National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 2 Solutions)

Basic Mastery Questions

1 **(i)** No. $0x+1\in S$, $0x-1\in S$, but $(0x+1)+(0x-1)=0x+0\notin S$. (A1 fails)

(ii) Yes. (To verify non-empty, A1 and A6 as T is a subset of $\mathbb{R}^3$)

(iii) Yes. (To verify all the 10 axioms)

(iv) Yes. (To verify all the axioms)

(v) No. $1\otimes(1,0,0)=\mathbf{0}$, A10 fails unless $(1,0,0)$ is the zero vector.

$1\otimes(2,0,0)=\mathbf{0}$, A10 fail unless $(2,0,0)$ is the zero vector.

However, there is only one zero vector in the vector space, so it is not a vector space.

(vi) Yes. (To verify all the axioms)

2 (To verify non-emptiness, A1 and A6 as they are subsets of $\mathbb{R}^3$)

(i) $\{(1,-1,2)\}$. **(ii)** $\{(2,1,0),(-3,0,1)\}$.

3 (To verify non-emptiness, A1 and A6 as they are subsets of $\mathbf{P}_2$)

(i) $\{x,1+2x^2\}$ **(ii)** $\{x^2-x-2\}$

4 **(i), (ii), (iv) and (iv):** Yes

(iii) No. $(1,1,0)\in$ the set but $2(1,1,0)=(2,2,0)\notin$ the set as $2(2)-2+5(0)=2\neq 1$.

(v) No. $(0,1,1)\in$ the set but $2(0,1,1)=(0,2,2)\notin$ the set.

(vii) No. An empty set is not a vector space.

(viii) No. $(1,0,1)\in$ the set, $(1,0,-1)\in$ the set but $(1,0,1)+(1,0,-1)=(2,0,0)\notin$ the set.

5 To show linearly independent, let $a(1,0,-1)+b(0,2,0)+c(1,1,1)=(0,0,0)$.

$$\begin{cases}1a & +0b & +1c=0\\0a & +2b & +1c=0\\-1a & +0b & +1c=0\end{cases}\Rightarrow\begin{pmatrix}1&0&1\\0&2&1\\-1&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

. Since the determinant of the coefficient matrix is 4, the coefficient matrix is invertible and there is exactly one solution to the system which is the trivial solution. Thus, the set is linearly independent.

To show spanning, let $p(1,0,-1)+q(0,2,0)+r(1,1,1)=(x,y,z)$.

$$\begin{cases}1p & +0q & +1r=x\\0p & +2q & +1r=y\\-1p & +0q & +1r=z\end{cases}\Rightarrow\begin{pmatrix}1&0&1\\0&2&1\\-1&0&1\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

. Since the determinant of the coefficient matrix is 4, the coefficient matrix is invertible and there is exactly one solution to the system for any vector (x,y,z) in $\mathbb{R}^3$. Thus, the set spans $\mathbb{R}^3$.

(i) Yes

(ii) No. It is not linearly independent as $(0,0,0)=0(1,0,-1)+0(0,2,0)$

(iii) No. It is not linearly independent as $(0,2,0)=0(1,0,-1)+0(1,1,1)+2(0,1,0)$

(iv) No. $(0,0,1)=k(0,2,0)+l(1,1,1)=(l,2k+l,l)$ has no solution, it does not span $\mathbb{R}^3$

6 **(a)** Let $U=\{\mathbf{v}+\mathbf{w}:\mathbf{v}\in V,\mathbf{w}\in W\}$. Since $\mathbf{0}\in V$ and $\mathbf{0}\in W$, $\mathbf{0}+\mathbf{0}=\mathbf{0}\in U$. Thus U is non-empty. $\forall\,\mathbf{u}_1,\mathbf{u}_2\in U$, $\exists\mathbf{v}_1,\mathbf{v}_2\in V$ and $\exists\mathbf{w}_1,\mathbf{w}_2\in W$ s.t. $\mathbf{u}_1=\mathbf{v}_1+\mathbf{w}_1$, $\mathbf{u}_2=\mathbf{v}_2+\mathbf{w}_2$. $\mathbf{u}_1+\mathbf{u}_2=(\mathbf{v}_1+\mathbf{w}_1)+(\mathbf{v}_2+\mathbf{w}_2)=(\mathbf{v}_1+\mathbf{v}_2)+(\mathbf{w}_1+\mathbf{w}_2)$. Since $\mathbf{v}_1+\mathbf{v}_2\in V$ and $\mathbf{w}_1+\mathbf{w}_2\in W$, $\mathbf{u}_1+\mathbf{u}_2\in U$. U is closed under addition.

