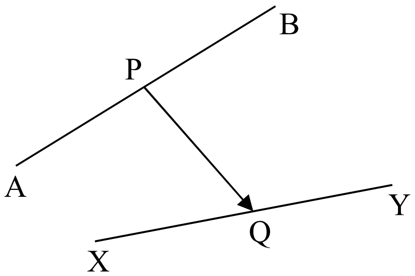
Tutorial 8A: Vectors I (Vectors in Two and Three Dimensions)

Basic Mastery Questions

1. If P is the mid-point of $\overrightarrow{AB}$ and Q is the mid-point of $\overrightarrow{XY}$, show that $2\overrightarrow{PQ} = \overrightarrow{AX} + \overrightarrow{BY}$.



$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AX} + \overrightarrow{XQ}$$
$$= \frac{1}{2}\overrightarrow{BA} + \overrightarrow{AX} + \frac{1}{2}\overrightarrow{XY}$$
$$2\overrightarrow{PQ} = \overrightarrow{BA} + 2\overrightarrow{AX} + \overrightarrow{XY}$$
$$= \overrightarrow{AX} + (\overrightarrow{BA} + \overrightarrow{AX} + \overrightarrow{XY})$$
$$= \overrightarrow{AX} + \overrightarrow{BY} \text{ (shown)}$$

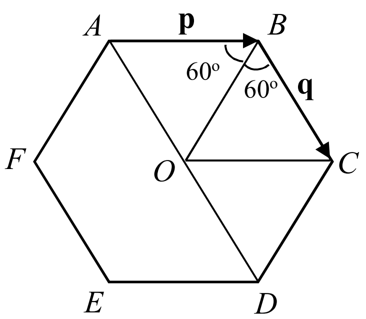
Or

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \frac{1}{2}\left(\overrightarrow{OX} + \overrightarrow{OY}\right) - \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$
$$2\overrightarrow{PQ} = \left(\overrightarrow{OX} + \overrightarrow{OY}\right) - \left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$
$$= \left(\overrightarrow{OX} - \overrightarrow{OA}\right) + \left(\overrightarrow{OY} - \overrightarrow{OB}\right)$$
$$= \overrightarrow{AX} + \overrightarrow{BY} \text{ (shown)}$$

Or

$$\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{PX} + \overrightarrow{PY}\right)$$
$$2\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AX} + \overrightarrow{PB} + \overrightarrow{BY}$$
$$= \overrightarrow{AX} + \overrightarrow{BY} \text{ (shown)}$$

2. $ABCDEF$ is a regular hexagon inscribed in a circle centred at O.
Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, express the following in terms of $\mathbf{p}$ and $\mathbf{q}$:
(i) $\overrightarrow{AD}$, (ii) $\overrightarrow{CD}$, (iii) $\overrightarrow{AE}$, (iv) $\overrightarrow{FA}$.



Each interior angle of the regular hexagon.

$$= \frac{(6-2)180^\circ}{6} = 120^\circ$$

(i) $\overrightarrow{AD} = 2\underset{\sim}{q}$

(ii) $\overrightarrow{CD} = \overrightarrow{BA} + \overrightarrow{AO}$
$= -\underset{\sim}{p} + \underset{\sim}{q}$

(iii) $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$
$= 2\underset{\sim}{q} - \underset{\sim}{p}$

(iv) $\overrightarrow{FA} = \overrightarrow{DC} = \underset{\sim}{p} - \underset{\sim}{q}$

