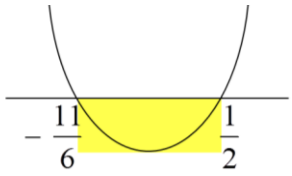
Paya Lebar Methodist Girls' School (Secondary)
Department of Mathematics
2017 Preliminary Examination
Additional Mathematics Paper 1 (4047/1) Worked Solutions

No.	**Answer**
1	State the values between which each of the following must lie (a) the principal value of $\tan^{-1} x$, Principal value of $\tan^{-1} x = -90° \le \tan^{-1} x \le 90°$ Principal value of $\tan^{-1} x = -\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$ (b) the principal value of $\cos^{-1} 2x$. Principal value of $\cos^{-1} 2x = 0° \le \cos^{-1} 2x \le 90°$ Principal value of $\cos^{-1} 2x = 0 \le \cos^{-1} 2x \le \frac{\pi}{2}$
2	The function f is defined, for all values of x, by $f(x) = (x + 3)(1 - 2x)^2$. Find the range of values of x for which f is a decreasing function. $f(x) = (x + 3)(1 - 2x)^2$ $= (x + 3)(1 - 4x + 4x^2)$ $= x - 4x^2 + 4x^3 + 3 - 12x + 12x^2$ $= 4x^3 + 8x^2 - 11x + 3$ $f'(x) = 12x^2 + 16x - 11$ For $f(x)$ to be a decreasing function, $f'(x) < 0$ $12x^2 + 16x - 11 < 0$ $(6x + 11)(2x - 1) < 0$

No.	Answer
	 Range of values of x: $-\frac{11}{6} < x < \frac{1}{2}$
3	In the expansion of $(2x-1)^2\left(1+\frac{p}{x}\right)^8$, where p is a positive constant, there is no term in $\frac{1}{x^3}$. Find the possible values of the constant p. $(2x-1)^2\left(1+\frac{p}{x}\right)^8$ $= \quad (4x^2-4x+1)\left[1+\ldots\binom{8}{3}(1)^5\left(\frac{p}{x}\right)^3+\binom{8}{4}(1)^4\left(\frac{p}{x}\right)^4+\binom{8}{5}(1)^3\left(\frac{p}{x}\right)^5+\ldots\right]$ $\binom{8}{3}p^3+(-4)\binom{8}{4}p^4+(4)\binom{8}{5}p^5=0$ $56p^3-280p^4+224p^5=0$ $56p^3(1-5p+4p^2)=0$ $56p^3(4p-1)(p-1)=0$ $p=0$ (N.A. as $p>0$) OR $p=\frac{1}{2}$ OR $p=1$

No.	Answer
4	A curve has the equation $y = 4x^2 - 24x + 30$.

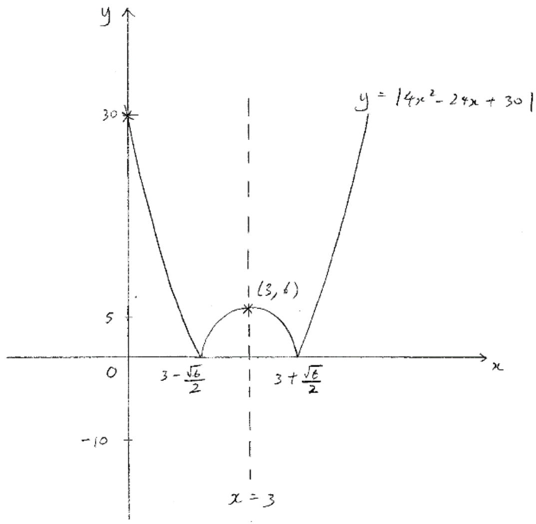
(i) Express $4x^2 - 24x + 30$ in the form $[a\,(x + h)]^2 + k$.

$$4x^2 - 24x + 30 = 4\left(x^2 - 6x + \frac{15}{2}\right)$$
$$= 4\left[(x-3)^2 + \frac{15}{2} - 9\right]$$
$$= 4\left[(x-3)^2 - \frac{3}{2}\right]$$
$$= 4\,(x-3)^2 - 6$$
$$= [2\,(x-3)]^2 - 6$$

(ii) Show that the minimum point of the curve has coordinates $(3, -6)$.