$\forall\,\mathbf{u}\in U$ and $\forall\,k\in\mathbb{R}$. $\exists\mathbf{v}\in V,\exists\mathbf{w}\in W$ s.t. $\mathbf{u}=\mathbf{v}+\mathbf{w}$. $k\mathbf{u}=k(\mathbf{v}+\mathbf{w})=k\mathbf{v}+k\mathbf{w}$. Since $k\mathbf{v}\in V$ and $k\mathbf{w}\in W$, $k\mathbf{u}\in U$. U is closed under scalar multiplication. Therefore U is a subspace of $\mathbb{R}^n$.

(b) No. Yes. Yes. Yes. (Solution omitted)

7 $U=\{(x,0):x\in\mathbb{R}\}$ and $W=\{(0,y):y\in\mathbb{R}\}$ are both subspaces of $\mathbb{R}^2$.

For $(1,0),(0,1)\in U\cup W$, $(1,0)+(0,1)=(1,1)\notin U\cup W$, so $U\cup W$ is not a vector space.

8(a) **(i)** The column space of $\begin{pmatrix}1&-1&1&0&0\\2&2&0&1&0\\1&1&0&0&1\end{pmatrix}$ is $\mathbb{R}^3$. Performing row operations,

$$\begin{pmatrix}1&-1&1&0&0\\2&2&0&1&0\\1&1&0&0&1\end{pmatrix}\to\begin{pmatrix}1&-1&1&0&0\\0&4&-2&1&0\\0&2&-1&0&1\end{pmatrix}\to\begin{pmatrix}1&-1&1&0&0\\0&1&-\frac{1}{2}&\frac{1}{4}&0\\0&0&0&-\frac{1}{2}&1\end{pmatrix}.$$

Thus a basis is $\{(1,2,1),(-1,2,1),(0,1,0)\}$ (1st, 2nd and 4th columns)

(ii) The column space of $\begin{pmatrix}1&1&1&1&0&0&0\\3&3&0&0&1&0&0\\1&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}$ is $\mathbb{R}^4$. Since the first 3 columns are linearly independent, the leading 1's in a row-echelon form are in the 1st, 2nd, 3rd and 7th columns. Thus, a basis is $\left\{\begin{pmatrix}1&3\\1&0\end{pmatrix},\begin{pmatrix}1&3\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$.

8(b) **(i)** Consider the corresponding spanning set for $\mathbb{R}^3$, $\{(1,1,0),(2,0,1),(0,2,1),(1,1,1)\}$.

The column space of $\begin{pmatrix}1&2&0&1\\1&0&2&1\\0&1&1&1\end{pmatrix}$ is $\mathbb{R}^3$. Performing row operations,

$$\begin{pmatrix}1&2&0&1\\1&0&2&1\\0&1&1&1\end{pmatrix}\to\begin{pmatrix}1&2&0&1\\0&-2&2&0\\0&1&1&1\end{pmatrix}\to\begin{pmatrix}1&2&0&1\\0&1&-1&0\\0&1&1&1\end{pmatrix}\to\begin{pmatrix}1&2&0&1\\0&1&-1&0\\0&0&2&1\end{pmatrix}.$$

Thus, a basis for $\mathbb{R}^3$ is $\{(1,1,0),(2,0,1),(0,2,1)\}$, correspondingly, a basis for $\mathbf{P}_2$ is $\{1+x,2+x^2,2x+x^2\}$.

(ii) $x-2y+z=0$, let $y=\lambda$ and $z=\mu$. Then $x=2\lambda-\mu$. The vector space can be rewritten as $\left\{\begin{pmatrix}2\lambda-\mu\\\lambda\\\mu\end{pmatrix}\right\}=\left\{\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$, so a basis can be $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$

Practice Questions

9 The 2×2 zero matrix $\mathbf{O}$ satisfies $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$, so $\mathbf{O}\in W$. Thus W is non-empty.

$\forall\ \mathbf{B},\mathbf{C}\in W$, $\mathbf{AB}=\mathbf{BA}$ and $\mathbf{AC}=\mathbf{CA}$.

We have $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}=\mathbf{BA}+\mathbf{CA}=(\mathbf{B}+\mathbf{C})\mathbf{A}$, so $\mathbf{B}+\mathbf{C}\in W$. Thus W is closed under addition.

$\forall\ \mathbf{D}\in W$ and $\forall\ k\in\mathbb{R}$, $\mathbf{AD}=\mathbf{DA}$.

