NEW TOWN SECONDARY SCHOOL
Preliminary Examination
Secondary 4 Express

NAME	Mark Scheme
CLASS	
INDEX NUMBER	

Additional Mathematics

4049/02

Paper 2

2 August 2021
09:05 – 11:20
2 hours 15 min

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class in the spaces provided above and on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use

Setter: Mr Alan Cheng

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 Express $\dfrac{x^2+2x-19}{(x-1)(x+3)^2}$ as a sum of three partial fractions. [5]

$$\frac{x^2+2x-19}{(x-1)(x+3)^2}=\frac{A}{(x-1)}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$$ [M1

$$x^2+2x-19=A(x+3)^2+B(x-1)(x+3)+C(x-1)$$ [M1]

Sub $x=1$, $-16=16A$
$A=-1$ [A1]

Sub $x=-3$, $-16=-4C$
$C=4$ [A1]

Sub $x=0$, $-19=9A-3B-C$
Sub $A=-1$, $C=4$, $-19=-9-3B-4$
$B=2$ [A1]

$$\frac{x^2+2x-19}{(x-1)(x+3)^2}=-\frac{1}{(x-1)}+\frac{2}{x+3}+\frac{4}{(x+3)^2}$$

2 The table below shows experimental values of two variables x and y.

x	2	4	6	8
y	8.48	5.99	4.90	4.24

It is known that x and y are connected by the equation $yx^n = k$, where k and n are constants.

(a) Plot $\ln y$ against $\ln x$, using a scale of 4 cm for 1 unit on both axes, for the given data and draw a straight line graph on the grid on page 5. [2]

(b) Use your graph to estimate the value of n and of k. [3]

$\ln yx^n = \ln k$

$\ln y + n\ln x = \ln k$

$\ln y = -n\ln x + \ln k$ [M1]

$-n = \dfrac{2.5-0.95}{0-3}$

$n = \dfrac{31}{60}$ Accept 0.517 ± 0.1 [B1]

$\ln k = 2.5$

$k = 12.2$ Accept $k = 12.2 \pm 1.2$ [B1]

(c) Use your graph to **estimate** the value of x when $y = e^2$. [2]

$\ln\left(e^2\right) = 2$

Draw $Y = 2$ [M1]

From graph, $\ln x = 1$

$x = e$ [A1]

(d) On the same diagram, draw the straight line representing the equation $y = x^3$ and hence find the value of x for which $x^{3+n} = k$ [3]

$y = x^3$

Draw $\ln y = 3\ln x$ [B1 – line]

$x^{3+n} = k$

$(3+n)\ln x = \ln k$

$3\ln x + n\ln x = \ln k$

$3\ln x = -n\ln x + \ln k$ [M1]