Since $[2\,(x-3)]^2 \geq 0$, the lowest value of
$y = [2\,(x-3)]^2 - 6$ is -6 and this occurs when
$[2\,(x-3)]^2 = 0$, when x has a value of 3. Hence the minimum point on the curve has coordinates $(3, -6)$.

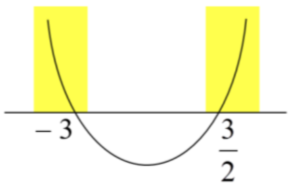
(iii) Sketch the graph of $y = |\,4x^2 - 24x + 30\,|$, indicating clearly the **exact** x-intercept(s) and y-intercept.



- Shape of curve and point $(3, -6)$
- y-intercept
- x-intercepts

No.	Answer
	A line of gradient m passes through the point (0, – 10). (iv) Given that $0 < m \leq 10$, determine the **exact** value of m, for which the line intersects the graph of $y = \| 4x^2 - 24x + 30 \|$ at one real and distinct point. For line to intersect curve at 1 real and distinct point only, the point of intersection is $\left(3+\frac{\sqrt{6}}{2}, 0\right)$. Gradient of line, $m = \frac{0-(-10)}{\left(3+\frac{\sqrt{6}}{2}\right)-0}$ $= 10 \div \left(\frac{6+\sqrt{6}}{2}\right)$ $= 10 \times \left(\frac{2}{6+\sqrt{6}}\right)$ $= \frac{20}{6+\sqrt{6}} \times \frac{6-\sqrt{6}}{6-\sqrt{6}}$ $= \frac{20\left(6-\sqrt{6}\right)}{30}$ $= \frac{2}{3}\left(6-\sqrt{6}\right) / \frac{12-2\sqrt{6}}{3} / 4-\frac{2}{3}\sqrt{6}$

No.	Answer
5	(i) Factorise completely $2x^3 + 7x^2 + 4x - 4$. Let $f(x) = 2x^3 + 7x^2 + 4x - 4$ $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) - 4$ $= 0$ $(2x - 1)$ is a factor of $f(x)$. $f(x) = 2x^3 + 7x^2 + 4x - 4$ $= (2x - 1)(x^2 + 4x + 4)$ $= (2x - 1)(x + 2)^2$ (ii) Express $\dfrac{12x^2 + 32x + 31}{2x^3 + 7x^2 + 4x - 4}$ in partial fractions. $\dfrac{12x^2 + 32x + 31}{(2x-1)(x+2)^2} = \dfrac{A}{2x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ $= \dfrac{A(x+2)^2 + B(2x-1)(x+2) + C(2x-1)}{(2x-1)(x+2)^2}$ $12x^2 + 32x + 31 = A(x + 2)^2 + B(2x - 1)(x + 2) + C(2x - 1)$ When $x = \frac{1}{2}$, $12\left(\frac{1}{2}\right)^2 + 32\left(\frac{1}{2}\right) + 31 = A\left(\frac{5}{2}\right)^2$ $\frac{25}{4}A = 50$ $A = 8$ When $x = -2$, $12(-2)^2 + 32(-2) + 31 = C(-5)$ $-5C = 15$ $C = -3$

