2x + \frac{\pi}{4}\right)$ exactly. [3]

(iii) Hence estimate the value of $\operatorname{cosec}\left(\frac{13\pi}{50}\right)\cot\left(\frac{13\pi}{50}\right)$, giving your answer in the form $\sqrt{2}\left(p + q\pi + r\pi^2\right)$, where p, q and r are rational constants to be determined. [2]

8 **(a)** The sum, S_n, of the first n terms of a sequence of numbers $u_1, u_2, u_3, \ldots,$ is given by

$$S_n = An^2 + Bn + 2^{n+1},$$

where A and B are non-zero constants. It is also given that the third term is 21 and the fifth term is 53. Find a simplified expression for u_n in terms of n. [4]

(b) **(i)** Use the method of differences to show that $\sum_{r=1}^{n} \ln\left(\frac{r(r+2)}{(r+1)^2}\right) = \ln\left(\frac{n+2}{n+1}\right) - \ln 2$. [3]

(ii) Hence, find the exact value of $\sum_{r=0}^{n} \ln\left(\frac{r^2 + 4r + 3}{(r+2)^2}\right)$ in terms of n. [2]

9 **(a)** Show that the curve with equation $y = \left(x^2 + cx\right)\mathrm{e}^{-x}$ has two stationary points for all real values of c. [3]

(b) The curves C_1 and C_2 have equations $x^2 + 4y^2 - 6x - 7 = 0$ and $y = \frac{2x-3}{x-1}$ respectively.

Write the equation of C_1 in the form $\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$. Sketch, on the same diagram, both C_1 and C_2, indicating clearly their key features as well as the coordinates of their points of intersection. [7]

10 The function f is defined by

$$f(x) = (x+1)|x+1|, \quad \text{for } x \in \mathbb{R}, -4 < x \leq 2.$$

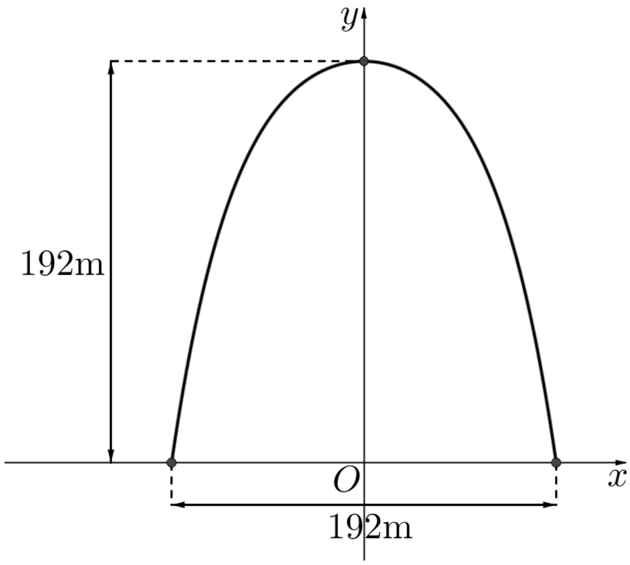
(i) Find f^{-1}. [4]

(ii) On the same diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = ff^{-1}(x)$, labelling clearly the coordinates of the end-points. [4]

(iii) Solve exactly the inequality $f(x) \leq f^{-1}(x)$. [3]

11 The diagram below shows the Gateway Arch, which is a monument in St. Louis, Missouri, United States. The arch stands at 192 metres tall and is 192 metres wide.

The arch can be modelled by part of a curve as shown in the diagram below.



The highest point of the curve lies on the y-axis and the curve is symmetrical about the y-axis. The two endpoints both lie on the x-axis. It is known that the curve satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = ak\sqrt{1+\left(\frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

for some constants a and k.

(i) Show that the substitution $p = \frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}$ reduces the differential equation to

$$\frac{\mathrm{d}p}{\mathrm{d}x} = a\sqrt{1+p^2}.$$ [1]

(ii) By using the substitution $p = \tan u$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$, to solve the reduced differential equation in part **(i)**, show that

$$p = \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{2}.$$ [8]

(iii) Given that $a = -0.0329$ and $k = 0.701$, find y in terms of x. [3]

12 The planes π_1 and π_2, which meet in the line l_1, have equations

$$\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 25 \text{ and } \pi_2 : x + ky - 2z = -15,$$

where k is a constant.

Another line l_2 has equation $\mathbf{r} = \begin{pmatrix} -15 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\beta \in \mathbb{R}$.

(i) Determine the position vector of the point on l_1 such that the coordinates of this point is independent of k. Hence find a vector equation of l_1. [4]

(ii) Determine the possible value(s) of k such that l_1 and l_2 are skew. [3]

Assume that $k = 4$ for the rest of this question.

(iii) Points A and B are on l_1 and l_2 respectively such that $\overrightarrow{AB}$ is perpendicular to both lines. Show that $\left|\overrightarrow{AB}\right| = \sqrt{\frac{p}{2}}$, where p is an integer to be determined. [3]

(iv) Find exactly the sine of the acute angle between l_2 and π_1. [2]

It is given further that l_2 lies on a third plane that is perpendicular to l_1, and l_2 intersects π_1 at point P.

(v) Deduce the shortest distance from P to l_1. [2]