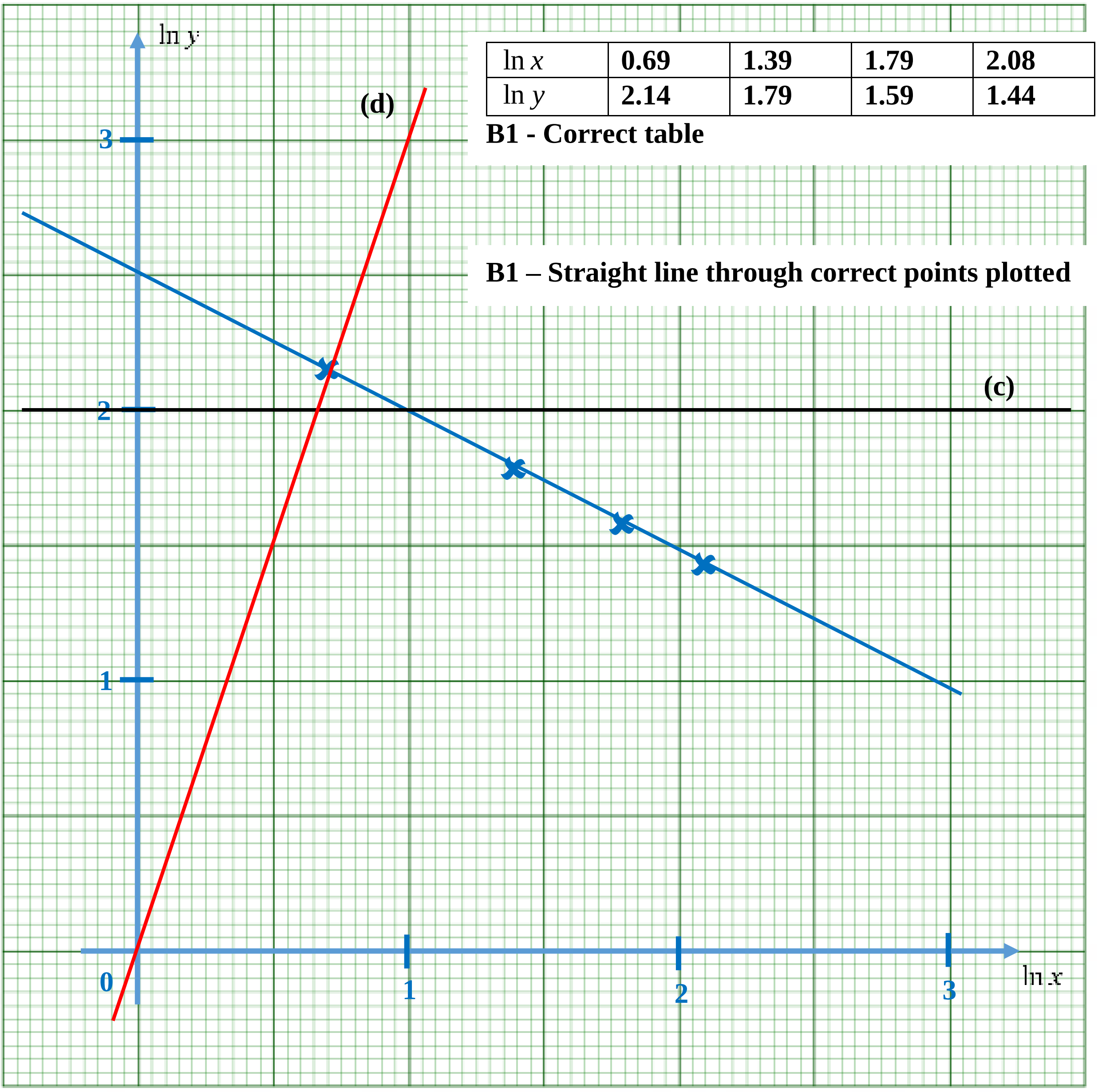
$3\ln x = \ln y$

$\therefore \ln x = 0.7 \Rightarrow x = e^{0.7}$ Accept 2.01 [A1]

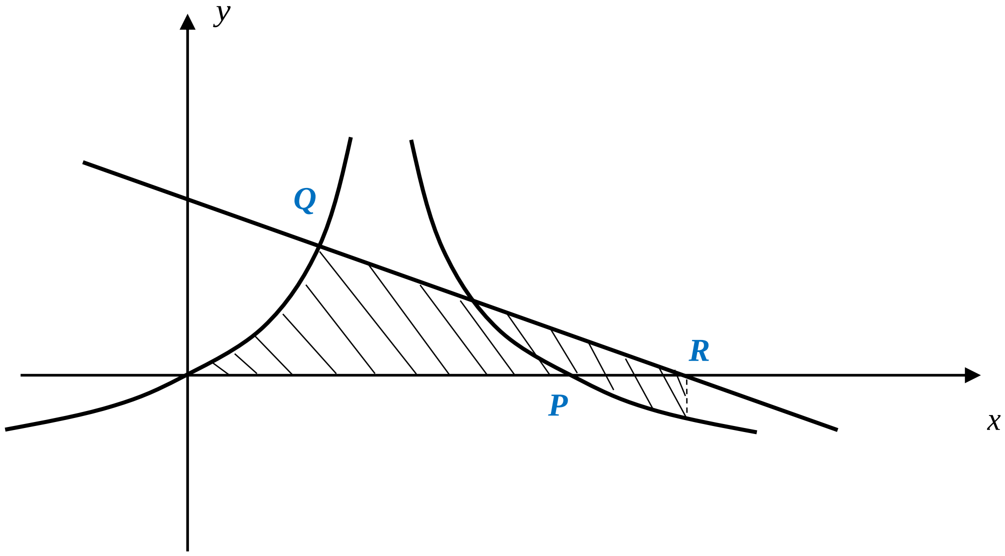
$\ln x$	**0.69**	**1.39**	**1.79**	**2.08**
$\ln y$	**2.14**	**1.79**	**1.59**	**1.44**