No.	Answer
	When $x = 0$, $12(0)^2 + 32(0) + 31 = 8(2)^2 + B(2)(-1) + (-3)(-1)$ $35 - 2B = 31$ $B = 2$ $\frac{12x^2 + 32x + 31}{(2x-1)(x+2)^2} = \frac{8}{2x-1} + \frac{2}{x+2} - \frac{3}{(x+2)^2}$
6	The equation of a curve is $y = ax^2 - 3x + 4 - a$, where a is a constant. (i) In the case where $a = -2$, find the set of values of x for which the curve lies completely below the line $y = -3$. When $a = -2$, $y = -2x^2 - 3x + 6$ If the curve lies completely below the line $y = -3$, $-2x^2 - 3x + 6 < -3$ $2x^2 + 3x - 6 > 3$ $2x^2 + 3x - 9 > 0$ $(2x-3)(x+3) > 0$  Set of values of x: $x < -3$ OR $x > \frac{3}{2}$

No.	Answer
	(ii) In the case where $a = 3$, show that the line $y = 3x - 2$ is a tangent to the curve. When $a = 3$, $y = 3x^2 - 3x + 1$ When the curve and line intersects, $3x^2 - 3x + 1 = 3x - 2$ $3x^2 - 6x + 3 = 0$ $x^2 - 2x + 1 = 0$ $b^2 - 4ac = (-2)^2 - 4(1)(1)$ $= 0$ Since discriminant = 0, the line $y = 3x - 2$ intersects the curve at only 1 real and distinct point. Hence the line is a tangent to the curve. (iii) Determine if there is any other value of a for which the line $y = 3x - 2$ intersects the curve at only one point. When the curve and line intersects, $ax^2 - 3x + 4 - a = 3x - 2$ $ax^2 - 6x + (6 - a) = 0$ For line to intersect curve at only 1 point, $b^2 - 4ac = 0$. $(-6)^2 - 4(a)(6 - a) = 0$ $4a^2 - 24a + 36 = 0$ $a^2 - 6a + 9 = 0$ $(a - 3)^2 = 0$ $a = 3$ [part (ii)] There is only 1 real and repeated root for a, i.e. $a = 3$. Hence there is no other value of a for which the line intersects the curve at one point only.

No.	Answer
7	(i) Prove that $\dfrac{\sin\theta}{(\sec\theta-\tan\theta)(\sin\theta+1)}=\tan\theta.$
	$\dfrac{\sin\theta}{(\sec\theta-\tan\theta)(\sin\theta+1)}$
	$=\dfrac{\sin\theta}{\left(\dfrac{1}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}\right)(\sin\theta+1)}$
	$=\dfrac{\sin\theta}{\left(\dfrac{1-\sin\theta}{\cos\theta}\right)(\sin\theta+1)}$
	$=\dfrac{\sin\theta\cos\theta}{1-\sin^2\theta}$
	$=\dfrac{\sin\theta\cos\theta}{\cos^2\theta}$
	$=\dfrac{\sin\theta}{\cos\theta}$
	$=\tan\theta$
	= RHS (proven)
	(ii) Find all the values of θ between 0 and π for which $\dfrac{\sin\theta}{(\sec\theta-\tan\theta)(\sin\theta+1)}=1-\sec^2\theta.$
	$\dfrac{\sin\theta}{(\sec\theta-\tan\theta)(\sin\theta+1)}=1-\sec^2\theta$
	$\tan\theta=1-\sec^2\theta$
	$\tan\theta=1-(1+\tan^2\theta)$
	$\tan^2\theta+\tan\theta=0$
	$\tan\theta(\tan\theta+1)=0$
	$\tan\theta=0$ (no solution) OR $\tan\theta=-1$