3. A, B, C, D are points with position vectors $\mathbf{j}+2\mathbf{k}$, $-\mathbf{i}-\mathbf{j}$, $4\mathbf{i}+\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ respectively. Prove that the triangle ABC is right-angled and that triangle ABD is isosceles. *[Note: The question can be solved using Dot Product (in Vectors II).]*

$$\underset{\sim}{a}=\begin{pmatrix}0\\1\\2\end{pmatrix},\ \underset{\sim}{b}=\begin{pmatrix}-1\\-1\\0\end{pmatrix},\ \underset{\sim}{c}=\begin{pmatrix}4\\0\\1\end{pmatrix},\ \underset{\sim}{d}=\begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}-\underset{\sim}{a} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix} \Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{1+4+4} = 3$$

$$\overrightarrow{BC} = \underset{\sim}{c}-\underset{\sim}{b} = \begin{pmatrix}5\\1\\1\end{pmatrix} \Rightarrow \left|\overrightarrow{BC}\right| = \sqrt{25+1+1} = \sqrt{27}$$

$$\overrightarrow{AC} = \underset{\sim}{c}-\underset{\sim}{a} = \begin{pmatrix}4\\-1\\-1\end{pmatrix} \Rightarrow \left|\overrightarrow{AC}\right| = \sqrt{16+1+1} = \sqrt{18}$$

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 3^2 + \sqrt{18}^2 = 9 + 18 = 27 = \left|\overrightarrow{BC}\right|^2$$

$\therefore$ triangle ABC is right-angled (shown)

$$\overrightarrow{AD} = \underset{\sim}{d}-\underset{\sim}{a} = \begin{pmatrix}3\\0\\0\end{pmatrix} \Rightarrow \left|\overrightarrow{AD}\right| = 3$$

Since $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right| \Rightarrow$ triangle ABD is isosceles (shown)

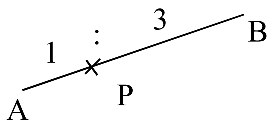
Tutorial Questions

1. The points A, B, C, D have coordinates $(0, 1, 3)$, $(4, 5, -5)$, $(-3, 0, -1)$ and $(7, 5, 4)$ respectively. Show that the point P which divides AB in the ratio $1 : 3$ also divides CD in the ratio $2 : 3$.

P divides AB in the ratio 1:3:

$$\Rightarrow \overrightarrow{OP} = \frac{1}{4}\left(3\overrightarrow{OA} + \overrightarrow{OB}\right)$$

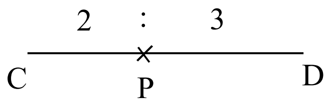
$$= \frac{1}{4}\left[3\begin{pmatrix}0\\1\\3\end{pmatrix} + \begin{pmatrix}4\\5\\-5\end{pmatrix}\right] = \begin{pmatrix}1\\2\\1\end{pmatrix}$$



$$\overrightarrow{CP} = \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}-3\\0\\-1\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix}$$

$$\overrightarrow{PD} = \begin{pmatrix}7\\5\\4\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}6\\3\\3\end{pmatrix}$$

$$\therefore \overrightarrow{PD} = \frac{3}{2}\overrightarrow{CP}$$



$$\Rightarrow \overrightarrow{PD} // \overrightarrow{CP}$$

Hence, P is on CD (since P is a common point) and $CP : PD = 2 : 3$ (shown)

2. $ABCD$ is a parallelogram with E as the midpoint of AB and F lies on DE such that $\overrightarrow{DE} = 3\overrightarrow{FE}$. Prove that A, F and C are collinear and $\overrightarrow{AC} = 3\overrightarrow{AF}$.

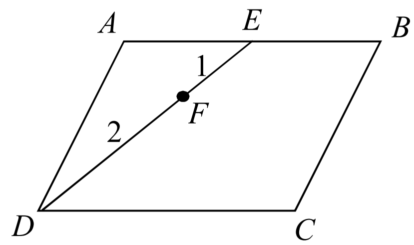
$$\overrightarrow{AF} = \frac{1}{3}\left(\overrightarrow{AD} + 2\overrightarrow{AE}\right)$$

$$= \frac{1}{3}\left(\overrightarrow{AD} + \overrightarrow{AB}\right)$$

$$= \frac{1}{3}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) \quad (\because \overrightarrow{AB} = \overrightarrow{DC})$$

$$= \frac{1}{3}\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AF} // \overrightarrow{AC}$$



$\therefore$ A, F and C are collinear since A is a common point, & $\overrightarrow{AC} = 3\overrightarrow{AF}$ (shown)