B1 - Correct table

B1 – Straight line through correct points plotted



3 The diagram shows part of the curve $y = \dfrac{9}{(3-x)^2} - 1$ which intersects the x-axis at the origin and at the point P. The normal at the point Q on the curve cuts the x-axis at R. The gradient of the curve at Q is 18.



(a) Find the coordinates of Q. [3]

$y = \dfrac{9}{(3-x)^2} - 1$ --- (1)

$\dfrac{dy}{dx} = \dfrac{18}{(3-x)^3}$ [M1]

$\dfrac{18}{(3-x)^3} = 18$ [M1]

$(3-x)^3 = 1$

$x = 2$

Sub $x = 2$ to (1)

$y = \dfrac{9}{(3-2)^2} - 1 = 8$

$Q(2,8)$ [A1]

(b) Find the coordinates of R. [3]

Gradient of $QR = -\frac{1}{18}$ [B1]

Sub $(2,8)$, $y-8=-\frac{1}{18}(x-2)$ [M1]

$y=-\frac{1}{18}x+\frac{73}{9}$

Sub $y=0$

$0=-\frac{1}{18}x+\frac{73}{9}$

$x=146$

$R(146,0)$ [A1]

(c) Find the area of the shaded region. [4]

Sub $y=0$, $0=\frac{9}{(3-x)^2}-1$

$(3-x)^2=9$

$x=3\pm 3$

$x=0,6$

$P(6,0)$ [B1]

$\int_0^2 \frac{9}{(3-x)^2}-1\ dx=\left[\frac{9}{3-x}-x\right]_0^2=4$ [B1]

$\frac{1}{2}(8)(146-2)=576$

$\int_6^{146} \frac{9}{(3-x)^2}-1\ dx=\left[\frac{9}{3-x}-x\right]_6^{146}=-\frac{19600}{143}$ [B1]

Shaded area $=4+576+\frac{19600}{143}=717$ unit^2 (3 s.f) [A1]

4 **(a)** The equation of a curve is $y=\left(b^2-2ac\right)x^2+4\left(a+c\right)x-8$.

(i) Show that the roots of the equation are real if *a*, *b* and *c* are real. [3]

$$\text{Discriminant}=\left[4\left(a+c\right)\right]^2-4\left(b^2-2ac\right)\left(-8\right) \qquad \text{[M1]}$$
$$=16a^2+32ac+16c^2+32b^2-64ac$$
$$=16a^2-32ac+16c^2+32b^2$$
$$=16\left(a^2-2ac+c^2\right)+32b^2$$
$$=16\left(a-c\right)^2+32b^2 \qquad \text{[A1]}$$

For all real values of *a*, *b* and *c*, $\left(a-c\right)^2\geq 0$, $b^2\geq 0$

Since Discriminant $=16\left(a-c\right)^2+32b^2\geq 0$, therefore roots are real.

[A1]

(ii) State the conditions that the roots are equal. [2]

Discriminant $=16\left(a-c\right)^2+32b^2=0$

$a-c=0$

$a=c$ [B1]

$32b^2=0$

$b=0$ [B1]

(b) The equation of a curve is $mx^2-4x+m-3$, where *m* is a constant.
Find the range of values of *m* for which $mx^2-4x+m-3$ is **not** negative for all real values of *x*. [5]

$b^2-4ac\leq 0$

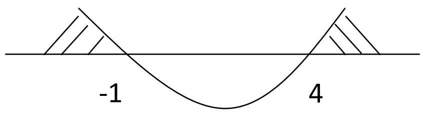
$\left(-4\right)^2-4\left(m\right)\left(m-3\right)\leq 0$ [M1]