We have $\mathbf{A}(k\mathbf{D})=k(\mathbf{AD})=k(\mathbf{DA})=(k\mathbf{D})\mathbf{A}$, so $k\mathbf{D}\in W$. Thus W is closed under scalar multiplication.

Therefore, W is a subspace of $\mathbf{M}_{2,2}(\mathbb{R})$.

10 **(a)** Other than the two extreme cases with dimensions 0 and 2 respectively, the dimension of any subspace of $\mathbb{R}^2$ must be 1, i.e. there is only one vector, say $\mathbf{u}$, in a basis. Thus the subspace is $\{k\mathbf{u}\}$, which is a line through the origin.

(b) Other than the two extreme cases with dimensions 0 and 3 respectively, the dimension of any subspace of $\mathbb{R}^3$ must be 1 or 2.
When there is only one vector in a basis of the subspace, the subspace is a line through the origin (following a similar argument from **(a)**).
When there are exactly two vectors in a basis of the subspace, say **u** and **v**, the subspace is $\{k\mathbf{u}+l\mathbf{v}\}$. As **u** and **v** are non-zero and linearly independent, the subspace is a plane through the origin.

11 $\left\{\begin{pmatrix}1\\a\\b\end{pmatrix},\begin{pmatrix}b\\1\\a\end{pmatrix},\begin{pmatrix}a\\b\\1\end{pmatrix}\right\}$ must not be linearly independent. (If so, it will become a basis as the dimension of $\mathbb{R}^3$ is 3). This means, $x\begin{pmatrix}1\\a\\b\end{pmatrix}+y\begin{pmatrix}b\\1\\a\end{pmatrix}+z\begin{pmatrix}a\\b\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ has non-trivial solution,

so we have $\begin{vmatrix}1 & b & a\\ a & 1 & b\\ b & a & 1\end{vmatrix}=1+a^3+b^3-ab-ab-ab=a^3-3ab+b^3+1=0$.

12 The vector space concerned is the row space of the matrix $\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -10 & 8\\ 7 & -7 & 8\end{pmatrix}$.

$\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -10 & 8\\ 7 & -7 & 8\end{pmatrix}\to\begin{pmatrix}1 & 2 & -1\\ 0 & -7 & 5\\ 0 & -14 & 10\\ 0 & -21 & 15\end{pmatrix}\to\begin{pmatrix}1 & 2 & -1\\ 0 & -7 & 5\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, so a basis is $\{(1,2,-1),(0,-7,5)\}$ and the dimension is 2. (If we wish to preserve the vectors, we may consider the column space of the transpose).

Let $(1,4,a)=\alpha(1,2,-1)+\beta(0,-7,5)$. Solve it, we have $\alpha=1,\beta=-\frac{2}{7},a=-\frac{17}{7}$.

13 $\alpha_1 \neq 0$. Consider the equation $k_1\mathbf{a} + k_2\mathbf{b}_2 + k_3\mathbf{b}_3 = \mathbf{0}$ (*)

Substituting $\mathbf{a} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3$, we have

$$k_1\left(\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \alpha_3\mathbf{b}_3\right) + k_2\mathbf{b}_2 + k_3\mathbf{b}_3 = \mathbf{0}$$

$$k_1\alpha_1\mathbf{b}_1 + \left(k_1\alpha_2 + k_2\right)\mathbf{b}_2 + \left(k_1\alpha_3 + k_3\right)\mathbf{b}_3 = \mathbf{0}$$

This equation has only trivial solution as $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent. i.e.

$$k_1\alpha_1 = k_1\alpha_2 + k_2 = k_1\alpha_3 + k_3 = 0.$$

When $\alpha_1 \neq 0$, $k_1 = 0$, then $0 + k_2 = 0 + k_3 = 0$, we have $k_1 = k_2 = k_3 = 0$. (*) has only the trivial solution thus $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent.

When $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent, suppose $\alpha_1 = 0$, then we can let

$$k_1 = 1, k_2 = -\alpha_2, k_3 = -\alpha_3.$$

This is a non-trivial solution to (*) which contradicts with the condition that $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ are linearly indepdent. Thus $\alpha_1 \neq 0$.

Therefore $\alpha_1 \neq 0$ is a necessary and sufficient condition for $\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3$ to be linearly independent.