No.	Answer
	Since $\tan\theta < 0$, θ lies in the 2nd quadrant. Consider $\tan\alpha = 1$, $\alpha = \dfrac{\pi}{4}$. $\theta = \pi - \dfrac{\pi}{4}$ $= \dfrac{3\pi}{4}$ rad. / 2.36 rad.
8	An auction house claimed that it is worthwhile to invest in their art pieces as the value of one of their art pieces has been increasing exponentially since it was produced. The value, $\$V$, of this art piece is related to t, the number of years since it was produced at the start of the year 1995. The variables V and t can be modelled by the equation $V = 10\,000 + ae^{kt}$, where a and k are constants. The table below gives values of V and t at the start of some of the years 2000 to 2015. <table><tr><td>Year</td><td>2000</td><td>2005</td><td>2010</td><td>2015</td></tr><tr><td>t years</td><td>5</td><td>10</td><td>15</td><td>20</td></tr><tr><td>$\$V$</td><td>16 000</td><td>20 260</td><td>27 545</td><td>40 000</td></tr></table> (i) Plot a suitable straight line graph to show that the model is valid for the years 2000 to 2015. $V = 10\,000 + ae^{kt}$ $V - 10\,000 = ae^{kt}$ $\ln(V - 10\,000) = \ln a + \ln e^{kt}$ $\ln(V - 10\,000) = kt + \ln a$ Vertical axis: $\ln(V - 10\,000)$ Horizontal axis: t Gradient: k Vertical axis-intercept: $\ln a$

No.	Answer

t years	5	10	15	20
$\$V$	16 000	20 260	27 545	40 000
ln (v – 10 000)	8.70	9.24	9.77	10.31

- $\ln (V - 10\,000) = kt + \ln a$
- Table of values
- Straight line graph with suitable scale

(ii) Estimate the value of a and of k.

$\ln a = 8.1875$

$a = e^{8.1875}$

≈ 3595.72

$= 3600$ (3 s.f.) OR 3596 (nearest whole no.)

$$k = \frac{9.625 - 8.25}{13.75 - 0.75}$$

$$= \frac{11}{104} \text{ / } 0.106 \text{ (3 s.f.)}$$

(iii) A claim was made that in the year 2065, this art piece will increase in value by 500 times from the time it was produced. Do you agree? Justify your answer.

$$V = 10\,000 + 3600e^{\frac{11}{104}t}$$

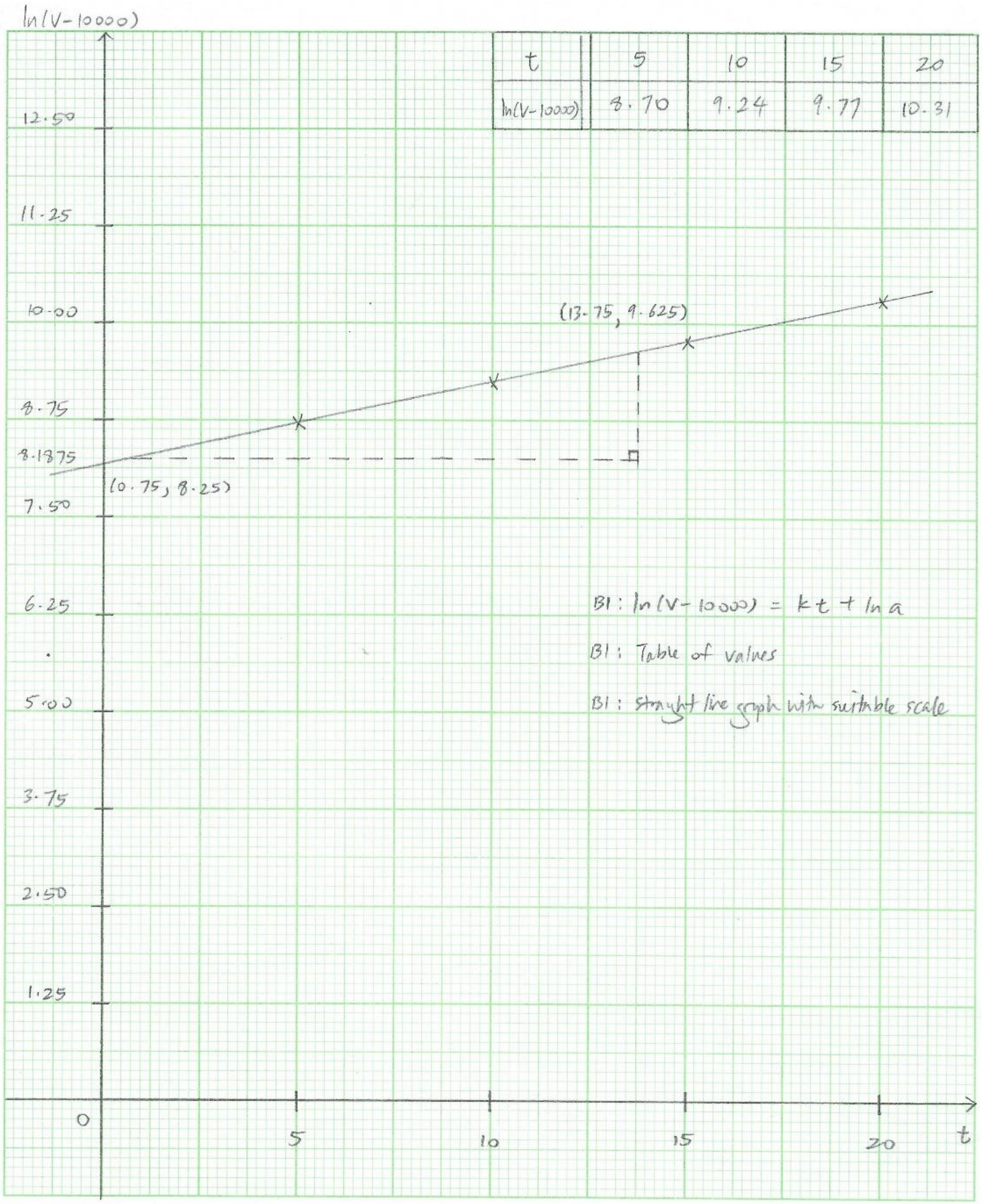
When $t = 70$, $V = 10\,000 + 3600e^{\frac{11}{104}(70)}$