$16-4m^2+12m\leq 0$

$4m^2-12m-16\geq 0$

$m^2-3m-4\geq 0$

$\left(m-4\right)\left(m+1\right)\geq 0$ [M1]



[M1]

$m\leq -1$ or $m\geq 4$ [M1]

Since curve is not negative, it must be U-shaped, hence *m* must be positive.
Therefore, $m\geq 4$ [A1]

5 **(a)** Show that $\dfrac{3x}{1-3x}=\dfrac{1}{1-3x}-1$ and hence find $\displaystyle\int \frac{3x}{1-3x}\,dx$. [4]

$$\frac{3x}{1-3x}=\frac{1}{1-3x}-1 \qquad \text{[B1]}$$

$$\int \frac{3x}{1-3x}\,dx=\int \frac{1}{1-3x}-1\,dx$$
$$=-\frac{1}{3}\ln(1-3x)-x+c \qquad \text{[B3 – minus 1 for each error, needs +c]}$$

(b) Given that $y=x\ln(1-3x)$, find an expression for $\dfrac{dy}{dx}$. [2]

$$\frac{dy}{dx}=x\left(\frac{-3}{1-3x}\right)+\ln(1-3x) \qquad \left[\text{M1 for correct } u\frac{dv}{dx} \text{ or } v\frac{du}{dx}\right]$$
$$=\frac{-3x}{1-3x}+\ln(1-3x) \qquad \text{[A1]}$$

(c) Using the results from (a) and (b), find $\displaystyle\int_{-1}^{0}\ln(1-3x)\,dx$. [4]

$$\frac{d}{dx}\left[x\ln(1-3x)\right]=\frac{-3x}{1-3x}+\ln(1-3x)$$
$$\int \ln(1-3x)\,dx=\int\frac{3x}{1-3x}\,dx+x\ln(1-3x) \qquad \text{[M1]}$$
$$\int \ln(1-3x)\,dx=-\frac{1}{3}\ln(1-3x)-x+x\ln(1-3x) \qquad \text{[M1]}$$
$$\int_{-1}^{0}\ln(1-3x)\,dx=\left[-\frac{1}{3}\ln(1-3x)-x+x\ln(1-3x)\right]_{-1}^{0}$$
$$=\left[0-\left(-\frac{1}{3}\ln 4+1+(-1)\ln 4\right)\right] \qquad \text{[M1]}$$
$$=\frac{4}{3}\ln 4-1 \qquad \text{[A1]}$$

6 In Figure 1, $AE = 3$ cm and angle $BAE = x$. Triangle ABE is cut and rotated clockwise about E until AE rests on ED. The resulting shape is shown in Figure 2 below.

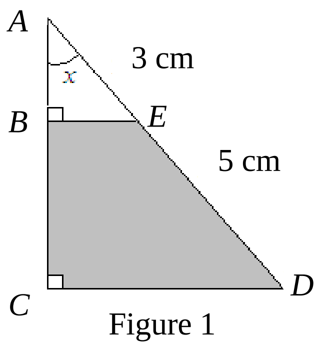


Figure 1

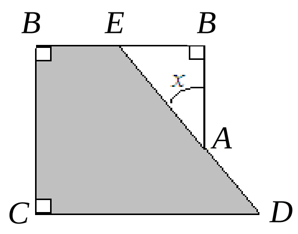


Figure 2

(a) Show that the perimeter, P cm of the resulting shape $BEBADCB$ is given by $P = 14\sin x + 8\cos x + 2$ [4]

$\sin x = \dfrac{BE}{3}$

$BE = 3\sin x$

$\cos x = \dfrac{AB}{3}$

$AB = 3\cos x$ [B1 – correct BE or AB]

$\sin x = \dfrac{CD}{3+5}$

$CD = 8\sin x$

$\cos x = \dfrac{AC}{3+5}$

$AC = 8\cos x$ [B1 – correct CD or AC]

$BC = 8\cos x - 3\cos x = 5\cos x$

$AD = 8 - 3 - 3 = 2$ [B1 – correct BC or AD]

$P = 3\sin x + 3\sin x + 3\cos x + 2 + 8\sin x + 5\cos x$

$\quad = 14\sin x + 8\cos x + 2$ (shown) [A1]

(b) Express $14\sin x + 8\cos x + 2$ in the form of $R\sin(x+\alpha)+2$, where $R>0$ and $0° \le \alpha \le 90°$. [4]
Hence find the maximum value of $14\sin x + 8\cos x + 2$.