Let $\mathbf{x} = \lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3 = \mu_1\mathbf{a} + \mu_2\mathbf{b}_2 + \mu_3\mathbf{b}_3 = \mu_1\alpha_1\mathbf{b}_1 + \left(\mu_1\alpha_2 + \mu_2\right)\mathbf{b}_2 + \left(\mu_1\alpha_3 + \mu_3\right)\mathbf{b}_3$, then $\left(\mu_1\alpha_1 - \lambda_1\right)\mathbf{b}_1 + \left(\mu_1\alpha_2 + \mu_2 - \lambda_2\right)\mathbf{b}_2 + \left(\mu_1\alpha_3 + \mu_3 - \lambda_3\right)\mathbf{b}_3 = \mathbf{0}$. Since $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent, the only solution to this equation is

$$\mu_1\alpha_1 - \lambda_1 = \mu_1\alpha_2 + \mu_2 - \lambda_2 = \mu_1\alpha_3 + \mu_3 - \lambda_3 = 0. \ (**)$$

We have $\mu_1\alpha_1 = \lambda_1 > 0$, so $\alpha_1 > 0$ since $\mu_1 \geq 0$.

From (**), we can see that $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\alpha_1 > 0$ implies $\mu_1 > 0$, but not $\mu_2, \mu_3 > 0$. For example, let $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = \alpha_1 = \alpha_2 = \alpha_3 = 1$,, the equation (**) becomes

$$\mu_1 - 2 = \mu_1 + \mu_2 - 1 = \mu_1 + \mu_3 - 1 = 0.$$

We have $\mu_1 = 2, \mu_2 = \mu_3 = -1 < 0$.

14 Let $U = V \cap W = \{\mathbf{u} : \mathbf{u} \in V \text{ and } \mathbf{u} \in W\}$. Since V and W are subspaces of $\mathbb{R}^n$, $\mathbf{0} \in V$ and $\mathbf{0} \in W$, so $\mathbf{0} \in U$. Thus U is non-empty.

$\forall\, \mathbf{u}_1, \mathbf{u}_2 \in U$, i.e. $\mathbf{u}_1, \mathbf{u}_2 \in V$ and $\mathbf{u}_1, \mathbf{u}_2 \in W$; $\mathbf{u}_1 + \mathbf{u}_2 \in V$ and $\mathbf{u}_1 + \mathbf{u}_2 \in W$. Thus $\mathbf{u}_1 + \mathbf{u}_2 \in U$.

$\forall\, \mathbf{u} \in U$ i.e. $\mathbf{u} \in V$ and $\mathbf{u} \in W$, and $\forall\, k \in \mathbb{R}$; $k\mathbf{u} \in V$ and $k\mathbf{u} \in W$. Thus $k\mathbf{u} \in U$.

Therefore U is a subspace of $\mathbb{R}^n$.

(i) $V \cap W = \left\{\left(x_1, x_2, x_3, x_4\right) \in \mathbb{R}^4 : x_1 = x_3 = 0, x_4 = 2x_2\right\}$. By letting $x_2 = t$, the vector space can be written as $\left\{t\left(0,1,0,2\right)\right\}$, so a basis is $\left\{\left(0,1,0,2\right)\right\}$.

(ii) $\left\{\left(14,-7,0,-6\right),\left(0,0,1,0\right)\right\}$ in a similar way as **(i)**.

(iii) By solving $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & -7 \\ -4 & -2 & 14 \end{pmatrix}\mathbf{x} = \mathbf{0}$, we obtain a basis $\left\{\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}\right\}$.

15 (Verify the 10 axioms, and do not forget to prove that the dimension is 3)

(i) The subset $\{ax^2+bx+c : c=0\}$ is a subspace with a basis $\{x^2, x\}$.

(ii) The subset $\{ax^2+bx+c : c=1\}$ is not a vector space. As the sum of two elements 1 and $x+1$, i.e. $x+2$, is not in this set.

(iii) The subset $\{ax^2+bx+c : a+b+c=0\}$ is a subspace with a basis $\{x^2-1, x-1\}$.

(iv) The subset $\{ax^2+bx+c : b=0\}$ is a subspace with a basis $\{x^2, 1\}$.

16 **(i)** $\{\mathrm{e}^x, \mathrm{e}^{2x}\}$. (Either verify non-emptiness, A1 and A6 directly, or solve the DE)

(ii) It is not a vector space. $2x+3 \in S_2$, but $2(2x+3) = 4x+6 \notin S_2$.

(iii) It is not a vector space. $(0,1) \in S_3$ but $2(0,1) = (0,2) \notin S_3$.

(iv) $\{(-1,1,0), (-1,0,1)\}$. (Either verify non-emptiness, A1 and A6 directly, or solve it)

(v) $\{1, \mathbf{i}\}$. (Verify non-emptiness, A1 and A6).