$= 5\,922\,240$ (nearest whole no.)

Original value $= 10\,000 + 3600e^{\frac{11}{104}(0)}$

$= \$13\,600$

No.	Answer
	Increase in value from the time art piece was produced $= \dfrac{5\,922\,240 - 13\,600}{13\,600}$ $=$ 434× (nearest whole no.) No, I do not agree as the increase is less than 500×.



t	5	10	15	20
ln(V−10000)	8.70	9.24	9.77	10.31

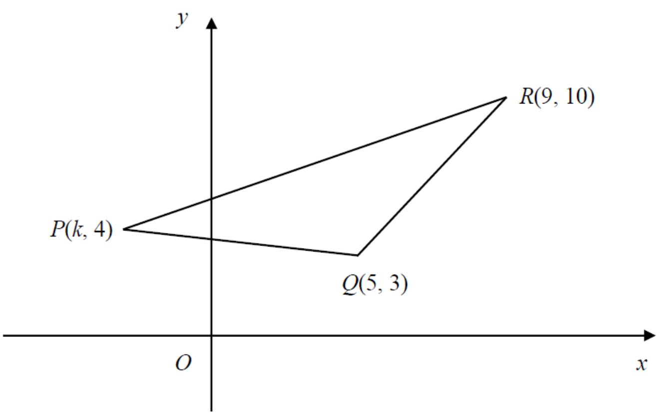
B1 : ln(V − 10000) = kt + ln a

B1 : Table of values

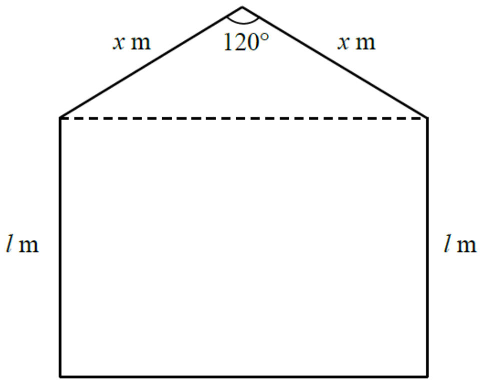
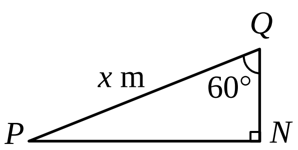
B1 : Straight line graph with suitable scale