$14\sin x + 8\cos x + 2 = R\sin(x+\alpha)+2$

$R = \sqrt{(14)^2 + (8)^2} = \sqrt{260}$ [B1]

$\tan\alpha = \dfrac{8}{14}$ [M1]

$\alpha = 29.745° = 29.7°$ (1d.p)

$14\sin x + 8\cos x + 2 = \sqrt{260}\sin(x+29.7°)+2$ [A1]

Max $14\sin x + 8\cos x + 2$ when $\sin(x+29.7°)=1$

Max $14\sin x + 8\cos x + 2 = \sqrt{260}+2$ [B1]

(c) Find the value of x for which P = 15 cm. [2]

$\sqrt{260}\sin(x+29.745°)+2=15$

$\sin(x+29.745°) = \dfrac{13}{\sqrt{260}}$ [M1]

Basic angle $= 53.729°$

$x + 29.745° = 53.729°, 180° - 53.729°$

$x = 24.0°, 96.5°$ (rej) [A1]

7 The equation of a curve is $y = 4x^2e^{-3x}$.

(a) Find an expression for $\frac{dy}{dx}$ and obtain the exact value of the coordinates of the stationary points of the curve. [6]

$$\frac{dy}{dx} = 4x^2\left(-3e^{-3x}\right) + 8xe^{-3x} \quad \text{[M1]}$$
$$= e^{-3x}\left(8x - 12x^2\right) \quad *$$
$$= 4xe^{-3x}\left(2 - 3x\right) \quad * \text{ [A1 for either *]}$$

$$\frac{dy}{dx} = 0$$
$$4xe^{-3x}\left(2 - 3x\right) = 0 \quad \text{[M1]}$$
$$x = 0, x = \frac{2}{3} \quad \text{[A1]}$$

Sub $x = 0$, $y = 0$

Sub $x = \frac{2}{3}$, $y = 4\left(\frac{2}{3}\right)^2 e^{-3\left(\frac{2}{3}\right)} = \frac{16}{9e^2}$

Stationary points are $\left(0, 0\right)$ and $\left(\frac{2}{3}, \frac{16}{9e^2}\right)$ [A1A1]

(b) Find an expression for $\frac{d^2y}{dx^2}$ and hence the nature of these stationary points. [5]

$$\frac{d^2y}{dx^2} = e^{-3x}\left(8 - 24x\right) + \left(-3\right)e^{-3x}\left(8x - 12x^2\right) \quad \text{[M1]}$$
$$= 4e^{-3x}\left(2 - 12x + 9x^2\right) \quad \text{[A1]}$$

Sub $x = 0$, $\frac{d^2y}{dx^2} = 4e^{-3(0)}\left[2 - 12\left(0\right) + 9\left(0\right)^2\right] = 8 > 0$ [M1]

Sub $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = 4e^{-3\left(\frac{2}{3}\right)}\left[2 - 12\left(\frac{2}{3}\right) + 9\left(\frac{2}{3}\right)^2\right] = -\frac{2}{e^2} < 0$ [M1]

$\left(0, 0\right)$ is Minimum point and $\left(\frac{2}{3}, \frac{16}{9e^2}\right)$ is Maximum point. [A1]

8 $D(-6,4)$, $E(2,4)$ and F are three points on a circle with radius 5 units.
The normal to the circle at F passes through E.