17 Performing row operations on

$$\begin{pmatrix} 1 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & -2 & 3 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$, so a basis for S is

$\{(1,0,1,1), (5,0,2,2)\}$

Obvious, $\mathbf{0}$ cannot be in the basis and two of the vectors are scalar multiple of another, so a basis for T is $\{(1,-1,0,1), (0,1,1,0)\}$.

Let $\mathbf{u} \in S \cap T$, then $\mathbf{u} = \alpha(1,0,1,1) + \beta(5,0,2,2) = \lambda(1,-1,0,1) + \mu(0,1,1,0)$.

$$\begin{cases} \alpha + 5\beta - \lambda = 0 \\ \lambda - \mu = 0 \\ \alpha + 2\beta - \mu = 0 \\ \alpha + 2\beta - \lambda = 0 \end{cases}$$

Let $\lambda = \mu = k$, then $\alpha + 5\beta = k$ and $\alpha + 2\beta = k$, so $\alpha = k$ and $\beta = 0$.

Now $\mathbf{u} = \alpha(1,0,1,1)$, so a basis for $S \cap T$ is $\{(1,0,1,1)\}$.

Let $\mathbf{v} \in S+T$, then $\mathbf{v} = a(1,0,1,1) + b(5,0,2,2) + c(1,-1,0,1) + d(0,1,1,0)$.

Perform row operations on

$$\begin{pmatrix} 1 & 5 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -3 & -1 & 1 \\ 0 & -3 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so a basis for $S+T$ is $\{(1,0,1,1), (5,0,2,2), (1,-1,0,1)\}$.

Reducing a basis of $\mathbb{R}^4$ from the spanning set

$$\{(1,0,1,1), (5,0,2,2), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

Perform row operations on

$$\begin{pmatrix} 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

. A basis for U (4$^{\text{th}}$ and 5$^{\text{th}}$ columns) is $\{(0,1,0,0),(0,0,1,0)\}$.

Application Problems

18 **(i)** (Verify the 10 axioms, working is omitted)

(ii) Observe that $y=1$, $y=\cos x$ and $y=\sin x$ satisfy the equation, so a linearly independent set of vectors is $\{1,\cos x,\sin x\}$.

(iii) Now $\{1,\cos x,\sin x\}$ becomes a basis since $\dim(V)=3$, so all the solutions in V are in the form $y=C_1+C_2\cos x+C_3\sin x$.

(iv) $0=C_1+C_2\cos\pi+C_3\sin\pi \Rightarrow C_1-C_2=0$. We can verify that

$$W=\{C_1+C_1\cos x+C_3\sin x : C_1,C_3\in\mathbb{R}\}$$

is a subspace of V. (To verify non-emptiness, A1 and A6)

(v) We observe a particular solution $y_p=6x$, so the solution to the differential equation is $y=6x+C_1+C_2\cos x+C_3\sin x$.

19 **(i)** $(2,2)$, $(1,2)$, $(1,1)$.

(ii) $(r,q)=(2,2)$ implies that the two planes intersect in a line.

$(r,q)=(1,2)$ implies that the two planes are parallel.

$(r,q)=(1,1)$ implies that the two planes coincide.

20 **(i)** $(3,3)$, $(2,3)$, $(2,2)$, $(1,2)$, $(1,1)$.

(ii) $(r,q)=(3,3)$ implies that the three planes intersect at exactly one common point, corresponding to Diagram (I).

$(r,q)=(2,2)$ implies that the three planes intersect in exactly one common line, corresponding to Diagrams (III) and (IV).
To further distinguish the cases, we look at whether there exists a row that is a multiple of another in the <u>augmented matrix</u>. It corresponds to Diagram (IV) if so, or to Diagram (III) otherwise.

$(r,q)=(1,1)$ implies that the three planes coincide, corresponding to Digram (VIII).

$(r,q)=(3,2)$ implies that two of the planes intersect the remaining plane in a line each, corresponding to Diagrams (II) and (VI).
To further distinguish the cases, we look at whether there exists a row that is a multiple of another in the coefficient matrix. It corresponds to Diagram (VI) if so, or to Diagram (II) otherwise.
(*We say the 3 planes form a triangular prism for Diagram (II))

$(r,q)=(2,1)$ implies that all planes are parallel and they have no point in common, corresponding to Diagrams (V) and (VII).
To further distinguish the cases, we look at whether there exists a row that is a multiple of another in the augmented matrix. It corresponds to Diagram (V) if so, or to Diagram (VII) otherwise.