2 mm Square 20 cm x 24 cm 40s

No.	Answer
9	The point A lies on the curve $y = x \ln x^2$, $x > 0$. The tangent to the curve at A is parallel to the line $y - 3x = 1$. (i) Find the **exact** coordinates of A. $y = x \ln x^2, x > 0$ $\frac{dy}{dx} = x\left(\frac{1}{x^2}\right)(2x) + \ln x^2\ (1)$ $= 2 + \ln x^2$ Gradient of tangent @ A = gradient of line $= 3$ When $\frac{dy}{dx} = 3$, $2 + \ln x^2 = 3$ $\ln x^2 = 1$ $x^2 = e$ $x = \sqrt{e}\ /\ e^{\frac{1}{2}}\ (x > 0)$ When $x = e$, $y = e^2 \ln e$ $= e^2$ $A = \left(\sqrt{e}, \sqrt{e}\right) / \left(e^{\frac{1}{2}}, e^{\frac{1}{2}}\right)$ The normal to the curve $y = x \ln x^2$ at A meets the line $y - 3x = 1$ at B. (ii) Show that the x-coordinate of B can be expressed in the form $\frac{1}{10}(p\sqrt{e} + q)$, where p and q are integers to be found. Gradient of tangent @ A = 3 Gradient of normal @ A = $-\frac{1}{3}$

No.	Answer
	Equation of normal @ A is $\dfrac{y-\sqrt{e}}{x-\sqrt{e}}=-\dfrac{1}{3}$ $y-\sqrt{e}=-\dfrac{1}{3}x+\dfrac{1}{3}\sqrt{e}$ $y=-\dfrac{1}{3}x+\dfrac{4}{3}\sqrt{e}$ When the normal @ A and the line $y-3x=1$ intersects, $-\dfrac{1}{3}x+\dfrac{4}{3}\sqrt{e}=3x+1$ $\dfrac{10}{3}x=\dfrac{4}{3}\sqrt{e}-1$ $x=\left(\dfrac{4}{3}\sqrt{e}-1\right)\div\dfrac{10}{3}$ $=\dfrac{2}{5}\sqrt{e}-\dfrac{3}{10}$ $=\dfrac{1}{10}\left(4\sqrt{e}-3\right)$
10	The diagram shows an isosceles triangle PQR in which $PQ = QR$. The vertices of the triangle are at the points $P(k, 4)$, $Q(5, 3)$ and $R(9, 10)$. 

No.	Answer
	(i) Find the value of k. Given $PQ = QR$, $\sqrt{(k-5)^2+(4-3)^2}=\sqrt{(9-5)^2+(10-3)^2}$ $(k-5)^2+1=65$ $k^2-10k-39=0$ $(k+3)(k-13)=0$ $k=-3$ OR $k=13$ (N.A. as $k<0$) A line is drawn from Q to cut the y-axis at S such that $PS = SR$. (ii) Find the equation of QS and the coordinates of S. Gradient of $PR = \dfrac{10-4}{9-(-3)}$ $= \dfrac{1}{2}$ Gradient of $QS = -2$ Equation of QS is $\dfrac{y-7}{x-3}=-2$ $y-7=-2x+6$ $y=-2x+13$ $S = (0, 13)$ (iii) Find the ratio of the area of triangle PQR to the area of quadrilateral $PQRS$. Area of triangle $PQR = \dfrac{1}{2}\begin{vmatrix} 5 & 9 & -3 & 5 \\ 3 & 10 & 4 & 3 \end{vmatrix}$ $= \dfrac{1}{2}\left[77-17\right]$ $= 30$ units^2