(a) Explain why EF is a diameter of the circle. [1]

EF is the normal at F. Hence EF is perpendicular to the tangent at F.
Since rad $\perp$ tan and E lies on the circle, hence EF is a diameter of circle. [B1]

(b) The centre of the circle is above DE. Find the equation of the circle. [4]

Midpoint of $DE = \left(\frac{-6+2}{2}, \frac{4+4}{2}\right) = (-2,4)$ [M1]

x-coordinate of centre of circle $= -2$

Let coordinates of centre of circle be $(-2, y)$

Radius $= 5$

$OE = 5$

$\sqrt{(2+2)^2 + (4-y)^2} = 5$ [M1]

$16 + 16 - 8y + y^2 = 25$

$y^2 - 8y + 7 = 0$

$(y-7)(y-1) = 0$

$y = 7, y = 1$ (rej since centre is above DE) [A1]

circle center be $C(-2,7)$

Equation of circle is $(x+2)^2 + (y-7)^2 = 25$ [A1]

(c) Find the coordinates of F. [2]

Let coordinates of F be (x, y)

$\left(\frac{x+2}{2}, \frac{y+4}{2}\right) = (-2,7)$ [M1]

$\frac{x+2}{2} = -2$

$x = -6$

$\frac{y+4}{2} = 7$

$y = 10$

$F(-6,10)$ [A1]

(d) Find the equation of the tangent at F. [3]

Gradient of $EF = \frac{10-4}{-6-2} = -\frac{3}{4}$ [M1]

Gradient of tangent at $F = \frac{4}{3}$

Equation of tangent at F is $y-10 = \frac{4}{3}(x+6)$ [M1]

$y = \frac{4}{3}x + 18$ [A1]

(e) Find the coordinates of the two points on the circle which has zero gradient. [2]

The two points whose gradient is zero are at the extreme top of the circle and the extreme bottom of the circle => the *x*-coordinate of these two points is -2.

Hence: $(-2+2)^2 + (y-7)^2 = 25$

$$y-7 = \pm\sqrt{25}$$

$$y = 12, y = 2$$

Therefore the coordinates of the two required points are

$(-2,2)$, [B1]

$(-2,12)$. [B1]

9 **(a)** By using the substitutions $2^x = a$ and $5^x = b$, show that the equation $\dfrac{4^x - 25^x}{10^x + 4^x} = \dfrac{1}{2}$ can be simplified to $a = 2b$, where $a \neq -b$. [4]

$$\frac{\left(2^x\right)^2 - \left(5^x\right)^2}{\left(2^x\right)\left(5^x\right) + \left(2^x\right)^2} = \frac{1}{2} \qquad \text{[M1]}$$

$$\frac{a^2 - b^2}{ab + a^2} = \frac{1}{2}$$

$$2a^2 - 2b^2 = ab + a^2 \qquad \text{[M1]}$$

$$a^2 - ab - 2b^2 = 0$$

$$\left(a + b\right)\left(a - 2b\right) = 0 \qquad \text{[M1]}$$

$a = 2b$ (shown) or $a = -b$ (rej) [A1]

(b) Solve the equation $2\log_2 x - 2\log_x 8 = -1$. [5]

$$2\log_2 x - 2\log_x 8 = -1$$

$$2\log_2 x - \frac{2\log_2 8}{\log_2 x} = -1 \qquad \text{[M1]}$$

$$2\log_2 x - \frac{2(3)}{\log_2 x} = -1$$

Let $u = \log_2 x$

$$2u - \frac{6}{u} = -1 \qquad \text{[M1]}$$

$$2u^2 + u - 6 = 0$$

$$(2u - 3)(u + 2) = 0 \qquad \text{[M1]}$$

$u = \dfrac{3}{2}$ or $u = -2$

$\log_2 x = \dfrac{3}{2}$ or $\log_2 x = -2$ [M1]

$x = 2^{\frac{3}{2}}$ or $x = 2^{-2}$

$x = 2\sqrt{2}$ or $x = \dfrac{1}{4}$ [A1]

(c) Show that the equation $\log_2(2x-6)-\log_2(x-2)=2$ has no real solutions. [3]

$\log_2(2x-6)-\log_2(x-2)=2$

$\log_2 \frac{2x-6}{x-2}=\log_2 2^2$ [M1]

$\frac{2x-6}{x-2}=4$

$2x-6=4x-8$ [M1]

$x=1$

Sub $x=1$ to $\log_2(x-2)$, $\log_2(1-2)=\log_2(-1)=$ undefined.

Hence no real solution. [A1]

End of Paper