No.	Answer
	Area of quadrilateral $PQRS = \frac{1}{2}\begin{vmatrix} 0 & -3 & 5 & 9 & 0 \\ 13 & 4 & 3 & 10 & 13 \end{vmatrix}$ $= \frac{1}{2}[158 - 8]$ $= 75$ units^2 Ratio $= 30 : 75$ $= 2 : 5$
11	A farmer uses 160 m of fencing to enclose a plot of his land in a shape that comprises an isosceles triangle and a rectangle, with the dimensions shown.  (i) Show that the area of the plot is $\dfrac{320\sqrt{3}\,x - (3\sqrt{3} + 6)\,x^2}{4}$ m^2. In right-angled triangle PQN,  $\sin 60° = \dfrac{PN}{x}$ $PN = \dfrac{\sqrt{3}}{2}x$ Length of rectangular plot $= \sqrt{3}\,x$ m Perimeter of plot $= 2x + 2l + \sqrt{3}\,x$ m

No.	Answer
	$2x + 2l + \sqrt{3}\,x = 160$ $2l = 160 - 2x - \sqrt{3}\,x$ $l = 80 - x - \dfrac{\sqrt{3}}{2}x$ Area of triangular plot = $\dfrac{1}{2}(x)(x)(\sin 120°)$ $= \dfrac{\sqrt{3}}{4}x^2 \text{ m}^2$ Area of rectangular plot = $\left(\sqrt{3}\,x\right)\left(80 - x - \dfrac{\sqrt{3}}{2}x\right)$ $= 80\sqrt{3}\,x - \left(\sqrt{3} + \dfrac{3}{2}\right)x^2 \text{ m}^2$ Area of plot = $\dfrac{\sqrt{3}}{4}x^2 + 80\sqrt{3}\,x - \left(\sqrt{3} + \dfrac{3}{2}\right)x^2$ $= 80\sqrt{3}\,x - \left(\sqrt{3} + \dfrac{3}{2} - \dfrac{\sqrt{3}}{4}\right)x^2$ $= \dfrac{320\sqrt{3}\,x - \left(4\sqrt{3} + 6 - \sqrt{3}\right)x^2}{4}$ $= \dfrac{320\sqrt{3}\,x - \left(3\sqrt{3} + 6\right)x^2}{4}$ m^2 (shown) (ii) Given that x can vary, find the value of x for which the area of the plot is stationary. Let $A = \dfrac{320\sqrt{3}\,x - \left(3\sqrt{3} + 6\right)x^2}{4}$ $\dfrac{dA}{dx} = \dfrac{320\sqrt{3} - \left(3\sqrt{3} + 6\right)(2x)}{4}$

No.	Answer
	At stationary value, when $\frac{dA}{dx} = 0$, $\frac{320\sqrt{3} - \left(3\sqrt{3}+6\right)(2x)}{4} = 0$ $320\sqrt{3} - \left(3\sqrt{3}+6\right)(2x) = 0$ $\left(3\sqrt{3}+6\right)(2x) = 320\sqrt{3}$ $x = 320\sqrt{3} \div 2\left(3\sqrt{3}+6\right)$ $= \frac{160\sqrt{3}}{3\sqrt{3}+6}$ ≈ 24.752 = 24.8 (3 s.f.) (iii) Explain why this value of x gives the farmer the largest possible area for the plot. Find this area and give your answer correct to the nearest square metre. $\frac{dA}{dx} = \frac{320\sqrt{3} - \left(3\sqrt{3}+6\right)(2x)}{4}$ $\frac{d^2A}{dx^2} = \frac{-\left(3\sqrt{3}+6\right)(2)}{4}$ $= \frac{-\left(3\sqrt{3}+6\right)}{2}$ / – 5.60 (3 s.f.) Since $\frac{d^2A}{dx^2} < 0$, the value of x = 24.8 gives the largest possible are for the plot. Area of plot = $\frac{320\sqrt{3}\,(24.752) - \left(3\sqrt{3}+6\right)(24.752)^2}{4}$ ≈ 1714.8748 = 1715 m^2 (nearest m^2)